Effective Viscosity Properties of Dilute Suspensions of Arbitrarily Shaped Particles *

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Abstract

In this paper we derive high-order asymptotic expansions of the effective viscosity properties of a dilute periodic suspension composed of freely-suspended arbitrarily shaped particles dispersed in an incompressible Newtonian fluid. High-order terms are not only function of the viscous moment tensor but also of a distortion tensor that characterizes the periodic array.

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1 Introduction

In this paper we consider the derivation of macroscale properties of a dilute suspension composed of identical arbitrarily shaped viscous particles dispersed in an incompressible Newtonian fluid from knowledge of its microscopic properties. This problem becomes increasingly important in chemical engineering, polymer science, and biophysics. Analytic calculations of the macroscale properties can be performed for only a limited number of shapes and calculations for bodies of complex shape have required approximation.

In this paper we derive high-order asymptotic expansions of the effective and the intrinsic viscosities of the suspension in the dilute limit in terms of the volume fraction occupied by the particles and their viscous moment tensor. Our derivations follow the layer-potential approach developed in [10, 6, 8, 4] and are valid even when the viscosity of the particles differs significantly from that of the fluid. The concept of viscous moment tensor has been introduced in [3] in connection with small volume expansions for the solution to the modified Stokes system.

The effective viscosity of a suspension is defined to be the four-tensor that relates the average stress to the average rate of strain [34]. We determine the effective viscosity of

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a periodic array of particles in an incompressible Newtonian fluid. High-order asymptotic expansions for low concentrations are obtained for particles of arbitrary shape. These expansions are written in terms of the volume fraction occupied by the particles and their viscous moment tensor.

The effective scalar viscosity is obtained from the effective viscosity tensor after an angular averaging. The derivative of the effective scalar viscosity with respect to the volume fraction is called the intrinsic viscosity [17, 19, 22, 32]. It is invariant under suspension rotation. The mixture of randomly orientated particles behaves as a Newtonian viscous fluid with viscosity equals the effective scalar viscosity of a periodic suspension of the same particles. If in the expansions of the effective scalar viscosity we take spherical particles and their viscosity goes to $+\infty$ with the ambient viscosity fixed, we find the limiting case of a suspension of rigid spherical particles and the first-order approximation of the effective viscosity of the mixture is found to be in agreement with the Einstein formula. A justification of the analyticity of the effective scalar viscosity in the volume fraction together with the high-order asymptotic expansions derived in this paper would suggest that introducing Padé approximations would yield a very accurate algorithm for computing the effective scalar viscosity of suspensions of arbitrarily shaped particles for higher concentrations. The implementation of this algorithm is under consideration and numerical results will be reported elsewhere. In this connection, we refer to [16, 12] for empirical formulae which describe the effective scalar viscosity of a suspension of spherical particles as a function of its volume fraction by extrapolating the Einstein formula to higher concentrations.

The problem considered in this paper has a quite long history. Einstein [20], as part of his investigation of the molecular nature of the matter, first calculated the first-order expansion of the effective scalar viscosity of a dilute hard sphere suspension. Rayleigh [36], Goodier [23], and Hill and Power [27] pointed out a fundamental analogy between the hydrodynamics of suspensions and the elastostatics of incompressible solids with rigid inclusions. This implies that Einstein's expansion also approximates the effective shear modulus of a quasi-incompressible elastic medium containing stiff inclusions at low concentration. We shall revisit this analogy in this paper. Lévy and Sanchez-Palencia [30, formula (2.31)] obtained the first-term in the asymptotic expansion of the effective viscosity tensor for particles of arbitrary shape. Their derivations are based on the general homogenization method developed in [35]. See also the papers by Almog and Brenner [1, 2]. Douglas and Friedman observed an interesting relation between hydrodynamics and electrostatic problems [17]. Batchelor and Green [11] obtained the second-order term in the effective viscosity of a suspension of spherical particles.

One of the reasons of the limited progress in calculating high-order approximations of the effective scalar viscosity of a suspension is the surprising difficulty in solving not only the transmission problem but also the Dirichlet problem for the Stokes system with particles of general shape. Having in hands the concept of viscous moment tensor together with earlier derivations based on layer-potential techniques of high-order expansions of the effective properties of dilute elastic composites, it becomes now possible to solve this problem and find in a simple and systematic way high-order approximations to the effective viscosity properties of a suspension of viscous particles in terms of their volume fraction.

Indeed, our main contribution in this paper is to rigorously derive higher-order approximations of the effective viscosity of dilute suspensions of particles with arbitrary shape and viscosity. Our expansions are in terms of the viscosity moment tensor, a notion that has been recently introduced in [3]. The viscosity moment tensor is the limit of the shear part of the elastic moment tensor as the compression modulus goes to $+\infty$. This has allowed us to relate these calculations to previous ones on estimating effective properties of electrical conductive and elastic composites. We refer to our recent paper [3] for the derivation of some useful relations between the effective viscosity properties, the effective shear modulus, and the effective conductivity of a dilute suspension. High-order terms in our formulae are not only function of the viscous moment tensor, which is a function of the shape and the viscosity of the particle, but also of a distortion tensor that characterizes the periodic array.

The paper is organized as follows. We first recall the definition and useful properties of the elastic moment tensor. Section 3 is devoted to explanation of the notion of viscous moment tensor. In section 4 we provide explicit formulae for the viscous moment tensor for ellipses and ellipsoids and give the corresponding formulae for the intrinsic viscosity. In section 5 we present the derivation of the effective properties of periodic dilute two-phase elastic composites in terms of the elastic moment tensor and the volume fraction occupied by the inclusions in the three-dimensional case. Using the analogy between the hydrodynamics of suspensions and the elastostatics of incompressible solids with rigid inclusions, asymptotic expansions of the effective viscosity tensor for low concentrations can be derived from those for an elastic composite. However, using exactly the same approach as in [8], a more direct derivation of such expansions can be easily written down. This approach is briefly described section 6 and the main modifications to the arguments given in [8] are provided. In the final section we use our asymptotic formula for the effective viscosity to explain how the absorption can be amplified in a composite.

2 Elastic moment tensor

In this section we recall the definition and properties of the elastic moment tensor (EMT) which was first introduced in [9] in connection with the inverse problem of detecting small elastic inclusions from boundary measurements.

Let B be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2$ or 3. Suppose that B and $\mathbb{R}^d \setminus \overline{B}$ are the elastic bodies with the Lamé constants (λ, μ) and $(\widetilde{\lambda}, \widetilde{\mu})$, respectively. The elastostatic system corresponding to the Lamé constants (λ, μ) is defined by

$$\mathcal{L}_{\lambda,\mu}\mathbf{u} := \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u},$$

and the corresponding traction $\partial\mathbf{u}/\partial\nu$ on ∂B is defined to be

$$\frac{\partial\mathbf{u}}{\partial\nu} := \lambda(\nabla \cdot \mathbf{u})N + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)N \quad \text{on } \partial B, \tag{2.1}$$

where N is the outward unit normal to ∂B. Those corresponding to $(\widetilde{\lambda}, \widetilde{\mu})$ are denoted by $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}$ and $\dfrac{\partial}{\partial\widetilde{\nu}}$.

For $p, q = 1, \dots, d$, let $\mathbf{w}_{pq}$ be the solution to

$$\begin{cases} \mathcal{L}_{\lambda,\mu} \mathbf{w}_{pq} = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}} \mathbf{w}_{pq} = 0 \quad \text{in } B, \\ \mathbf{w}_{pq}\big|_{-} = \mathbf{w}_{pq}\big|_{+} \quad \text{on } \partial B, \\ \dfrac{\partial \mathbf{w}_{pq}}{\partial \widetilde{\nu}}\bigg|_{-} = \dfrac{\partial \mathbf{w}_{pq}}{\partial \nu}\bigg|_{+} \quad \text{on } \partial B, \\ \mathbf{w}_{pq}(x) - x_p \mathbf{e}_q = O(|x|^{1-d}) \quad \text{as } |x| \to +\infty, \end{cases} \tag{2.2}$$

where the subscripts $+$ and $-$ indicate the limits from outside and inside of B, respectively. Here and throughout this paper, $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ denotes the standard basis for $\mathbb{R}^d$. The elastic moment tensor $M = (m_{ijpq})$ associated with the domain B and the pairs of Lamé parameters $(\lambda, \widetilde{\lambda}; \mu, \widetilde{\mu})$ is defined by

$$m_{ijpq} = \int_{\partial B} \left[-\frac{\partial(\xi_i \mathbf{e}_j)}{\partial \nu} + \frac{\partial(\xi_i \mathbf{e}_j)}{\partial \widetilde{\nu}} \right] \cdot \mathbf{w}_{pq} \, d\sigma, \quad i, j, p, q = 1, \dots, d. \tag{2.3}$$

The notion of the EMT occurs naturally in various contexts such in asymptotic expansions of the perturbation of the displacement vector due to the presence of small elastic inclusions [9] and in the effective elastic moduli of dilute periodic composites [8]. The EMT enjoys several important properties: It is known that the EMT $M = (m_{ijpq})$ has the following symmetry:

$$m_{ijpq} = m_{pqij} = m_{qpij} = m_{pqji}, \quad i, j, p, q = 1, \dots, d, \tag{2.4}$$

which allows us to identify the EMT with a symmetric linear transformation on the space of symmetric $d \times d$ matrices. It also has a property of positivity (positive or negative definiteness). For the proofs of these properties we refer to [9, 4]. It is worth mentioning that these properties allow the EMT to be an (anisotropic) elasticity tensor after taking a proper sign.

3 Viscous moment tensor

In this section we review the notion of viscous moment tensor (VMT) which was introduced for the asymptotic expansion of the solution to the Stokes system [3]. While the notion of EMT corresponds to the Lamé system, that of VMT corresponds to the Stokes system. Since the Stokes system is a limiting case $(\lambda, \widetilde{\lambda} \to +\infty)$ of the Lamé system, it is natural to realize the VMT as a limit of the EMT, and the VMT inherits all the important properties of EMT.

Suppose that the particle B has the (scalar) viscosity $\widetilde{\mu}$ while the ambient fluid $\mathbb{R}^d \setminus \overline{B}$ has μ. For $x \in \mathbb{R}^d$, let $d(x) := \frac{1}{d} \sum_k x_k \mathbf{e}_k$ and let $\mathbf{v}_{pq}$, $p, q = 1, \dots, d$, be the solution to the following transmission problem for the Stokes system:

$$\begin{cases} \mu\Delta \mathbf{v}_{pq} + \nabla p = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \widetilde{\mu}\Delta \mathbf{v}_{pq} + \nabla p = 0 \quad \text{in } B, \\ \mathbf{v}_{pq}|_- - \mathbf{v}_{pq}|_+ = 0 \quad \text{on } \partial B, \\ (p\mathbf{N} + \widetilde{\mu}\dfrac{\partial \mathbf{v}_{pq}}{\partial \mathbf{N}})\Big|_- - (p\mathbf{N} + \mu\dfrac{\partial \mathbf{v}_{pq}}{\partial \mathbf{N}})\Big|_+ = 0 \quad \text{on } \partial B, \\ \nabla\cdot \mathbf{v}_{pq} = 0 \quad \text{in } \mathbb{R}^d, \\ \mathbf{v}_{pq}(x) - x_p\mathbf{e}_q + \delta_{pq}d(x) = O(|x|^{1-d}) \quad \text{as } |x| \to +\infty, \\ p(x) = O(|x|^{-d}) \quad \text{as } |x| \to +\infty, \end{cases} \tag{3.1}$$

where the conormal derivative $\frac{\partial \mathbf{v}}{\partial \mathbf{N}}$ is given by

$$\frac{\partial \mathbf{v}}{\partial \mathbf{N}} := (\nabla \mathbf{v} + \nabla \mathbf{v}^T)\mathbf{N}, \tag{3.2}$$

and δ_{pq} is the Kronecker's delta. Since $x_p\mathbf{e}_q - \delta_{pq}d(x)$ is divergence free, there is a unique solution to (3.1). The VMT $V = (v_{ijpq})_{i,j,p,q=1,\dots,d}$ associated with the particle B and the viscosity pair $(\mu, \widetilde{\mu})$ is defined by

$$v_{ijpq} = (\widetilde{\mu} - \mu)\int_B \nabla \mathbf{v}_{pq} : (\nabla(x_i\mathbf{e}_j) + \nabla(x_i\mathbf{e}_j)^T)\, dx. \tag{3.3}$$

It is helpful to notice the similarity between the definitions (3.3) and (2.3). Note that we have used the standard notation of the contraction: For matrices $a = (a_{ij})$ and $b = (b_{ij})$, $a : b = \sum_{i,j} a_{ij}b_{ij}$.

Let $\mathbf{w}_{pq}$ and $\mathbf{v}_{pq}$ be the solutions to (2.2) and (3.1), respectively. We have proved in [3] that

$$\mathbf{v}_{pq} = \lim_{\lambda, \widetilde{\lambda} \to +\infty} (\mathbf{w}_{pq} - \frac{\delta_{pq}}{d}\sum_k \mathbf{w}_{kk}), \tag{3.4}$$

in a proper (locally) H^1-norm where the limit is taken under the assumption that $\lambda/\widetilde{\lambda} = O(1)$. ($H^1$ standing for the usual Sobolev space of order one).

In particular, the following convergence holds:

$$\left\| \nabla\left(\mathbf{v}_{pq} - \mathbf{w}_{pq} + \frac{\delta_{pq}}{d}\sum_k \mathbf{w}_{kk} \right) \right\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \lambda, \widetilde{\lambda} \to +\infty. \tag{3.5}$$

The relation (3.4) basically says that the solution to the Stokes system is the limit of the solution to the Lamé system. When $p = q$, the correction on the right-hand side (3.4) is required since $\mathbf{w}_{pp}$ is not divergence free. Since

$$-\frac{\partial(x_i\mathbf{e}_j)}{\partial \nu} + \frac{\partial(x_i\mathbf{e}_j)}{\partial \widetilde{\nu}} = (\widetilde{\lambda} - \lambda)\nabla\cdot(x_i\mathbf{e}_j)\,\mathbf{N} + (\widetilde{\mu} - \mu)\frac{\partial(x_i\mathbf{e}_j)}{\partial \mathbf{N}},$$

it follows from the divergence theorem that

$$m_{ijpq} - \frac{\delta_{ij}}{d}\sum_k m_{kkpq} = (\widetilde{\mu} - \mu)\int_B \nabla \mathbf{w}_{pq} : (\nabla(x_i\mathbf{e}_j - \delta_{ij}d(x)) + \nabla(x_j\mathbf{e}_i - \delta_{ij}d(x))^T). \tag{3.6}$$

Combining (3.4) and (3.6), we immediately get

$$v_{ijpq} = \lim_{\lambda,\widetilde{\lambda}\to\ +\infty} \left(m_{ijpq} - \frac{\delta_{ij}}{d}\sum_{k=1}^{d} m_{kkpq} - \frac{\delta_{pq}}{d}\sum_{s=1}^{d} m_{ijss} + \frac{\delta_{ij}\delta_{pq}}{d^2}\sum_{k,s=1}^{d} m_{kkss} \right) \tag{3.7}$$

for $i, j, p, q = 1, \ldots, d$, as was done in [3]. The limit is still taken under the assumption that $\lambda/\widetilde{\lambda} = O(1)$.

Formula (3.7) can be written in a condensed form: Let $P = (P_{ijpq})$ be the orthogonal projection from the space of symmetric $d \times d$ matrices onto the space of symmetric matrices of trace zero, *i.e.,*

$$P_{ijpq} = \frac{1}{2}(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) - \frac{1}{d}\delta_{ij}\delta_{pq}. \tag{3.8}$$

Then (3.7) is equivalent to

$$V = \lim_{\lambda,\widetilde{\lambda}\to+\infty} PMP. \tag{3.9}$$

We can also see that

$$V = \lim_{\lambda=\widetilde{\lambda}\to+\infty} PM = \lim_{\lambda=\widetilde{\lambda}\to+\infty} PM. \tag{3.10}$$

The following properties of the VMT are inherited from those of the EMT and (3.9).

Theorem 3.1 (i) *$v_{ijpq} = v_{pqij} = v_{qpij} = v_{pqji}$ for $i, j, p, q = 1, \ldots, d$.*

(ii) *$\sum_p v_{ijpp} = 0$ for all i, j and $\sum_i v_{iipq} = 0$ for all p, q, or equivalently $V = PVP$.*

(iii) *V is positive (negative, resp.) definite on the space of symmetric matrices of trace zero if $\widetilde{\mu} > \mu$ ($\widetilde{\mu} < \mu$, resp.)*

Let us now derive a rotation formula for the VMT which might be useful in dealing with suspensions of Brownian particles [22]. Let R be a unitary transformation and let $\hat{B} = R(B)$. We seek a relation between the VMT V of B and the VMT $\hat{V}$ of $\hat{B}$.

Let X be the space of $d \times d$ symmetric matrices of trace zero. Then Theorem 3.1 says that V is a nonsingular symmetric linear transformation on X. Let $d^* = \frac{d(d+1)}{2} - 1$. There are eigen-matrices $A_1, \ldots, A_{d^*}$ such that

$$V = \sum_{j=1}^{d^*} A_j \otimes A_j. \tag{3.11}$$

The following lemma holds.

Lemma 3.2 *Let $\hat{B} = R(B)$, where R is a unitary transformation. Let V and $\hat{V}$ be the VMTs associated with B and $\hat{B}$, respectively, and the same viscosity $\widetilde{\mu}$. We have*

$$\hat{V} = \sum_{j=1}^{d^*} RA_jR^T \otimes RA_jR^T, \tag{3.12}$$

where $A_j, j = 1, \ldots, d^*$, are defined by (3.11).

Since (3.12) holds for EMTs (see [9, 4]), (3.12) for VMTs follows from (3.9).

4 VMT and the intrinsic viscosity for ellipses and ellipsoids

We now write down the VMTs for ellipses and ellipsoids. In [7], EMTs for ellipses and ellipsoids are computed explicitly using a layer potential technique. Thus using (3.9), we can calculate the corresponding VMTs. If B is an ellipse of the form

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad a \geq b > 0, \tag{4.1}$$

then the VMT for B is given by

$$\begin{cases} v_{1111} = v_{2222} = -v_{1122} = -v_{2211} = |B| \dfrac{2\mu(\tilde{\mu} - \mu)}{\mu + \tilde{\mu} - (\tilde{\mu} - \mu)m^2}, \\ v_{1212} = v_{2112} = v_{1221} = v_{2121} = |B| \dfrac{2\mu(\tilde{\mu} - \mu)}{\mu + \tilde{\mu} + (\tilde{\mu} - \mu)m^2}, \\ \text{the remaining terms are zero,} \end{cases} \tag{4.2}$$

where the volume of B, $|B| = \pi ab$ and $m = \frac{a-b}{a+b}$.

If B is a ball in three dimensions, we can find the VMT for any $\widetilde{\mu}$:

$$\begin{cases} v_{iiii} = \dfrac{20\mu|B|}{3} \dfrac{\widetilde{\mu} - \mu}{2\widetilde{\mu} + 3\mu}, \quad v_{iijj} = -\dfrac{10\mu|B|}{3} \dfrac{\widetilde{\mu} - \mu}{2\widetilde{\mu} + 3\mu} \ (i \neq j), \\ v_{ijij} = v_{ijji} = 5\mu|B| \dfrac{\widetilde{\mu} - \mu}{2\widetilde{\mu} + 3\mu}, \ (i \neq j), \\ \text{the remaining terms are zero.} \end{cases} \tag{4.3}$$

The VMT for ellipsoids can be computed from the EMT computed in [7], but the formula involves too many terms to be written down. Instead, we write down the VMT for ellipsoids when $\widetilde{\mu} = +\infty$, which means that the viscosity of the particle is infinite and hence the particle is solid. Before writing down the formula for the VMT, let us briefly mention about the transmission conditions (the third and fourth equations in (3.1)) when $\widetilde{\mu} = +\infty$. In this case, the fourth condition in (3.1) becomes $\frac{\partial \mathbf{v}_{pq}}{\partial \mathbf{N}}|_- = 0$, and hence $\mathbf{v}_{pq} = \mathbf{a} \times x + \mathbf{b}$ in B for some constant vectors $\mathbf{a}$ and $\mathbf{b}$. Then by the third equation, we have the following boundary condition for the exterior problem:

$$\mathbf{v}_{pq} = \mathbf{a} \times x + \mathbf{b} \quad \text{on } \partial B. \tag{4.4}$$

Suppose that B is an ellipsoid given by

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} = 1,$$

and its viscosity $\widetilde{\mu} = +\infty$. Let, for $i, j = 1, \ldots, d$,

$$s_{ij} = -\frac{a_1 a_2 a_3 (a_i^2 + a_j^2)}{2} \int_0^{+\infty} \frac{ds}{(a_i^2 + s)(a_j^2 + s) g(s)},$$

$$t_i = a_1 a_2 a_3 \int_0^{+\infty} \frac{(a_i^2 + s) s ds}{[g(s)]^3},$$

where $g(s) = \sqrt{(a_1^2+s)(a_2^2+s)(a_3^2+s)}$. Then the VMT for B is given by

$$\begin{cases} v_{iiii} = \dfrac{4\mu|B|}{9}\dfrac{(t_1+t_2+t_3)+3t_i}{t_1t_2+t_1t_3+t_2t_3}, \\ v_{iijj} = \dfrac{4\mu|B|}{9}\dfrac{(t_1+t_2+t_3)-3t_i-3t_j}{t_1t_2+t_1t_3+t_2t_3}, \quad i \neq j, \\ v_{ijij} = v_{ijji} = -\mu|B|s_{ij}^{-1}, \quad i \neq j, \\ \text{the remaining terms are zero.} \end{cases} \tag{4.5}$$

We note that formula (4.5) was also obtained in [13].

Before completing this section, let us compute the intrinsic viscosity. We denote the intrinsic viscosity by $[\mu]_{B,\widetilde{\mu}}$ to signify its dependence on B and $\widetilde{\mu}$. When $\widetilde{\mu} = +\infty$, we simply denote it by $[\mu]_B$. The definition of the intrinsic viscosity will be given in the last section where we compute the intrinsic viscosity of a dilute suspension of particles. Our computation will show that

$$[\mu]_{B,\widetilde{\mu}} = \frac{\mathrm{tr}(V)}{2\mathrm{tr}(P)} \tag{4.6}$$

where V is the VMT associated with B (and $\mu, \widetilde{\mu}$). Recall that $\mathrm{tr}(C)$ for a 4-tensor $C = (C_{ijpq})$ is given by $\mathrm{tr}(C) = \sum_{i,j=1}^d C_{ijij}$. Note that $\mathrm{tr}(P) = 2$ in two dimensions and $\mathrm{tr}(P) = 5$ in three dimensions.

Assuming $|B| = 1$, we obtain the following list of intrinsic viscosities from (4.2), (4.5), and (4.3).

- If B is an ellipse:
$$[\mu]_{B,\widetilde{\mu}} = \frac{\mu(\widetilde{\mu}-\mu)(\widetilde{\mu}+\mu)}{(\widetilde{\mu}+\mu)^2 - (\widetilde{\mu}-\mu)^2 m^4}. \tag{4.7}$$
In particular,
$$[\mu]_B = \frac{\mu}{1-m^4}. \tag{4.8}$$
- If B is an ellipsoid:
$$[\mu]_B = \frac{\mu}{10}\left[\frac{8}{3}\frac{t_1+t_2+t_3}{t_1t_2+t_1t_3+t_2t_3} - \sum_{i\neq j}\frac{1}{s_{ij}}\right]. \tag{4.9}$$
- If B is a ball:
$$[\mu]_{B,\widetilde{\mu}} = 5\mu\frac{\widetilde{\mu}-\mu}{2\widetilde{\mu}+3\mu}. \tag{4.10}$$
In particular,
$$[\mu]_B = \frac{5\mu}{2}. \tag{4.11}$$

Note that the intrinsic viscosity for a spherical particle (4.11) coincides with the one derived by Einstein [20]. We also note that there is discrepancy between (4.8) and the one derived in [17]. This discrepancy occurs because in [17] the Brownian motion effect is included and hence the boundary condition is set to be zero instead of (4.4), which is the correct one.

5 Effective elasticity tensor

In this section we consider the effective property of a periodic elastic dilute composite, which was obtained for two dimensions in [8]. We derive a similar formula for three dimensions using the same method as in the afore mentioned paper. We will be brief and only indicate main differences.

The periodic structure is given by the unit cell $Y =]-1/2, 1/2[^d$ and the inclusion $D = \rho B$, where B is a reference Lipschitz bounded domain containing 0 whose volume, $|B|$, is 1. Denote $f = \rho^d$ the volume fraction of the inclusion. Suppose that both D and $Y \setminus D$ are isotropic whose Lamé constants are $(\widetilde{\lambda}, \widetilde{\mu})$ and (λ, μ), respectively. So, the elasticity tensor for $Y \setminus D$ and D are given by $\tilde{C}$ and C^0, respectively, where

$$C^0_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$
$$\tilde{C}_{ijkl} = \widetilde{\lambda}\delta_{ij}\delta_{kl} + \widetilde{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Moreover, the elasticity tensor C for the composite is periodic with the periodic cell Y and given on Y by

$$C = C^0\chi(Y \setminus D) + \tilde{C}\chi(D). \tag{5.1}$$

where $\chi(D)$ is the characteristic function of D.

We consider the problem of determining the effective elastic properties of the composite with the elasticity tensor $C(x/\epsilon)$ as $\epsilon \to 0$. The effective property is defined as follows: Let $\mathbf{w}_{lk}$ be the solution to the following equation:

$$\begin{cases} \nabla \cdot (C\mathcal{E}(\mathbf{w}_{lk})) = 0 \quad \text{in } Y, \\ \mathbf{w}_{lk} - x_k\mathbf{e}_l \quad \text{is periodic,} \\ \displaystyle\int_Y (\mathbf{w}_{lk} - x_k\mathbf{e}_l)dx = 0, \end{cases} \tag{5.2}$$

where $\mathcal{E}(\mathbf{w}_{lk}) := \frac{1}{2}(\nabla\mathbf{w}_{lk} + \nabla{\mathbf{w}_{lk}}^T)$. The effective elasticity tensor $C^* = (C^*_{ijpq})$ is then defined by (see, for example, [28])

$$C^*_{ijpq} = \int_Y (C\mathcal{E}(\mathbf{w}_{ij}))_{pq}dx, \quad i,j,p,q = 1,\ldots,d. \tag{5.3}$$

In [8], the following asymptotic expansion of the effective elasticity tensor in two dimensions was derived:

$$C^* = C^0 + fM(I - fS^eM)^{-1} + O(f^3), \tag{5.4}$$

where M is the EMT associated with B and $S^e = (S^e_{ijkl})$ is a 4-tensor given by

$$\begin{cases} S^e_{1111} = S^e_{2222} = c_e + \dfrac{1}{2(\lambda+2\mu)}, \quad S^e_{1122} = S^e_{2211} = -c_e, \\ S^e_{1212} = S^e_{1221} = S^e_{2121} = S^e_{2112} = -c_e + \dfrac{1}{4\mu}, \\ \text{the other entries are zero.} \end{cases} \tag{5.5}$$

Here c_e is a constant given by

$$c_e = \frac{\lambda+\mu}{4\mu(\lambda+2\mu)}\left(\frac{2\pi}{3} + 16\pi^2 \sum_{n=1}^{+\infty} \frac{n^2 e^{-2\pi n}}{(1-e^{-2\pi n})^2} - 8\pi \sum_{n=1}^{+\infty} \frac{n e^{-2\pi n}}{1-e^{-2\pi n}}\right). \tag{5.6}$$

The tensor S^e, which we call *the array tensor*, is of particular interest. The superscript e is to emphasize its role in elasticity. We will make some remark on S^e at the end of this section. [1]

We now extend (5.4) to the three-dimensional case. Since the procedure of derivation of the three-dimensional formula is identical to the two dimensional case except the analysis for the periodic Green function, we will make an expansion of the periodic Green function and then write down the final formula for the effective elasticity tensor without details of derivation. One is referred to [8] for details of the derivation.

If we define $\mathbf{G} = (G_{pq})_{p,q=1,2,3}$ by

$$G_{pq}(x) = \frac{1}{4\pi^2\mu} \sum_{n\in\mathbb{Z}^3\setminus 0} \left[-\frac{\delta_{pq}}{|n|^2} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{n_p n_q}{|n|^4}\right] e^{i2\pi n\cdot x}, \tag{5.7}$$

then $\mathbf{G}$ is periodic and satisfies

$$\mathcal{L}_{\lambda,\mu}\mathbf{G}(x) = \sum_{n\in\mathbb{Z}^3} \delta_0(x+n)I - I, \quad x\in\mathbb{R}^3. \tag{5.8}$$

The following lemma was obtained in [26] using Ewald's technique.

Lemma 5.1 *Let us define*

$$S_1 = \frac{1}{\pi} \sum_{n\in\mathbb{Z}^3\setminus 0} \frac{e^{i2\pi n\cdot x}}{|n|^2}, \tag{5.9}$$

$$S_2 = -\frac{1}{4\pi^3} \sum_{n\in\mathbb{Z}^3\setminus 0} \frac{e^{i2\pi n\cdot x}}{|n|^4}. \tag{5.10}$$

Then

$$S_1 = \frac{1}{|x|} - c_1 + \frac{2\pi}{3}|x|^2 + O(|x|^4), \tag{5.11}$$

$$S_2 = \frac{|x|}{2} - c_2 - \frac{c_1}{6}|x|^2 + \frac{\pi}{30}|x|^4 + O(|x|^6), \tag{5.12}$$

for some constant c_1, c_2.

Since

$$G_{pq} = -\frac{\delta_{pq}}{4\pi\mu} S_1 + \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\partial_p\partial_q S_2, \tag{5.13}$$

we have the following lemma.

[1] It should be noted that the sign for S^e in [8] is incorrect and the array tensor in this paper has the sign opposite to that in [8].

Lemma 5.2 *There exists a smooth function* $\mathbf{R}$ *such that*

$$\mathbf{G}(x) = \mathbf{\Gamma}(x) + \mathbf{R}(x), \quad x \in Y.$$

The function $\mathbf{R} = (R_{pq}),\ p, q = 1, 2, 3,$ *has the following Taylor expansion at* 0

$$R_{pq}(x) = R_{pq}(0) - \frac{4\lambda + 9\mu}{30\mu(\lambda + 2\mu)}|x|^2\delta_{pq} + \frac{\lambda + \mu}{15\mu(\lambda + 2\mu)}x_p x_q + O(|x|^4). \tag{5.14}$$

We can follow the same process as in the derivation of (5.4) in [8] to obtain the following Theorem. Uniformity in λ and $\widetilde{\lambda}$ of the asymptotic formula in the theorem is proved in Appendix A.

Theorem 5.3 *Let* $S^e = (S^e_{ijkl})$ *be the 4-tensor given by*

$$\begin{cases} S^e_{iiii} = \dfrac{2\lambda + 7\mu}{15\mu(\lambda + 2\mu)}, \\ S^e_{iijj} = -\dfrac{\lambda + \mu}{15\mu(\lambda + 2\mu)}, \quad S^e_{ijij} = \dfrac{3\lambda + 8\mu}{30\mu(\lambda + 2\mu)}, \quad i \neq j, \\ \text{the other entries are zero.} \end{cases} \tag{5.15}$$

Then the following asymptotic expansion for the effective elasticity tensor C^* *holds*

$$C^* = C^0 + fM(I - fS^eM)^{-1} + O(f^{\frac{8}{3}}), \tag{5.16}$$

where M *is the EMT associated with* B*. Moreover, the projected remainder term* $PO(f^{\frac{8}{3}})P$ *is uniformly bounded as* λ *and* $\widetilde{\lambda}$ *tend to* ∞ *with* $\lambda/\widetilde{\lambda} = O(1)$*.*

We now make some important remarks on formulae (5.4) and (5.16).

(i) The array tensor S^e is derived as follows: Let $\mathbf{G}$ be the periodic Green function for the Lamé system. If

$$\mathbf{G}(x) - \mathbf{\Gamma}(x) = c - \sum_{i,j,p,q} a_{ijpq} x_i x_j \mathbf{e}_p \otimes \mathbf{e}_q + O(|x|^4) \tag{5.17}$$

in a neighborhood of 0 where $a_{ijpq} = a_{jipq}$, then S^e is given by

$$S_{ijkl} = \frac{1}{2}(a_{ipjq} + a_{iqjp} + a_{jpiq} + a_{jqip}). \tag{5.18}$$

Thus S^e is independent of the inclusion. If other kind of periodic structure or array, other than the orthogonal square or cubic array we are dealing with in this paper, then the periodic Green function $\mathbf{G}$ is different and hence S^e will be different. This is why we have called it the array tensor.

(ii) The formula (5.4) and (5.16) can be written as

$$C^* = C^0 + fM + f^2 MS^eM + O(f^{\frac{2(d+1)}{d}}). \tag{5.19}$$

This formula shows that the first-order term is independent of the periodic structure and depending only on the single particle. On the other hand, the second-order term depends on both the particle and the periodic structure and the term MS^eM yields a clear decomposition of this dependency.

(iii) If $\lambda \to +\infty$, then S^e converges to S^v where in two dimensions

$$\begin{cases} S^v_{iiii} = c_v, \quad S^v_{iijj} = -c_v, \quad S^v_{ijij} = -c_v + \dfrac{1}{4\mu}, \quad i \neq j, \\ \text{the other entries are zero.} \end{cases} \tag{5.20}$$

with

$$c_v = \frac{1}{4\mu}\left(\frac{2\pi}{3} + 16\pi^2 \sum_{n=1}^{+\infty} \frac{n^2 e^{-2\pi n}}{(1-e^{-2\pi n})^2} - 8\pi \sum_{n=1}^{+\infty} \frac{n e^{-2\pi n}}{1-e^{-2\pi n}}\right), \tag{5.21}$$

and in three dimensions

$$\begin{cases} S^v_{iiii} = \dfrac{2}{15\mu}, \quad S^v_{iijj} = -\dfrac{1}{15\mu}, \quad S^v_{ijij} = \dfrac{1}{10\mu}, \quad i \neq j, \\ \text{the other entries are zero.} \end{cases} \tag{5.22}$$

The tensor S^v will play the role of the array tensor for the effective viscosity tensor which we will compute in the next section. It should be noted that $\sum_i S^v_{iikl} = \sum_i S^v_{klii} = 0$ for all k, l, which amount to saying

$$S^v = P S^v P. \tag{5.23}$$

(iv) If $\lambda \to +\infty$, then the periodic Green function $\mathbf{G}$ defined by (5.7) converges to the periodic Green function of the Stokes system given by Hasimoto in [26].

6 Effective viscosity tensor

In this section, we will derive an asymptotic formula for the effective viscosity properties of the material containing a square or a cubic array of viscous particles occupying a small volume fraction.

We first note that making use of remarks (iii) and (iv) in the previous section together with the representations formulae for the solutions to the Stokes system provided in our recent paper [3], we can easily derive, by following exactly the same arguments as those developed in [8], the effective viscosity properties of the mixture. In this section, we rather make use of the analogy between hydrodynamics and elasticity.

Let $Y =]-\frac{1}{2}, \frac{1}{2}[^d$ and $D = \rho B$, where B is a smooth bounded domain containing the origin with the volume $|B|1$. We set $\widetilde{\mu}$ and μ to be the viscosity of D and $Y \setminus \bar{D}$, respectively and extend it to the whole space periodically with the periodic cell Y, *i.e.*,

$$\eta := \mu \chi(\mathbb{R}^d \setminus \overline{D^\#}) + \widetilde{\mu}\chi(D^\#)$$

where $D^\# := \bigcup_{n \in \mathbb{Z}^d}(D + n)$. Then we will consider the problem to determine the effective viscous properties of the material with viscous constants $\eta(x/\epsilon)$ as $\epsilon \to 0$.

We define the effective viscosity tensor to be the rate of the averaged viscous stress to the averaged strain [34]:

$$\langle \sigma \rangle = \hat{C} \langle \epsilon \rangle. \tag{6.1}$$

Since the stress and the strain are of trace zero, the fourth rank viscosity tensor can be understood as an operator from the space of symmetric matrices of trace zero to itself, which means that the viscosity tensors are not uniquely determined. This can be remedied by posing the following restriction:

$$\begin{cases} \hat{C}_{ijkl} = \hat{C}_{jikl} = \hat{C}_{ijlk} \quad i,j,k,l = 1,\dots d, \\ \displaystyle\sum_i \hat{C}_{iikl} = 0, \ \sum_i \hat{C}_{klii} = 0, \end{cases} \tag{6.2}$$

or equivalently, $\hat{C} = P\hat{C}P$.

For $k, l = 1, \dots, d$, let $\mathbf{v}_{kl}$ be the solution to

$$\begin{cases} \mu \Delta \mathbf{v}_{kl} + \nabla p = 0 \quad \text{in } Y \setminus \overline{D}, \\ \widetilde{\mu} \Delta \mathbf{v}_{kl} + \nabla p = 0 \quad \text{in } D, \\ \mathbf{v}_{kl}|_- - \mathbf{v}_{kl}|_+ = 0 \quad \text{on } \partial D, \\ (p\mathbf{N} + \widetilde{\mu}\dfrac{\partial \mathbf{v}_{kl}}{\partial \mathbf{N}})|_- - (p\mathbf{N} + \mu\dfrac{\partial \mathbf{v}_{kl}}{\partial \mathbf{N}})|_+ = 0 \quad \text{on } \partial D, \\ \nabla \cdot \mathbf{v}_{kl} = 0 \quad \text{in } Y, \\ \mathbf{v}_{kl}(x) - x_l \mathbf{e}_k + \delta_{kl} d(x), \ p(x) \text{ are periodic}, \\ \displaystyle\int_Y (\mathbf{v}_{kl} - x_l \mathbf{e}_k + \delta_{kl} d(x)) dx = 0. \end{cases} \tag{6.3}$$

Lemma 6.1 *Let $\mathbf{w}_{kl}$ be the solution to (5.2). Then, we have*

$$\left\| \mathbf{w}_{kl} - \frac{\delta_{kl}}{d} \sum_i \mathbf{w}_{ii} - \mathbf{v}_{kl} \right\|_{H^1(Y)} \to 0 \ \textit{as } \lambda, \widetilde{\lambda} \to \infty \quad (\textit{with } \frac{\widetilde{\lambda}}{\lambda} = O(1)). \tag{6.4}$$

Proof. Define $\mathbf{e}$ in Y by

$$\mathbf{e} = \mathbf{w}_{pq} - \frac{\delta_{pq}}{d} \sum_k \mathbf{w}_{kk} - \mathbf{v}_{pq}.$$

By using (6.3) and (5.2) we have

$$\begin{cases} \mathcal{L}_{\lambda,\mu} \mathbf{e} = \nabla p \quad \text{in } Y \setminus \overline{B}, \\ \mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}} \mathbf{e} = \nabla p \quad \text{in } B, \\ \mathbf{e}\big|_+ = \mathbf{e}\big|_- \quad \text{on } \partial B, \\ \dfrac{\partial \mathbf{e}}{\partial \nu}\Big|_+ = \dfrac{\partial \mathbf{e}}{\partial \widetilde{\nu}}\Big|_- + \Big(p\big|_+ - p\big|_-\Big)\mathbf{N} \quad \text{on } \partial B \\ \mathbf{e}, \ p \text{ are periodic}, \\ \displaystyle\int_Y \mathbf{e}\, dx = 0. \end{cases}$$

Using integration by parts we obtain the following energy identity:

$$\int_Y (\lambda\chi(Y\setminus B) + \widetilde{\lambda}\chi(B))|\nabla\cdot\mathbf{e}|^2 + \frac{1}{2}\int_Y (\mu\chi(Y\setminus B) + \widetilde{\mu}\chi(B))|\nabla\mathbf{e} + (\nabla\mathbf{e})^T|^2 = \int_Y p\nabla\cdot\mathbf{e}\,.$$

Then

$$\inf(\mu,\widetilde{\mu})\|\mathbf{e}\|^2_{H^1(Y)} + \inf(\lambda,\widetilde{\lambda})\|\nabla\cdot\mathbf{e}\|^2_{L^2(Y)} \leq C\|\nabla\cdot\mathbf{e}\|_{L^2(Y)},$$

from which it follows that

$$\|\nabla\cdot\mathbf{e}\|_{L^2(Y)} \leq C\left(\frac{1}{\lambda} + \frac{1}{\widetilde{\lambda}}\right),$$

and therefore

$$\inf(\mu,\widetilde{\mu})\|\mathbf{e}\|^2_{H^1(Y)} \leq C\left(\frac{1}{\lambda} + \frac{1}{\widetilde{\lambda}}\right),$$

as desired. This completes the proof. □

Then the following asymptotic expansion for the effective viscosity tensor $\hat{C}$ holds.

Theorem 6.2 *Let $f = \rho^d$ and let S^v be the array tensor given in (5.20) and (5.22). Then*

$$\hat{C} = 2\mu P + fV + f^2 V S^v V + O(f^{\frac{2(d+1)}{d}}), \tag{6.5}$$

where V is the VMT associated with B and the pair $(\mu,\widetilde{\mu})$.

Proof. From the definition (6.1) of the effective viscosity tensor we obtain that

$$\int_Y 2(\mu + (\widetilde{\mu}-\mu)\chi(B))\mathcal{E}(\mathbf{v}_{lk})_{ij}dx = \hat{C}_{ijmn}\int_Y \mathcal{E}(\mathbf{v}_{lk})_{mn}dx. \tag{6.6}$$

By the periodicity of $\mathbf{v}_{kl}(x) - x_l\mathbf{e}_k + \delta_{kl}d(x)$, we have

$$\int_Y \mathcal{E}(\mathbf{v}_{lk}) = \int_Y \mathcal{E}(x_l\mathbf{e}_k - \delta_{kl}d(x)) = \frac{1}{2}(\mathbf{e}_l\otimes\mathbf{e}_k + \mathbf{e}_k\otimes\mathbf{e}_l) - \frac{1}{d}I,$$

which, combined with the condition (6.2), gives

$$\hat{C}_{ijmn}\int_Y \mathcal{E}(\mathbf{v}_{lk})_{mn}dx = \hat{C}_{ijkl}. \tag{6.7}$$

Because $\mathrm{Tr}\mathcal{E}(\mathbf{v}_{lk}) = 0$, we also have

$$PC\mathcal{E}(\mathbf{v}_{lk}) = 2(\mu + (\widetilde{\mu}-\mu)\chi(B))\mathcal{E}(\mathbf{v}_{lk}). \tag{6.8}$$

By (6.6), (6.7), and (6.8), we get

$$\hat{C}_{ijkl} = \int_Y (PC\mathcal{E}(\mathbf{v}_{lk}))_{ij}dx.$$

On the other hand, by (6.4), it follows that

$$\int_Y (PC\mathcal{E}(\mathbf{v}_{lk}))_{ij}dx = \lim_{\lambda,\widetilde{\lambda}\to\infty}\int_Y\left(PC\mathcal{E}\left(\mathbf{w}_{lk} - \frac{\delta_{lk}}{d}\sum_i \mathbf{w}_{ii}\right)\right)_{ij}dx.$$

Therefore, from (5.3), we conclude that

$$\hat{C} = \lim_{\lambda,\widetilde{\lambda}\to\infty} PC^*P. \tag{6.9}$$

Using (5.19), (3.7), and the fact that $PC^0 = 2\mu P$ we obtain (6.5). □

The quantity $\frac{\mathrm{tr}(\hat{C})}{2\mathrm{tr}(P)}$ is called the *effective scalar viscosity* and denoted by μ^*. It is defined as follows the angular averaging of the effective viscosity tensor $\hat{C}$. The scalar coefficient μ^* is the viscosity of a Newtonian fluid that behaves as the mixture of randomly oriented particles with shape B and viscosity μ. It is then invariant under suspension rotation.

By taking the trace of identity (6.5) we get

$$\mu^* = \mu + f\frac{\mathrm{tr}(V)}{2\mathrm{tr}(P)} + f^2\frac{\mathrm{tr}(VS^vV)}{2\mathrm{tr}(P)} + O(f^{\frac{2(d+1)}{d}}). \tag{6.10}$$

Formula (6.10) is our main contribution in this paper.

Note that in view of the property of positivity of the viscous moment tensor V in Lemma 3.1, adding particles with higher viscosity than the one of the ambient fluid increases the effective viscosity of the suspension while the addition of particles with lower viscosity particles decreases the overall viscosity properties of the fluid. Note also that the second-order term in (6.10) depends on the array tensor S^v, which means that the second-order correction would be different for other periodic arrays than the orthogonal square or cubic array considered here.

The derivative the the effective scalar viscosity with respect to the volume fraction f is called the *intrinsic viscosity* which we denoted by $[\mu]_{B,\widetilde{\mu}}$ or $[\mu]_B$. One can now see that (4.6) follows from (6.10).

Using the formula (4.2) and (4.5), one can explicitly compute the effective viscosity for ellipses and ellipsoids. It is worth mentioning that the EMT for inclusions of general shape can be computed numerically by solving certain boundary integral equations (see [4]). Then one can compute the VMT using (3.9), and hence the effective viscosity tensor and the intrinsic viscosity can be computed. We emphasize that this process works not only for the solid particle ($\widetilde{\mu} = \infty$), but also for the soft particles ($\widetilde{\mu} < \infty$)

Before concluding this section, let us write down the formula for the effective scalar viscosity when the inclusion is a ball. Using (4.3), (4.10), and (6.10), we have

$$\mu^* = \mu + 5\mu\frac{\widetilde{\mu}-\mu}{2\widetilde{\mu}+3\mu}f + \frac{11\mu}{2}\left(\frac{\widetilde{\mu}-\mu}{2\widetilde{\mu}+3\mu}\right)^2 f^2 + O(f^{\frac{8}{3}}). \tag{6.11}$$

It should be noted that (6.11) holds for soft particles as well as solid ones. In particular, if $\widetilde{\mu} = +\infty$, then

$$\mu^* = \mu + \frac{5\mu}{2}f + \frac{11\mu}{8}f^2 + O(f^{\frac{8}{3}}). \tag{6.12}$$

Note that the first-order term (the intrinsic viscosity) in (6.12) was derived by Einstein [20] while the second-order term seems to be new as far as we are aware of.

To conclude this section, we point out that if the particles orient accordingly to a probability distribution $\Psi(\theta)$ then, from Lemma 3.2 and formula (6.10), the effective scalar viscosity of the suspension (say, to fix ideas, in the two-dimensional case) is given by

$$\mu^* = \mu + \frac{f}{2\mathrm{tr}(P)}\int_0^{2\pi}\Psi(\theta)\mathrm{tr}(\hat{V}_\theta)\,d\theta + O(f^2),$$

where

$$\hat{V}_\theta = \sum_{j=1}^{d^*} R(\theta) A_j R(\theta)^T \otimes R(\theta) A_j R(\theta)^T,$$

with $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta \in [0, 2\pi]$, and $A_j, j = 1, \ldots, d^*$, given by $V = \sum_{j=1}^{d^*} A_j \otimes A_j$.

A similar formula holds in the three-dimensional case.

7 Apparent absorption

In this final section, we use the asymptotic formula we derived for the effective viscosity to explain how the absorption can be amplified in a composite.

When studying composites, both theoretically and experimentally, one occasionally notices that the effective viscosity could have a nonzero imaginary part even though no absorption term was input in the model. It means that even if the microstructure is non dissipative or of negligible dissipation, one could observe an effective loss of energy, yielded by the complex value of the effective viscosity μ^*. This is called the "apparent absorption"; see [29]. In the framework of a periodic setting, we encounter a similar behavior for the effective viscosity. We set a dilute periodic collection of ellipses in 2D with a shear modulus μ for the background and $\widetilde{\mu}$ for the inclusions. Then we study the case where $\mu \in \mathbb{R}$ and $\widetilde{\mu} = k\mu(1 + i\delta)$, that is, elastic inclusions that are slightly dissipative are scattered over a purely elastic background.

We recall from (4.6), (4.7), and (6.10) that in this case

$$\mu^* - \mu = f\frac{\mu}{2}\Big[\frac{1}{\frac{k^*+1}{k^*-1} - m^2} + \frac{1}{\frac{k^*+1}{k^*-1} + m^2}\Big] + O(f^2),$$

where $k^* = \frac{\widetilde{\mu}}{\mu} = k(1 + i\delta)$ and m is a parameter of excentricity of the ellipse. Assuming $\delta << 1$, this leads to the following expression for the viscosity amplification:

$$\frac{Im(\mu^*)}{Im(\widetilde{\mu})} = \mathbb{A}(k) + O(\delta) + O(f^2),$$

where

$$\mathbb{A}(k) = \frac{2f}{(k+1)^2}\frac{1 + m^4\alpha^2}{(1 - m^4\alpha^2)^2}, \quad \alpha = \frac{k-1}{k+1}.$$

The amplification factor $\mathbb{A}(k)$ is maximal at $k = 0$ and decays as $k \to \infty$. It is worth noticing that although the term $(1 + m^4\alpha^2)/(1 - m^4\alpha^2)^2$ is invariant under $k \to 1/k$ and thus obeys a reciprocity principle, $\mathbb{A}$ is not invariant under $k \to 1/k$. If one would like to investigate efficient absorption, one should focus on the maximum of amplification:

$$\mathbb{A}_{max} = \mathbb{A}(k = 0) = 2f\frac{1 + m^4}{(1 - m^4)^2}.$$

This means that if the microstructure is anisotropic enough then one could observe a significant amplification of the absorption coefficient. That is, even if the microstructure is marginally absorbing, the effective medium could have a nonzero absorbing coefficient $Im(\mu^*)$. In other words, by putting an imaginary part of negligible size in the model, we observe absorption amplification, this amplification seems related to the apparent absorption in the sense that the more heterogenous and anisotropic the microstructure is, the more absorption we observe on the macroscopic level.

This computation is done in the framework of the present article, in particular in the framework of dilute materials, that is neglecting any interactions between the inclusions. Interacting particles should not differ radically from our situation, at least we expect the apparent absorption effect to be still present and even enhanced by a "multiple scattering" effect. Moreover, we expect inclusions of low viscosity to produce still more apparent absorption than those of high viscosity.

A A proof of uniformity

This section is devoted to the proof of uniformity of the projected remainder of the asymptotic formula (5.16).

To make notation short put $X := L^2(\partial D)$ (actually $X := L^2(\partial D)^3$) and $Y := H^1(\partial D)^3$. Let X_0 and Y_0 be subspaces of X and Y, respectively, with the zero flux through ∂D, namely, $X_0 = \{ \phi \in X \mid \int_{\partial D} \phi \cdot \mathbf{N} \, d\sigma = 0 \}$. Let $\mathcal{P} : X \to X_0$ ($Y \to Y_0$) be the projection defined by

$$\mathcal{P}[\phi] := \phi - \frac{\int_{\partial D} \phi \cdot \mathbf{N} \, d\sigma}{|\partial D|} \mathbf{N}.$$

Let $\mathcal{S}_D$ and $\tilde{\mathcal{S}}_D$ be the single layer potentials corresponding to the Lamé system with Lamé constants (λ, μ) and $(\widetilde{\lambda}, \widetilde{\mu})$, respectively, and let $\mathcal{G}_D$ be the periodic single layer potential corresponding to Lamé constants (λ, μ). We refer to [4] for the definition of the single layer potentials. The following theorems are essential ingredients for derivation of the asymptotic formula (5.16) in [8]; the first one was obtained in [21], and the second one in [8].

Theorem A.1 *Suppose that* $(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0$ *and* $0 < \widetilde{\lambda}, \widetilde{\mu} < \infty$. *For any given* $(f, g) \in Y \times X$, *there exists the unique solution* $(\phi, \psi) \in X \times X$ *such that*

$$\begin{cases} \tilde{\mathcal{S}}_D[\phi]|_- - \mathcal{S}_D[\psi]|_+ = f \\ \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_D[\psi]\Big|_- - \dfrac{\partial}{\partial \nu} \mathcal{S}_D[\psi]\Big|_+ = g, \end{cases} \tag{A.1}$$

where $|_+$ *and* $|_-$ *indicate the limit along the normal direction to* ∂D *from outside* D *and inside* D, *respectively. Moreover, there exists a constant* C *such that*

$$\|\phi\|_X + \|\psi\|_X \leq C \left(\|f\|_Y + \|g\|_X \right). \tag{A.2}$$

Theorem A.2 *Suppose that* $(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0$ *and* $0 < \widetilde{\lambda}, \widetilde{\mu} < \infty$. *For any given* $(f, g) \in Y \times X$, *there exists the unique solution* $(\phi, \psi) \in X \times X$ *such that*

$$\begin{cases} \tilde{\mathcal{S}}_D[\phi]|_- - \mathcal{G}_D[\psi]|_+ = f \\ \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_D[\psi]\Big|_- - \dfrac{\partial}{\partial \nu} \mathcal{G}_D[\psi]\Big|_+ = g. \end{cases} \tag{A.3}$$

Moreover, there exists a constant C *such that*

$$\|\phi\|_X + \|\psi\|_X \leq C \left(\|f\|_Y + \|g\|_X \right). \tag{A.4}$$

The constant C in both (A.1) and (A.3) depends on μ, $\widetilde{\mu}$, λ, and $\widetilde{\lambda}$ in general, which makes the asymptotic formula (5.16) obtained in [8] depend on the same parameters. In order to prove (A.3) as claimed in Theorem 5.3, we need to clarify the dependency of the constants C on λ, and $\widetilde{\lambda}$. For that we obtain the following theorems.

Theorem A.3 *Suppose that* $(\lambda-\widetilde{\lambda})(\mu-\widetilde{\mu})\geq 0$ *and* $0<\widetilde{\lambda},\widetilde{\mu}<\infty$. *If* λ, $\widetilde{\lambda}$ *are sufficiently large with* $\widetilde{\lambda}/\lambda = O(1)$, *then there exists a constant* C, *independent of* λ *(and hence of* $\widetilde{\lambda}$*), such that for any solution* $(\phi,\psi)\in X\times X$ *to (A.1)*

$$\|\phi\|_X + \|\mathcal{P}\psi\|_X \leq C\left(\|f\|_Y + \|g\|_X\right) \tag{A.5}$$

$$\|(I-\mathcal{P})\psi\|_X \leq C\left(\|\mathcal{P}f\|_Y + \lambda\|(I-\mathcal{P})f\|_Y + \|g\|_X\right). \tag{A.6}$$

Theorem A.4 *Under the same hypotheses as in Theorem A.3, there exists a constant* C, *independent of* λ *(and hence of* $\widetilde{\lambda}$*), such that for any solution* $(\phi,\psi)\in X\times X$ *to (A.3)*

$$\|\phi\|_X + \|\mathcal{P}\psi\|_X \leq C\left(\|f\|_Y + \|g\|_X\right) \tag{A.7}$$

$$\|(I-\mathcal{P})\psi\|_X \leq C\left(\|\mathcal{P}f\|_Y + \lambda\|(I-\mathcal{P})f\|_Y + \|g\|_X\right). \tag{A.8}$$

Theorems A.3 and A.4 show that

$$\|\phi\|_X + \|\psi\|_X \leq C\left(\|\mathcal{P}f\|_Y + \lambda\|(I-\mathcal{P})f\|_Y + \|g\|_X\right). \tag{A.9}$$

So, if $f\in Y_0$ (zero flux), then $\|\phi\|_X+\|\psi\|_X$ is bounded independently of λ and $\widetilde{\lambda}$. In general, $\|\phi\|_X+\|\psi\|_X$ may not be bounded, but $\|\phi\|_X+\|\mathcal{P}\psi\|_X$ is bounded. One can use Theorem A.3 and A.4 to derive (A.3) in Theorem 5.3 by simply following the same lines of derivation in [8].

The rest of this section is devoted to the proof of Theorem A.3. Theorem A.4 can be proved in the same way.

The integral equation (A.1) can be rewritten as

$$\mathcal{A}\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix} f\\ g\end{bmatrix} \tag{A.10}$$

where the operator $\mathcal{A}: X\times X\to Y\times X$ is defined by

$$\mathcal{A} := \begin{bmatrix} \tilde{\mathcal{S}}_D & -\mathcal{S}_D \\ \dfrac{\partial}{\partial\tilde{\nu}}\tilde{\mathcal{S}}_D\Big|_- & -\dfrac{\partial}{\partial\nu}\mathcal{S}_D\Big|_+ \end{bmatrix}. \tag{A.11}$$

Let (Γ^s, F) be the fundamental solution to the Stokes equation with viscosity μ, *i.e.*,

$$\Gamma^s_{ij}(x) = -\frac{1}{8\mu\pi}\left(\frac{\delta_{ij}}{|x|}+\frac{x_ix_j}{|x|^3}\right) \quad \text{and} \quad F(x) = \frac{1}{4\pi}\frac{x}{|x|^3}.$$

Define the single layer potential $\mathcal{S}^s_D$ and a boundary integral operator $(\mathcal{K}^s_D)^*$ associated with the Stokes system by

$$\begin{aligned}
\mathcal{S}^s_D[\phi]_i(x) &:= \int_{\partial D}\Gamma^s_{ij}(x-y)\phi_j(y)d\sigma(y), \quad x\in\mathbb{R}^3\\
(\mathcal{K}^s_D)[\phi]_i(x) &:= \text{p.v.}\int_{\partial D}\mu\left(\frac{\partial\Gamma^s_{ij}(x-y)}{\partial x_l}+\frac{\partial\Gamma^s_{lj}(x-y)}{\partial x_i}\right)N_l(x)\phi_j(y)d\sigma(y)\\
&\quad + \text{p.v.}\int_{\partial D}F_j(x-y)N_i(x)\phi_j(y)d\sigma(y), \quad x\in\partial D,
\end{aligned}$$

for $i = 1, 2, 3$. Here p.v. stands for the Cauchy principal value. We denote the same operators for viscosity $\widetilde{\mu}$ by $\tilde{\mathcal{S}}_D^s$ and $(\tilde{\mathcal{K}}_D^s)^*$. It is worth emphasizing that $(\mathcal{K}_D^s)^*$ is related to $\mathcal{S}_D^s$ by the formula

$$\mu \frac{\partial}{\partial \mathbf{N}} \mathcal{S}_D^s[\phi]\Big|_+ = \left(\frac{1}{2} I + (\tilde{\mathcal{K}}_D^s)^*\right)[\phi] - \left(\int_{\partial D} F(x-y) \cdot \phi(y) d\sigma(y)\right)\Big|_+ \mathbf{N}.$$

We observe that the operator $\mathcal{A}$ can be decomposed as follows

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1, \tag{A.12}$$

where

$$\mathcal{A}_0 := \begin{bmatrix} \tilde{\mathcal{S}}_D^s & -\mathcal{S}_D^s \\ -\frac{1}{2} I + \tilde{\mathcal{K}}_D^{s*} & -\left(\frac{1}{2} I + \mathcal{K}_D^{s*}\right) \end{bmatrix} \tag{A.13}$$

and

$$\mathcal{A}_1 := \begin{bmatrix} \frac{1}{8\pi(2\widetilde{\mu} + \widetilde{\lambda})} \mathcal{B} & -\frac{1}{8\pi(2\mu + \lambda)} \mathcal{B} \\ \frac{\widetilde{\mu}}{4\pi(2\widetilde{\mu} + \widetilde{\lambda})} \mathcal{C} & -\frac{\mu}{4\pi(2\mu + \lambda)} \mathcal{C} \end{bmatrix} \tag{A.14}$$

in which the operators $\mathcal{B}$ and $\mathcal{C}$ are defined by

$$\begin{aligned} \mathcal{B}[\phi]_i &:= \int_{\partial D} \left(-\frac{\phi_i(y)}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)\phi_j(y)}{|x-y|^3}\right) d\sigma(y), \\ \mathcal{C}[\phi]_i &:= \text{p.v.} \int_{\partial D} \left(\frac{(x_j - y_j)\phi_i(y) N_j(x) + (x_i - y_i)\phi_j(y) N_j(x) - (x_j - y_j)\phi_j(y) N_i(x)}{|x-y|^3}\right) d\sigma(y) \\ &\quad - \text{p.v.} \int_{\partial D} \frac{3(x_i - y_i)(x_j - y_j)(x_l - y_l)\phi_j(y) N_l(x)}{|x-y|^5} d\sigma(y) \end{aligned}$$

for $i = 1, 2, 3$.

Note that $\mathcal{C}$ is a Calderón-Zygmund operator and hence bounded on X. On the other hand, $\nabla \mathcal{B}$ is a Calderón-Zygmund operator and hence $\mathcal{B}$ is bounded from X into Y. Thus we have

$$\left\| \mathcal{A}_1 \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{Y \times X} \leq \frac{C}{\lambda} \left(\|\phi\|_X + \|\psi\|_X\right) \tag{A.15}$$

for some constant C independent of λ. The operator $\mathcal{A}_0$ arises from the transmission problem for the Stokes' system and is the limit of $\mathcal{A}$ as λ and $\widetilde{\lambda}$ tend to ∞. It is proved in [3] that $\mathcal{A}_0$ is bounded from $X \times X$ into $Y \times X$ and invertible as an operator from $X \times X_0$ onto $Y_0 \times X$, and the null space of $\mathcal{A}_0$ on $X \times X$ is one dimensional and generated by $\begin{bmatrix} 0 \\ \mathbf{N} \end{bmatrix}$, in other words,

$$\mathcal{A}_0 \begin{bmatrix} 0 \\ (I - \mathcal{P})\psi \end{bmatrix} = 0 \tag{A.16}$$

for any $\psi \in X$. By (A.1) and (A.16), we have

$$\mathcal{A}_0 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} + \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} + \mathcal{A}_1 \begin{bmatrix} 0 \\ (I - \mathcal{P})\psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{A.17}$$

Suppose $f \in Y_0$. We get from (A.15) and (A.17)

$$\left\| \mathcal{A}_0 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \right\|_{Y \times X} \leq \|f\|_Y + \|g\|_X + \frac{C}{\lambda} \left(\|\phi\|_X + \|\mathcal{P}\psi\|_X + \|(I - \mathcal{P})\psi\|_X \right). \tag{A.18}$$

Since $\mathcal{A}_0 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix}$ and $\begin{bmatrix} f \\ g \end{bmatrix}$ belong to $Y_0 \times X$, we have from (A.17) that

$$\begin{aligned} 0 &= \int_{\partial D} \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} + \int_{\partial D} \mathcal{A}_1 \begin{bmatrix} 0 \\ (I - \mathcal{P})\psi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} \\ &= \int_{\partial D} \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} + \frac{1}{|\partial D|} \int_{\partial D} \psi \cdot \mathbf{N} d\sigma \int_{\partial D} \mathcal{A}_1 \begin{bmatrix} 0 \\ \mathbf{N} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix}. \end{aligned}$$

By divergence theorem we have

$$\begin{aligned} \int_{\partial D} \mathcal{A}_1 \begin{bmatrix} 0 \\ \mathbf{N} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} &= -\frac{1}{8\pi(2\mu + \lambda)} \int_{\partial D} \mathcal{B}[\mathbf{N}] \cdot \mathbf{N} \\ &= \frac{1}{4\pi(2\mu + \lambda)} \int_{\partial D} \int_{\partial D} \frac{\mathbf{N}(x) \cdot \mathbf{N}(y)}{|x - y|} d\sigma(x) d\sigma(y) = \frac{|D|}{2\mu + \lambda}. \end{aligned}$$

Therefore we get

$$\int_{\partial D} \psi \cdot \mathbf{N} = -\frac{|\partial D|(2\mu + \lambda)}{|D|} \int_{\partial D} \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix}.$$

It then follows (A.15) that

$$\|(I - \mathcal{P})\psi\|_X \leq C\lambda \left\| \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \right\|_{Y \times X} \leq C \left(\|\phi\|_X + \|\mathcal{P}\psi\|_X \right). \tag{A.19}$$

Since $\mathcal{A}_0$ is an invertible operator from $X \times X_0$ onto $Y_0 \times X$, we get from (A.18)

$$\|\phi\|_X + \|\mathcal{P}\psi\|_X \leq C \left(\|f\|_Y + \|g\|_X \right) + \frac{C}{\lambda} \left(\|\phi\|_X + \|\mathcal{P}\psi\|_X \right).$$

Thus, if λ is sufficiently large, then we have

$$\|\phi\|_X + \|\mathcal{P}\psi\|_X \leq C \left(\|f\|_Y + \|g\|_X \right). \tag{A.20}$$

We finally obtain from (A.19) and (A.20)

$$\|\phi\|_X + \|\psi\|_X \leq C \left(\|f\|_Y + \|g\|_X \right). \tag{A.21}$$

Now suppose that $f = \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix}$ and $g = 0$. Arguing as before we get

$$\int_{\partial D} \psi \cdot \mathbf{N} = -\frac{|\partial D|(2\mu + \lambda)}{|D|} \left(\int_{\partial D} \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} - |\partial D| \right), \tag{A.22}$$

which yields

$$\|(I - \mathcal{P})\psi\|_X \leq C(\lambda + \|\phi\|_X + \|\mathcal{P}\psi\|_X). \tag{A.23}$$

On the other hand, we have from (A.17) and (A.22)

$$\mathcal{A}_0 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} + \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} = \frac{2\mu+\lambda}{|D|} \left(\int_{\partial D} \mathcal{A}_1 \begin{bmatrix} \phi \\ \mathcal{P}\psi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} - |\partial D| \right) \mathcal{A}_1 \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix},$$

which yields

$$\|\phi\|_X + \|\mathcal{P}\psi\|_X \le C + \frac{C}{\lambda} \left(\|\phi\|_X + \|\mathcal{P}\psi\|_X \right).$$

Thus, if λ is sufficiently large, then we have

$$\|\phi\|_X + \|\mathcal{P}\psi\|_X \le C. \tag{A.24}$$

This completes the proof of Theorem A.3.

References

[1] Y. Almog and H. Brenner, Ensemble-average suspension-scale Cauchy continuum-mechanical definitions of stress in polarized suspensions: Global homogenization of a dilute suspension of dipolar particles, Physics of Fluids, 11 (1999), 268–273.

[2] Y. Almog and H. Brenner, Global homogenization of a dilute suspension of spheres. Suspension rheology, preprint.

[3] H. Ammari, P. Garapon, H. Kang, and H. Lee, A Method of Biological Tissues Elasticity Reconstruction Using Magnetic Resonance Elastography Measurements, Quart. Appl. Math., 66 (2008), 139–175.

[4] H. Ammari and H. Kang, *Polarization and Moment Tensors: with Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences Series, Vol. 162, Springer-Verlag, New York, 2007.

[5] H. Ammari and H. Kang, Properties of the generalized polarization tensors, Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal, 1 (2003), 335–348.

[6] H. Ammari, H. Kang, and K. Kim, Polarization tensors and effective properties of anisotropic composite materials, J. Differ. Equat., 215 (2005), 401–428.

[7] H. Ammari, H. Kang, and H. Lee, A Boundary integral method for computing elastic moment tensors for ellipses and ellipsoids, J. Comp. Math. 25 (2007), 2–12.

[8] H. Ammari, H. Kang, and M. Lim, Effective parameters of elastic composites, Indiana Univ. Journal, 55 (2006), 903–921.

[9] H. Ammari, H. Kang, G. Nakamura and K. Tanuma, Complete asymptotic expansions of solutions of the system of elastostatics in the presence of inhomogeneities of small diameter, Journal of Elasticity, 67 (2002), 97–129.

[10] H. Ammari, H. Kang, and K. Touibi, Boundary layer techniques for deriving the effective properties of composite materials, Asymp. Anal., 41 (2005), 119–140.

[11] G.K. Batchelor and J.T. Green, The determination of the bulk stress is suspension of spherical particles to order c^2, J. Fluid. Mech. 56 (1972), 401-427.

[12] C.W.J. Beenakker, The effective viscosity of a concentrated suspension of spheres (and its relation to diffusion), Physica 128 A (1984), 48–81.

[13] H. Brenner, Suspension rheology in the presence of rotary Brownian motion and external couples: elongational flow of dilute suspensions, Chem. Eng. Sc., 27(1972), 1069–1107.

[14] Y. Capdeboscq and H. Kang, Improved Hashin-Shtrikman bounds for elastic moment tensors and an application, Applied Math. Optimization 57 (2008), 263-288.

[15] R.E. Collin, *Field Theory of Guided Waves*, Second Edition, IEEE Press, New York, 1991.

[16] A. Costa, Viscosity of high crystal content melts: dependence on solid fraction, Geophys. Research lett., 20 (2005), DOI:10.1029.

[17] J.F. Douglas and A. Friedman, Coping with complex boundaries, IMA Series on Mathematics and its Applications, Vol. 67, 166–185, Springer, New York, 1995.

[18] J.F. Douglas and E.J. Garboczi, Hydrodynamic friction of arbitrarily shaped Brownian particles, Phys. Review E., 47 (1993), R2983.

[19] J.F. Douglas and E.J. Garboczi, Intrinsic viscosity and polarizability of particles having a wide range of shapes, Adv. Chem. Phys., 91 (1995), 85–153.

[20] A. Einstein, Eine neue Bestimmung der Moleküldimensionen, Annalen der Physik, 19 (1906), 289–306.

[21] L. Escauriaza and J.K. Seo, Regularity properties of solutions to transmission problems, Trans. Amer. Math. Soc., 338(1) (1993), 405-430.

[22] E.J. Garboczi and J.F. Douglas, Intrinsic conductivity of objects having arbitrary shape and conductivity, Physical Review E 53 (1996), 6169–6180.

[23] J.N. Goodier, Concentration of stress around spherical and cylindrical inclusions and flaws, J. Appl. Mech. A, 55 (1933), 39-44.

[24] S. Haber and H. Brenner, Rheological properties of dilute suspensions of centrally symmetric Brownian particles at small shear rates, J. Coll. Inter. Sci., 97 (1984), 496-514.

[25] Z. Hashin, Analysis of composite materials-A survey, J. Appl. Mech., 50 (1983), 481–505.

[26] H. Hasimoto, On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres, J. Fluid Mech., 5 (1959), 317–328.

[27] R. Hill and G. Power, Extremum principles for slow viscous flow and the approximate calculation of drag, Q. J. Mech. and Appl. Math., 9 (1956), 313-319.

[28] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, New York, 1994.

[29] F.C. Karal and J. B. Keller, Elastic, Electromagnetic and other waves in a random medium, J. Math. Physics, 5 (1964), 537–547.

[30] T. Lévy and E. Sánchez-Palencia, Einstein-like approximation for homogenization with small concentration. II. Navier-Stokes equation, Nonlinear Anal., 9 (1985), 1255–1268.

[31] R. Lipton, On the effective elasticity of a two-dimensional homogenized incompessible elastic composite, Proc. Roy. Soc. Edinburgh Sect. A, 110 (1988), 45–61.

[32] M.L. Mansfield, J.F. Douglas, and E.J. Garboczi, Intrinsic viscosity and electrical polarizability of arbitrary shaped objects, Physical Review E, 64 (2001), 061401.

[33] H-M. Nguyen and M. Vogelius, A representation formula for the voltage perturbations caused by diametrically small conductivity inhomogeneities. Proof of uniform validity, Annales de l'Institut Henri Poincaré, 26 (2009), 2283–2315.

[34] K.C. Numan and J.B. Keller, Effective elasticity of a periodic composite, J. Mech. Phys. Solids, 32 (1984), 259–280.

[35] E. Sánchez-Palencia, Einstein-like approximation for homogenization with small concentration. I. Elliptic problems, Nonlinear Anal., 9 (1985), 1243–1254.

[36] J.W. Strutt (Lord Rayleigh), *Theory of Sound*, Vol. 2, Dover, New York, 1945.

[37] M.F. Thorpe and I. Jasiuk, New results in the theory of elasticity for two-dimensional composites, Proc. R. Soc. Lond. A, 438 (1992), 531–544.