Citations
From References: 37
From Reviews: 0

MR2168949 (2006k:35295) 35R30 (31B20 35-02 74G75 78A70 92C55)

Ammari, Habib (F-POLY-AM); **Kang, Hyeonbae** (KR-SNU-SM)

★Reconstruction of small inhomogeneities from boundary measurements.

Lecture Notes in Mathematics, 1846.
Springer-Verlag, Berlin, 2004. *x*+238 *pp*. \$59.95. *ISBN* 3-540-22483-1

The aim of electrical impedance tomography is to reconstruct the electrical conductivity of a body from boundary measurements of currents and voltages. The problem is strongly ill-posed. In order to make it tractable, prior knowledge about the unknown conductivity is required. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain which represents the body and $D_s = \varepsilon B_s + z_s \subset \Omega$, $1 \leq s \leq m$, be subdomains which neither intersect each other nor touch $\partial\Omega$. The D_s represent inhomogeneities centered at z_s, of size ε, shape B_s and constant conductivity k_s, $0 < k_s \neq 1 < +\infty$, $\forall s$. As a consequence, conductivity $k[\cdot]$ is assumed to be a function of the type $k[x] = \chi[\Omega \setminus \bigcup_{s=1}^m \overline{D}_s] + \sum_{s=1}^m k_s \chi[D_s]$. The goal becomes to determine the tuple $\{\varepsilon, z_s, B_s, k_s\}$ instead of the precise geometry and value of conductivity everywhere in Ω.

The key steps which lead to the reconstruction methods presented in the book are:

(1) Decomposition of the potential resulting from inhomogeneities into the sum of a harmonic part and a refraction part.

(2) Definition of polarization tensors generalized from [G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Stud., 27, Princeton Univ. Press, Princeton, NJ, 1951; MR0043486 (13,270d)] and the investigation of their properties.

(3) Derivation of complete asymptotic expansions [H. Kang and J. K. Seo, Inverse Problems **12** (1996), no. 3, 267–278; MR1391539 (97d:35242)] of boundary potential (“output voltage”) perturbations due to inclusions and consequent design of direct reconstruction algorithms.

This paradigm applies to three instances: direct current (DC) regime (Part I), linear elastostatics (Part II) and alternate current (AC) regime (Part III).

Part I addresses DC problems. Chapter 2 deals with the transmission problem and provides the related solvability results based on potential theory. Let there be one inclusion, i.e., $D = \varepsilon B + z$ and $m = 1$. Throughout the book it is assumed that ∂D is Lipschitz. Then the voltage potential u, which solves the transmission problem, is uniquely represented on $\partial\Omega$ by the sum of a harmonic part, $H[x]$, and a refraction part, $\mathcal{S}_D[\varphi[x]]$ (Theorem 2.17). Here $\mathcal{S}_D[\cdot]$ is the single layer potential on ∂D and $\varphi[\cdot]$ solves a boundary integral equation (BIE) on ∂D controlled by the conductivity contrast.

Chapter 3 introduces generalized polarization tensors (GPTs), which extend the definition of the Schiffer-Pólya-Szegő tensor, M [M. Schiffer and G. Szegő, Trans. Amer. Math. Soc. **67** (1949), 130–205; MR0033922 (11,515d); G. Pólya and G. Szegő, op. cit.]. If there is a single inclusion supported in $B \subset \Omega$, of constant conductivity k, then the knowledge of all GPTs uniquely determines the Dirichlet-to-Neumann map, hence [V. Isakov, Comm. Pure Appl. Math. **41** (1988), no. 7, 865–877; MR0951742 (90f:35205)] the pair $\{B, k\}$ (Theorem 3.4). Symmetry and positivity of GPTs are proved; bounds for the entries of M are provided. The multiple inclusion case is addressed by $\S 3.6$. In particular, if k is known, then M can be numerically inverted to estimate the ellipse equivalent to the inclusions.

Chapter 4 deals with full asymptotic formulae. Theorem 4.1 provides, in terms of the GPTs, the pointwise asymptotic expansion (Equation 4.2) on $\partial\Omega$ of the output voltage u which solves the Neumann boundary value problem of Equation 4.1 when there is one inclusion $D = \varepsilon B + z$. The corresponding formula for many closely spaced small inclusions is stated in $\S 4.3$.

Chapter 5 is devoted to algorithms derived from asymptotic expansions which identify the location, z, and shape of inclusions from a finite number of measurements. Estimates, $\varepsilon^\star$ and $M^\star$, of ε and M are obtained by a singular value decomposition method applied to surface integrals of $H[e_p \cdot \nu]$, where $H[\cdot]$ is the above-mentioned harmonic function, e_p, $1 \le p \le d$, are mutually orthogonal unit vectors of $\mathbb{R}^d$ and ν is the outward unit normal. Next, z is estimated from $M^\star$ (Theorem 5.2) and k is said to be recoverable from GPTs. Other algorithms are presented [O. Kwon, J. K. Seo and J.-R. Yoon, Comm. Pure Appl. Math. **55** (2002), no. 1, 1–29; MR1857878 (2002g:78026); H. Ammari, S. Moskow and M. S. Vogelius, ESAIM Control Optim. Calc. Var. **9** (2003), 49–66 (electronic); MR1957090 (2004j:78025); M. Brühl, M. Hanke and M. S. Vogelius, Numer. Math. **93** (2003), no. 4, 635–654; MR1961882 (2004b:65169); D. L. Colton and A. Kirsch, Inverse Problems **12** (1996), no. 4, 383–393; MR1402098 (97d:35032)]. Differences in the amounts of input data required by each algorithm are pointed out. A stability estimate is provided in $\S 5.2$.

Part II addresses the inverse inclusion problems of linear elastostatics under the same hypotheses, that the small inclusion or crack perturbs an originally homogeneous body. Chapter 6 begins with a review of the layer potentials of the Lamé system $(\S 6.1)$. The single inclusion (D, as above) transmission problem with Neumann jump and boundary conditions is stated (Equation 6.21). Its solvability (Theorem 6.13) relies on [L. Escauriaza and J. K. Seo, Trans. Amer. Math. Soc. **338** (1993), no. 1, 405–430; MR1149120 (93j:35039)]. Displacement, $\mathbf{u}$, in, e.g., $\Omega \setminus \overline{D}$ is decomposed into the sum of $\mathbf{H}[x]$, a function which derives from the boundary data $\mathbf{u}|_{\partial\Omega}$ and $(\frac{\partial \mathbf{u}}{\partial \nu})|_{\partial\Omega}$, and $\mathcal{S}_D[\psi[x]]$, where ψ solves a BIE on ∂D (Theorem 6.15).

Chapter 7, in analogy to Chapter 3, introduces the elastic moment tensors (EMTs) of [H. Ammari et al., J. Elasticity **67** (2002), no. 2, 97–129 (2003); MR1985444 (2004c:74005)], which are the counterpart of GPTs and extend the early notion given by V. G. Maz′ya and S. Nazarov [Trudy Moskov. Mat. Obshch. **50** (1987), 79–129, 261; MR0912054 (89a:35034)]. Symmetry and positive definiteness (Theorem 7.6) are proved; the behaviour under linear transformations is provided $(\S 7.3)$. The entries of EMTs for ellipses and elliptic holes are expressed in computable form (§§7.4, 7.5, Theorems 7.14–7.18).

Chapter 8 is devoted to full asymptotic expansions for the $\mathbf{u}$, which results from a single inclusion, in terms of the Lamé reference constants, the location of the inclusion and its geometry. The expansion (Theorem 8.3) represents $\mathbf{u}$ on $\partial\Omega$ up to a prescribed approximation order, n, as the background solution $\mathbf{U}$ plus a sum, the terms of which are controlled by powers of ε up to $n + d - 1$ and depend on EMTs.

Chapter 9 carries explicit formulae for estimating z and the EMTs of the inclusion from a finite number of $\{\mathbf{g}, \mathbf{f}\}$, i.e., $\{$traction, displacement$\}$ pairs measured on $\partial\Omega$. The function $\mathbf{H}[\mathbf{g}][x] :=$

$-\mathcal{S}_\Omega[\mathbf{g}][x] + \mathcal{D}_\Omega[\mathbf{f}][x]$, where $\mathcal{D}_\Omega[\cdot]$ is the double layer potential on $\partial\Omega$, admits a representation in terms of the EMTs (Theorem 9.1). The latter can in turn be reconstructed from $\mathbf{H}[\cdot]$ (Theorem 9.3) modulo $O[\varepsilon^2 d]$. The "linear" and the "quadratic" methods which detect z, as well as the numerics, are described in §§9.3, 9.4.

Part III addresses AC problems and is mainly based on [H. Ammari and H. Kang, Inverse Problems **19** (2003), no. 1, 63–71; MR1964251 (2004c:78015); J. Math. Anal. Appl. **296** (2004), no. 1, 190–208; MR2070502 (2005c:35054)]. With obvious bounds, the piecewise magnetic permeability μ_δ equals μ_s, $1 \leq s \leq m$, in the inclusions, and μ_0 elsewhere. Electric permittivity ε_δ is defined analogously. The electromagnetic field is governed by the 2D or 3D (scalar) Helmholtz equation of the type $\nabla \cdot (\frac{1}{\mu_\delta} + \omega^2 \varepsilon_\delta u) = 0$ in Ω.

Chapter 10 relies on [M. S. Vogelius and D. Volkov, M2AN Math. Model. Numer. Anal. **34** (2000), no. 4, 723–748; MR1784483 (2001f:78024)] to prove existence and uniqueness for the m-inclusion transmission problem (Equation 10.1), provided $k_0^2 := \omega^2 \varepsilon_0 \mu_0$ is not a Dirichlet eigenvalue of $-\Delta$ in $L^2(\Omega)$.

Chapter 11 deals with the single inclusion transmission problem (Equation 11.8) and arrives at the representations of u in Ω (Theorem 11.5) and of $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$ (Theorem 11.7), which are structurally analogous to those of Chapter 2 and Chapter 7. E.g., $u = \mathbf{H}[x] + \mathcal{S}_D^{(k_0)}[\psi[x]]$ in $\Omega \setminus \overline{D}$ (Equation 11.14), where $\mathcal{S}_D^{(k_0)}[\cdot]$ corresponds to $k = k_0$ everywhere.

Chapter 12, in analogy with Chapter 4 and Chapter 8, provides asymptotic expansions for $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$ based on the moments of $\{\psi_i\}$ (Theorems 12.2-3, 12.5-6). Here i is a d-index such that $0 \leq |i| \leq n$; the $\{\psi_i\}$ solve a system of BIEs on ∂B, where the known terms are x^i and $\frac{\partial x^i}{\partial \nu}$. Leading order terms were derived in the above-cited paper by Vogelius and Volkov.

Chapter 13 presents reconstruction algorithms and numerical results for multiple inclusions from a limited number of voltage-current pairs. Applications of the inverse Fourier transform, based on [H. Ammari, E. N. Iakovleva and S. Moskow, SIAM J. Math. Anal. **34** (2003), no. 4, 882–900 (electronic); MR1969606 (2004a:78022)], and of MUSIC [C. W. Therrien, *Discrete random signals and statistical signal processing*, Prentice Hall, Englewood Cliffs, NJ, 1992; Zbl 0747.94004; M. Cheney, Inverse Problems **17** (2001), no. 4, 591–595; MR1861470] are illustrated.

{Reviewer's comments: The smooth background conductivity and the tuple $\{\varepsilon, z_s, B_s, k_s\}$ represent the "structure" of the body and of its inclusions respectively, whereas the precise support geometry and conductivity function would represent "texture". The indisputable success of the inversion strategy presented by the authors resides in separating structure from texture and focusing on structure for regularization. Technically, the goal is achieved by means of expansions based on GPTs, which in turn derive from layer potential theory and the related addition theorems (as can be seen from the proof of Theorem 3.4).}

Reviewed by *Giovanni F. Crosta*