

# Direct algorithms for thermal imaging of small inclusions

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## Abstract

The goal of this paper is to reconstruct a collection of small inclusions inside a homogeneous object by applying a heat flux and measuring the induced temperature on its boundary. Taking advantage of the smallness of the inclusions, we design efficient non-iterative algorithms for locating the inclusions from boundary measurements of the temperature. We illustrate the feasibility and the viability of our algorithms by numerical examples.

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## 1 Introduction

Thermal imaging is a technique of wide utility in nondestructive testing and evaluation. To apply this procedure one uses a heat source to apply a heat flux to the boundary of an object and then observes the resulting temperature response on the object's surface. From this information one attempts to determine internal thermal conductivity of the object. Thermal imaging has been significantly investigated as a method for damage and corrosion detection [22, 18, 7]. For mathematical considerations of thermal imaging we refer to [17, 14, 5, 6, 10, 11] and the references therein.

In its most general form thermal imaging is severely ill-posed and nonlinear [17]. If, however, in advance we have additional structural information about the object, then we may be able to determine specific features about the internal thermal conductivity of the object with a good resolution. One such type of knowledge could be that the object consists of a smooth background containing a number of unknown small inclusions with a significantly different thermal conductivities. In this case thermal imaging seeks to recover the unknown inclusions. Due to the smallness of the inclusions the associated temperatures measured on the surface of the object are very close to the temperatures corresponding to the object without inclusions. So unless one knows exactly what patterns to look for, noise will largely dominate the information contained in the measured temperature. Furthermore, in applications it is often not necessary to reconstruct precisely the thermal properties or the geometry of the inclusions. The information of real interest is their positions.

Our goal in this paper is to develop, in a mathematically rigorous way, a numerical method to detect the location of the inclusions from boundary measurements of the temperature. We find an asymptotic expansion of solutions to the transmission problem for the heat equation in terms of the geometry of the

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inclusion. This formula describes the perturbation of the solution caused by the presence of an anomaly of small diameter. Based on this asymptotic expansion we derive formulae to find the location of the inclusions with good accuracy. The formulae are explicit and can be easily implemented numerically. The general approach we will take is in the spirit of that developed for reconstructing small conductivity inclusions from boundary measurements, see [16, 9, 4, 20, 1, 8, 2] and the references therein.

The paper is organized as follows. In the next section we formulate the problem and state the asymptotic expansion of the boundary perturbations that are due to a collection of conductivity inclusions. The rigorous derivation of this formula is explained in sections 3 and 4. In section 5 we use our asymptotic expansion to design direct reconstruction algorithms for locating the inclusions. Numerical examples given in this section illustrate the feasibility and the viability of our direct procedures.

## 2 Problem formulation

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ . Assume that  $\Omega$  contains a finite number  $m$  of small inclusions  $\{D_s\}_{s=1}^m$ , each of the form  $D_s = \varepsilon B_s + z_s$ , where  $B_s$ ,  $s = 1, \dots, m$ , is a bounded Lipschitz domain in  $\mathbb{R}^d$  containing the origin. We denote the collection of inclusions by  $D = \cup_{s=1}^m D_s$ . We assume that the inclusions  $D_s$ ,  $s = 1, \dots, m$ , are well-separated from each other and from  $\partial\Omega$ . More precisely, we assume that there exists a constant  $d_0 > 0$  such that

$$\inf_{x \in D_s} \text{dist}(x, \partial\Omega) > d_0 \quad \text{and} \quad \text{dist}(D_i, D_j) > d_0, \quad i \neq j. \quad (2.1)$$

We also assume that the background is homogeneous with thermal conductivity  $\gamma_0$  and the inclusion  $D_s$  has thermal conductivity  $\gamma_s$ , where  $\gamma_s$  and  $\gamma_0$  are positive constants,  $\gamma_s \neq \gamma_0$ ,  $s = 1, \dots, m$ . The thermal conductivity profile  $\gamma_\varepsilon$  of  $\Omega$  is the piecewise constant function

$$\gamma_\varepsilon = \begin{cases} \gamma_0 & \text{for } x \in \Omega \setminus \overline{D}, \\ \gamma_s & \text{for } x \in D_s, s = 1, \dots, m. \end{cases}$$

In this paper we consider the following transmission problem for the heat equation:

$$\begin{cases} \partial_t u - \nabla \cdot (\gamma_\varepsilon \nabla u) = 0 & \text{in } \Omega \times (0, T), \\ u(x, 0) = \theta(x) & \text{for } x \in \Omega, \\ \gamma_0 \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (2.2)$$

where the Neumann boundary data  $g \in L^2(0, T; L^2(\partial\Omega))$  and the initial data  $\theta(x) \in C^1(\overline{\Omega})$  are subject to the compatibility condition:

$$\gamma_0 \frac{\partial \theta}{\partial \nu} = g(\cdot, 0) \quad \text{on } \partial\Omega. \quad (2.3)$$

Here and throughout this paper, the vector  $\nu$  denotes the unit outward normal to  $\partial\Omega$ . We will also use the following notation:

$$\Omega_T = \Omega \times (0, T), \quad D_{sT} = D_s \times (0, T), \quad \partial\Omega_T = \partial\Omega \times (0, T), \quad \partial D_{sT} = \partial D_s \times (0, T).$$

Let  $U$  be the background solution defined as the solution of

$$\begin{cases} \partial_t U - \gamma_0 \Delta U = 0 & \text{in } \Omega_T, \\ U(x, 0) = \theta(x) & \text{for } x \in \Omega, \\ \gamma_0 \frac{\partial U}{\partial \nu} = g & \text{on } \partial\Omega_T. \end{cases}$$

Introduce  $\psi_j^{(s)}$  for  $j = 1, \dots, d$  and  $s = 1, \dots, m$ , to be the unique solution of

$$\begin{cases} \Delta \psi_j^{(s)}(x) = 0 & \text{in } \mathbb{R}^d \setminus \partial B_s, \\ \psi_j^{(s)}|_+(x) - \psi_j^{(s)}|_-(x) = x_j & \text{on } \partial B_s, \\ \gamma_0 \frac{\partial \psi_j^{(s)}}{\partial \nu} \Big|_+(x) - \gamma_s \frac{\partial \psi_j^{(s)}}{\partial \nu} \Big|_-(x) = \gamma_0 \frac{\partial x_j}{\partial \nu} & \text{on } \partial B_s, \\ \psi_j^{(s)}(x) = O(|x|^{1-d}) & \text{as } |x| \rightarrow \infty. \end{cases} \quad (2.4)$$

Here  $\nu$  denotes the outward unit normal to  $\partial B_s$ ; superscripts  $+$  and  $-$  indicate the limiting values as we approach  $\partial B_s$  from outside  $B_s$  and from inside  $B_s$ .

Let

$$\Psi_j^{(s)}(x) := \begin{cases} \psi_j^{(s)}(x), & x \in \mathbb{R}^d \setminus \overline{B_s}, \\ \psi_j^{(s)}(x) + x_j, & x \in B_s. \end{cases} \quad (2.5)$$

Then it follows from (2.4) and (2.5) that

$$\Psi_j^{(s)}|_+ - \Psi_j^{(s)}|_- = 0, \quad \gamma_0 \frac{\partial \Psi_j^{(s)}}{\partial \nu} \Big|_+ - \gamma_s \frac{\partial \Psi_j^{(s)}}{\partial \nu} \Big|_- = (\gamma_0 - \gamma_s) \frac{\partial x_j}{\partial \nu} \quad \text{on } \partial B_s. \quad (2.6)$$

Define the  $d \times d$  matrix  $M^{(s)} = (m_{ij}^{(s)})$  for  $s = 1, \dots, m$ , by

$$m_{ij}^{(s)} = \int_{\partial B_s} y_i \frac{\partial \psi_j^{(s)}}{\partial \nu} \Big|_-(y) d\sigma_y,$$

where  $y_i$  is the  $i$ -th component of  $y$ . The matrix  $M^{(s)}$  is the Pólya-Szegő polarization tensor associated with the domain  $B_s$  and the conductivity  $\gamma_s$  [23]. Its properties were extensively studied in [2]. Recall that if  $B_s$  is a disk, then the polarization tensor  $M^{(s)}$  has the following explicit form [9]:

$$M^{(s)} = -\frac{d\gamma_0|B_s|}{(\gamma_s + (d-1)\gamma_0)} I_d, \quad (2.7)$$

where  $I_d$  is the  $d \times d$  identity-matrix.

The following asymptotic expansion holds.

**Theorem 2.1** *Let  $\Phi \in C^\infty(\overline{\Omega}_T)$  satisfy  $(\partial_t + \gamma_0 \Delta)\Phi(x, t) = 0$  in  $\Omega_T$  with  $\Phi(x, T) = 0$  for  $x \in \Omega$ . Define the weighted boundary measurements*

$$I_\Phi(T) := \int_0^T \int_{\partial \Omega} \gamma_0(u - U)(x, t) \frac{\partial \Phi}{\partial \nu}(x, t) d\sigma_x dt.$$

Then the following asymptotic expansion for  $I_\Phi(T)$  holds as  $\varepsilon \rightarrow 0$ :

$$I_\Phi(T) = -\varepsilon^d \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \nabla U(z_s, t) \cdot M^{(s)} \nabla \Phi(z_s, t) dt + \begin{cases} O(\varepsilon^4) & \text{if } d = 3, \\ O(\varepsilon^3 |\log \varepsilon|) & \text{if } d = 2. \end{cases} \quad (2.8)$$

By making appropriate choices of test functions  $\Phi$  and background solutions  $U$  we will develop from the asymptotic formula (2.8) efficient location search algorithms for detecting the inclusions  $D_s$ ,  $s = 1, \dots, m$ , from measurements of  $u - U$  on  $\partial \Omega_T$ .

### 3 Preliminary results

For the transmission problem for the heat equation there are convenient functional spaces which we recall below. The first one is formed of all functions  $v(x, t)$  on  $\Omega_T$  with finite norm

$$\|v\|_{L^2(0,T;H(\Omega))} = \left( \int_0^T \|v(\cdot, t)\|_{H(\Omega)}^2 dt \right)^{1/2},$$

where  $H(\Omega) = L^2(\Omega)$  or  $H^1(\Omega)$ , and the second one of all functions  $v(x, t)$  on  $\Omega_T$  with finite norm

$$|v|_{\Omega_T} = \sup_{0 \leq t \leq T} \|v(\cdot, t)\|_{2,\Omega} + \|\nabla v\|_{2,\Omega_T},$$

where

$$\|v(\cdot, t)\|_{2,\Omega} := \left( \int_{\Omega} |v(x, t)|^2 dx \right)^{1/2}, \quad \|v\|_{2,\Omega_T} = \left( \int_0^T \int_{\Omega} |v(x, t)|^2 dx dt \right)^{1/2}.$$

We also recall that the Lipschitz character of  $\Omega$  is the  $L^\infty$ -norm of the gradient of the function parametrizing the boundary  $\partial\Omega$ . See for example [2]. It is clear that the Lipschitz character of the domain  $t\Omega$ ,  $t > 0$ , is independent of  $t$ . The following regularity result on the transmission problem for the heat equation will be of use to us.

**Lemma 3.1** *Let  $v$  be the solution of the following transmission problem for the heat equation:*

$$\left\{ \begin{array}{l} \partial_t v(x, t) - \gamma_0 \Delta v(x, t) = F(x, t) \quad \text{in } \Omega \setminus \overline{D} \times (0, T), \\ \partial_t v(x, t) - \gamma_s \Delta v(x, t) = F(x, t) \quad \text{in } D_{sT}, \quad s = 1, \dots, m, \\ v|_+(x, t) - v|_-(x, t) = 0 \quad \text{on } \partial D_{sT}, \\ \gamma_0 \frac{\partial v}{\partial \nu} \Big|_+(x, t) - \gamma_s \frac{\partial v}{\partial \nu} \Big|_-(x, t) = f(x, t) \quad \text{on } \partial D_{sT}, \\ v(x, 0) = v_0(x) \quad \text{for } x \in \Omega, \\ \gamma_0 \frac{\partial v}{\partial \nu}(x, t) = h(x, t) \quad \text{on } \partial\Omega_T. \end{array} \right.$$

Then the following regularity estimate holds:

$$|v|_{\Omega_T} \leq C \left[ \|v_0\|_{2,\Omega} + T^{1/2} \left( \|F\|_{L^2(0,T;L^2(\Omega))} + \|h\|_{L^2(0,T;L^2(\partial\Omega))} + \|f\|_{L^2(0,T;L^2(\partial D))} \right) \right], \quad (3.1)$$

where the constant  $C$  depends only on the Lipschitz characters of  $\Omega$  and  $B_s$ ,  $s = 1, \dots, m$ , and the constants  $\gamma_s$ ,  $s = 0, \dots, m$ .

*Proof.* We first recall the trace estimate

$$\|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)},$$

where the constant  $C$  depends only on the Lipschitz character of  $\Omega$ . See [15] and [21]. Through integrations by parts over  $\Omega \times (0, t)$  for  $0 \leq t \leq T$  and straightforward calculations, we have

$$\begin{aligned} \int_0^t \int_{\Omega} \gamma_{\varepsilon} |\nabla v|^2 &= \int_0^t \int_{\Omega \setminus D} \gamma_0 \nabla v \cdot \nabla v + \sum_{s=1}^m \int_0^t \int_{D_s} \gamma_s \nabla v \cdot \nabla v \\ &= \int_0^t \int_{\partial \Omega} \gamma_0 \frac{\partial v}{\partial \nu} v + \sum_{s=1}^m \int_0^t \int_{\partial D_s} \left( \gamma_s \frac{\partial v}{\partial \nu} \Big|_{-} - \gamma_0 \frac{\partial v}{\partial \nu} \Big|_{+} \right) v - \int_0^t \int_{\Omega} (\partial_t v - F)v \\ &= \int_0^t \int_{\partial \Omega} h v - \int_0^t \int_{\partial D} f v - \int_0^t \int_{\Omega} (\partial_t v - F)v. \end{aligned}$$

It then follows from the above trace estimate that

$$\begin{aligned} \int_0^t \int_{\Omega} \frac{1}{2} \partial_t (v^2) + \gamma_{\varepsilon} |\nabla v|^2 &= \int_0^t \int_{\Omega} F v + \int_0^t \int_{\partial \Omega} h v - \int_0^t \int_{\partial D} f v \\ &\leq \int_0^t \left[ \|F(\cdot, \tau)\|_{L^2(\Omega)} \|v(\cdot, \tau)\|_{L^2(\Omega)} + \|h(\cdot, \tau)\|_{L^2(\partial \Omega)} \|v(\cdot, \tau)\|_{L^2(\partial \Omega)} \right. \\ &\quad \left. + \|f(\cdot, \tau)\|_{L^2(\partial D)} \|v(\cdot, \tau)\|_{L^2(\partial D)} \right] d\tau \\ &\leq \|F\|_{L^2(0, T; L^2(\Omega))} \|v\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; L^2(\partial \Omega))} \|v\|_{L^2(0, T; L^2(\partial \Omega))} \\ &\quad + \|f\|_{L^2(0, T; L^2(\partial D))} \|v\|_{L^2(0, T; L^2(\partial D))} \\ &\leq \|F\|_{L^2(0, T; L^2(\Omega))} \|v\|_{L^2(0, T; L^2(\Omega))} + C \|h\|_{L^2(0, T; L^2(\partial \Omega))} \|v\|_{L^2(0, T; H^1(\Omega))} \\ &\quad + C \|f\|_{L^2(0, T; L^2(\partial D))} \|v\|_{L^2(0, T; H^1(D))} \\ &\leq CT^{1/2} \left( \|F\|_{L^2(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; L^2(\partial \Omega))} + \|f\|_{L^2(0, T; L^2(\partial D))} \right) |v|_{\Omega_T}, \end{aligned}$$

for some constant  $C$  depending only on the Lipschitz characters of  $\Omega$  and  $B_s$ ,  $s = 1, \dots, m$ .

Let

$$\mu = \min \left\{ \frac{1}{2}, \min_{0 \leq s \leq m} \gamma_s \right\}. \quad (3.2)$$

Then, by using Young's inequality, we obtain that

$$\mu |v|_{\Omega_T}^2 \leq \frac{1}{2} \|v_0\|_{2, \Omega}^2 + \frac{CT}{2\delta} \left( \|F\|_{L^2(0, T; L^2(\Omega))}^2 + \|h\|_{L^2(0, T; L^2(\partial \Omega))}^2 + \|f\|_{L^2(0, T; L^2(\partial D))}^2 \right) + \frac{3\delta}{2} |v|_{\Omega_T}^2,$$

which gives the desired estimate by taking  $\delta$  small enough. This completes the proof.  $\square$

We start the derivation of the asymptotic formula (2.8) for  $u$  with the following estimate of  $u - U$ .

**Lemma 3.2** *There exists a constant  $C$  such that*

$$|u - U|_{\Omega_T} \leq C \varepsilon^{\frac{d}{2}},$$

where  $C = \frac{1}{\mu} T^{1/2} \sup_{(x, t) \in \Omega_T} |\nabla U(x, t)| \sum_{s=1}^m |\gamma_0 - \gamma_s| |B_s|$  and  $\mu$  is given by (3.2).

*Proof.* Let  $W = u - U$ . Then  $W$  is the solution of

$$\begin{cases} \partial_t W - \nabla \cdot (\gamma_\varepsilon \nabla W) = \sum_{s=1}^m (\gamma_s - \gamma_0) \left[ \chi_{D_s} \Delta U - \delta_{\partial D_s} \frac{\partial U}{\partial \nu} \right] & \text{in } \Omega_T, \\ W(x, 0) = 0 & \text{for } x \in \Omega, \\ \gamma_0 \frac{\partial W}{\partial \nu} = 0 & \text{on } \partial \Omega_T. \end{cases} \quad (3.3)$$

Here  $\chi_{D_s}$  is the characteristic function of the domain  $D_s$  and  $\delta_{\partial D_s}$  is the Dirac-delta function on the boundary of  $D_s$ . The function  $W$  also satisfies

$$\begin{cases} W|_+ - W|_- = 0 & \text{on } \partial D_{sT}, \\ \gamma_0 \frac{\partial W}{\partial \nu} \Big|_+ - \gamma_s \frac{\partial W}{\partial \nu} \Big|_- = \sum_{s=1}^m (\gamma_s - \gamma_0) \frac{\partial U}{\partial \nu} & \text{on } \partial D_{sT}. \end{cases}$$

Let

$$f = \sum_{s=1}^m (\gamma_s - \gamma_0) \left[ \chi_{D_s} \Delta U - \delta_{\partial D_s} \frac{\partial U}{\partial \nu} \right].$$

Multiplying the first equation in (3.3) by  $W$ , integrating by parts over  $\Omega \times (0, t)$  and using the divergence theorem and (3.3), we obtain

$$\begin{aligned} \int_0^t \int_\Omega f W \, dx d\tau &= \int_0^t \int_\Omega \left( \partial_t W - \nabla \cdot (\gamma_\varepsilon \nabla W) \right) W \, dx d\tau \\ &= \int_0^t \int_\Omega \frac{1}{2} \partial_t (W^2) \, dx d\tau - \int_0^t \int_{\partial \Omega} \gamma_\varepsilon \frac{\partial W}{\partial \nu} W \, d\sigma_x d\tau + \int_0^t \int_\Omega \gamma_\varepsilon \nabla W \cdot \nabla W \, dx d\tau \\ &= \frac{1}{2} \|W(\cdot, t)\|_{2, \Omega}^2 + \int_0^t \int_\Omega \gamma_\varepsilon |\nabla W|^2 \, dx d\tau. \end{aligned}$$

On the other hand, making use of the divergence theorem and Hölder's inequality, we readily get

$$\begin{aligned} \left| \int_0^t \int_\Omega f W \, dx d\tau \right| &= \left| \sum_{s=1}^m \left[ \int_0^t \int_{D_s} (\gamma_s - \gamma_0) \Delta U W + \int_0^t \int_{\partial D_s} (\gamma_0 - \gamma_s) \frac{\partial U}{\partial \nu} W \right] \right| \\ &= \left| \sum_{s=1}^m \int_0^t \int_{D_s} (\gamma_0 - \gamma_s) \nabla U \cdot \nabla W \, dx d\tau \right| \\ &\leq \sum_{s=1}^m \int_0^T \|(\gamma_0 - \gamma_s) \nabla U(\cdot, t)\|_{2, D_s} \|\nabla W(\cdot, t)\|_{2, D_s} \, d\tau \\ &\leq \sum_{s=1}^m \|(\gamma_0 - \gamma_s) \nabla U\|_{2, D_{sT}} \|\nabla W\|_{2, D_{sT}} \\ &\leq \sum_{s=1}^m |\gamma_0 - \gamma_s| \sup_{(x,t) \in \Omega_T} |\nabla U(x, t)| |D_s|^{1/2} T^{1/2} |W|_{\Omega_T} \\ &= c\varepsilon^{\frac{d}{2}} |W|_{\Omega_T}, \end{aligned}$$

where  $c = T^{1/2} \sup_{(x,t) \in \Omega_T} |\nabla U(x,t)| \sum_{s=1}^m |\gamma_0 - \gamma_s| |B_s|$ . Combining the above estimates we obtain that

$$\frac{1}{2} \sup_{0 \leq t \leq T} \|W(\cdot, t)\|_{2,\Omega}^2 + \int_0^T \int_{\Omega} \gamma_{\varepsilon} |\nabla W|^2 dx d\tau \leq c\varepsilon^{\frac{d}{2}} |W|_{\Omega_T},$$

and hence

$$\mu |W|_{\Omega_T}^2 \leq c\varepsilon^{\frac{d}{2}} |W|_{\Omega_T},$$

which completes the proof.  $\square$

**Lemma 3.3** *Let  $u, U$  and  $\Phi$  be the functions defined in Theorem 2.1. Then the following identity holds:*

$$I_{\Phi}(T) = \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \int_{D_s} \nabla u(x,t) \cdot \nabla \Phi(x,t) dx dt. \quad (3.4)$$

*Proof.* Set  $W = u - U$ . Then integrating by parts and using the conditions  $\Phi|_{t=T} = 0$  and  $W|_{t=0} = 0$  we obtain that

$$\begin{aligned} \int_0^T \int_{\Omega} \left( \partial_t W \Phi + \gamma_0 \nabla W \cdot \nabla \Phi \right) &= \int_{\Omega} \left[ W(x,T) \Phi(x,T) - W(x,0) \Phi(x,0) \right] dx \\ &\quad - \int_0^T \int_{\Omega} W \partial_t \Phi + \gamma_0 \int_0^T \int_{\partial\Omega} W \frac{\partial \Phi}{\partial \nu} - \gamma_0 \int_0^T \int_{\Omega} W \Delta \Phi \\ &= \gamma_0 \int_0^T \int_{\partial\Omega} W \frac{\partial \Phi}{\partial \nu}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_0^T \int_{\Omega} \left( \partial_t W \Phi + \gamma_0 \nabla W \cdot \nabla \Phi \right) &= \int_0^T \int_{\partial(\Omega \setminus D)} \gamma_0 \frac{\partial u}{\partial \nu} \Phi + \sum_{s=1}^m \int_0^T \int_{\partial D_s} \gamma_s \frac{\partial u}{\partial \nu} |_{-\Phi} \\ &\quad + \sum_{s=1}^m \int_0^T \int_{D_s} (\gamma_0 - \gamma_s) \nabla u \cdot \nabla \Phi - \int_0^T \int_{\partial\Omega} \gamma_0 \frac{\partial U}{\partial \nu} \Phi \\ &= \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \int_{D_s} \nabla u \cdot \nabla \Phi. \end{aligned}$$

The proof of identity (3.4) is now complete.  $\square$

## 4 Proof of the asymptotic expansion formula

In this section we prove Theorem 2.1. Recall that  $D = \cup_{s=1}^m D_s = \cup_{s=1}^m (\varepsilon B_s + z_s)$ . Let

$$V(x,t) = u(x,t) - U(x,t) + \varepsilon \sum_{s=1}^m \sum_{j=1}^d \partial_j U(z_s,t) \Psi_j^{(s)} \left( \frac{x - z_s}{\varepsilon} \right), \quad (4.1)$$

where  $\Psi_j^{(s)}$  is defined by (2.5).

**Lemma 4.1** *The following estimate holds*

$$\|\nabla V\|_{2,\Omega_T} \leq C \begin{cases} \varepsilon^{5/2} & \text{if } d = 3, \\ \varepsilon^2 |\log \varepsilon| & \text{if } d = 2, \end{cases} \quad (4.2)$$

for some constant  $C$  independent of  $\varepsilon$ .

*Proof.* We suppose that there is only a single inclusion  $D = \varepsilon B$  ( $z = 0$ ) with conductivity  $\gamma$ . The general case of multiple inclusion follows by iteration of the arguments that we will present for the case  $m = 1$ . In other words, we may develop asymptotic formulae involving the difference between the fields  $u$  and  $U$  with  $s$  inclusions and those with  $s - 1$  inclusions,  $s = m, \dots, 1$ , and then at the end essentially form the sum of those  $m$  formulae.

Observe that  $V$  satisfies

$$\begin{aligned} \partial_t V - \gamma_0 \Delta V &= \varepsilon \sum_{j=1}^d \partial_t \partial_j U(0, t) \Psi_j \left( \frac{x}{\varepsilon} \right) := F(x, t) \quad \text{in } (\Omega \setminus \overline{D}) \times (0, T), \\ \partial_t V - \gamma \Delta V &= (\gamma_\varepsilon - \gamma_0) \chi_D \Delta U + \varepsilon \sum_{j=1}^d \partial_t \partial_j U(0, t) \Psi_j \left( \frac{x}{\varepsilon} \right) := F(x, t) \quad \text{in } D \times (0, T). \end{aligned}$$

We also have

$$\begin{aligned} V|_+ - V|_- &= 0 \quad \text{on } \partial D \times (0, T), \\ \gamma_0 \frac{\partial V}{\partial \nu} \Big|_+ - \gamma \frac{\partial V}{\partial \nu} \Big|_- &= (\gamma - \gamma_0) \left[ \frac{\partial U}{\partial \nu} - \sum_{j=1}^d \partial_j U(0, t) \frac{\partial \Psi_j}{\partial \nu} \right] := G(x, t) \quad \text{on } \partial D \times (0, T), \end{aligned}$$

and

$$\begin{aligned} V(x, 0) &= \varepsilon \sum_{j=1}^d \partial_j \theta(0) \Psi_j \left( \frac{x}{\varepsilon} \right) \quad \text{in } \Omega, \\ \gamma_0 \frac{\partial V}{\partial \nu} &= \gamma_0 \sum_{j=1}^d \partial_j U(0, t) \frac{\partial \Psi_j}{\partial \nu} \left( \frac{x}{\varepsilon} \right) := H(x, t) \quad \text{on } \partial \Omega \times (0, T). \end{aligned}$$

Let  $\tilde{\Omega} = \varepsilon^{-1} \Omega$  and

$$v(x, t) := V(\varepsilon x, \varepsilon^2 t), \quad x \in \tilde{\Omega}, \quad 0 \leq t \leq T/\varepsilon^2.$$

Then,

$$\|\nabla V\|_{2,\Omega_T}^2 = \varepsilon^{2+d} \int_0^{T/\varepsilon^2} \|\nabla v(\cdot, t)\|_{2,\Omega}^2 dt,$$

and hence it suffices to show that

$$\int_0^{T/\varepsilon^2} \|\nabla v(\cdot, t)\|_{2,\Omega}^2 dt \leq C \quad (4.3)$$

for some constant  $C$  independent of  $\varepsilon$ .

Observe that  $v$  satisfies

$$\begin{cases} \partial_t v - \gamma_0 \Delta v = \varepsilon^2 F(\varepsilon x, \varepsilon^2 t) := f(x, t) & \text{in } (\tilde{\Omega} \setminus \overline{B}) \times (0, T/\varepsilon^2), \\ \partial_t v - \gamma \Delta v = \varepsilon^2 F(\varepsilon x, \varepsilon^2 t) := f(x, t) & \text{in } B \times (0, T/\varepsilon^2), \\ v|_+ - v|_- = 0 & \text{on } \partial B \times (0, T/\varepsilon^2), \\ \gamma_0 \frac{\partial v}{\partial \nu} \Big|_+ - \gamma \frac{\partial v}{\partial \nu} \Big|_- = \varepsilon G(\varepsilon x, \varepsilon^2 t) := g(x, t) & \text{on } \partial B \times (0, T/\varepsilon^2), \\ v(x, 0) = \varepsilon \sum_{j=1}^d \partial_j \theta(0) \Psi_j(x) & \text{in } \tilde{\Omega}, \\ \gamma_0 \frac{\partial v}{\partial \nu} = \varepsilon H(\varepsilon x, \varepsilon^2 t) := h(x, t) & \text{on } \partial \tilde{\Omega} \times (0, T/\varepsilon^2). \end{cases}$$

Since  $\Psi_j(x) = O(|x|^{1-d})$  as  $|x| \rightarrow \infty$  and  $\text{diam}(\tilde{\Omega}) = O(\varepsilon^{-1})$ , we have

$$\int_{\tilde{\Omega}} |\Psi_j(x)|^2 dx = \begin{cases} O(\log \frac{1}{\varepsilon}) & \text{if } d = 2, \\ O(1) & \text{if } d = 3, \end{cases} \quad (4.4)$$

and therefore

$$\int_{\tilde{\Omega}} |v(x, 0)|^2 dx \leq C \begin{cases} \varepsilon^2 & \text{if } d = 3, \\ \varepsilon^2 |\log \varepsilon| & \text{if } d = 2. \end{cases}$$

Moreover, we claim that

$$\int_0^{T/\varepsilon^2} \|f(\cdot, t)\|_{2, \tilde{\Omega}}^2 dt \leq C\varepsilon^2, \quad \int_0^{T/\varepsilon^2} \|g(\cdot, t)\|_{2, \partial B}^2 dt \leq C\varepsilon^2, \quad \int_0^{T/\varepsilon^2} \|h(\cdot, t)\|_{2, \partial \tilde{\Omega}}^2 dt \leq C\varepsilon^2, \quad (4.5)$$

and hence (4.3) follows from Lemma 3.1.

We now prove (4.5). By definition of  $f$  and  $F$ , we get

$$\begin{aligned} \int_0^{T/\varepsilon^2} \|f(\cdot, t)\|_{2, \tilde{\Omega}}^2 dt &\leq C\varepsilon^4 \int_0^{T/\varepsilon^2} \int_B |\Delta U(\varepsilon x, \varepsilon^2 t)|^2 dx dt \\ &\quad + C\varepsilon^6 \sum_{j=1}^d \int_0^{T/\varepsilon^2} |\partial_t \partial_j U(0, \varepsilon^2 t)|^2 dt \int_{\tilde{\Omega}} |\Psi_j(x)|^2 dx. \end{aligned}$$

Since  $U$  is a smooth function, the first inequality in (4.5) follows from (4.4).

To prove the second inequality in (4.5), we note that

$$\frac{\partial U}{\partial \nu}(x, t) - \sum_{j=1}^d \partial_j U(0, t) \frac{\partial x_j}{\partial \nu} = O(|x|)$$

for all  $(x, t) \in \partial D \times (0, T)$ . Therefore,

$$g(x, t) = \varepsilon G(\varepsilon x, \varepsilon^2 t) = O(\varepsilon^2),$$

and hence the second inequality in (4.5) holds.

For the third inequality, we have

$$\int_0^{T/\varepsilon^2} \|h(\cdot, t)\|_{2, \partial\tilde{\Omega}}^2 dt \leq \varepsilon^2 \sum_{j=1}^d \int_0^{T/\varepsilon^2} |\partial_j U(0, t)|^2 dt \int_{\partial\tilde{\Omega}} \left| \frac{\partial \Psi_j}{\partial \nu}(x) \right|^2 dx.$$

Since  $|\nabla \Psi_j(x)| = O(|x|^{-d})$  as  $|x| \rightarrow \infty$ , we have  $|\nabla \Psi_j(x)| = O(\varepsilon^d)$  for all  $x \in \partial\tilde{\Omega}$ . Thus we get

$$\int_0^{T/\varepsilon^2} \|h(\cdot, t)\|_{2, \partial\tilde{\Omega}}^2 dt \leq C\varepsilon^{d+1}.$$

This completes the proof of (4.5) and Lemma 4.1.  $\square$

We are now ready to prove Theorem 2.1. Since  $\nabla \Psi_j^{(s)}(x) = O(|x|^{-d})$  as  $|x| \rightarrow \infty$  and the inclusions are well-separated, it follows from (3.4) and (4.2) that

$$\begin{aligned} & \left| I_\Phi(T) - \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \int_{D_s} [\nabla U(x, t) - \sum_{j=1}^d \partial_j U(z_s, t) \nabla \Psi_j^{(s)}(\frac{x - z_s}{\varepsilon})] \cdot \nabla \Phi(x, t) dx dt \right| \\ &= \left| \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \int_{D_s} \nabla V(x, t) \cdot \nabla \Phi(x, t) dx dt \right| \\ &\leq C\varepsilon^{d/2} \|\nabla V\|_{2, \Omega_T} = \begin{cases} O(\varepsilon^4) & \text{if } d = 3, \\ O(\varepsilon^3 |\log \varepsilon|) & \text{if } d = 2. \end{cases} \end{aligned}$$

Since  $\nabla U(x, t) = \sum_{j=1}^d \partial_j U(z_s, t) \nabla x_j + O(\varepsilon)$  and  $\nabla \Phi(x, t) = \sum_{j=i}^d \partial_i \Phi(z_s, t) \nabla x_i + O(\varepsilon)$  for all  $x \in D_s$ , we get the following modulo  $O(\varepsilon^4)$  if  $d = 3$  and  $O(\varepsilon^3 |\log \varepsilon|)$  if  $d = 2$ :

$$\begin{aligned} I_\Phi(T) &= \varepsilon^d \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \sum_{j=1}^d \sum_{i=1}^d \partial_j U(z_s, t) \partial_i \Phi(z_s, t) \int_{B_s} \nabla(x_j - \Psi_j^{(s)}(x)) \cdot \nabla x_i dx dt \\ &= -\varepsilon^d \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \sum_{j=1}^d \sum_{i=1}^d \partial_j U(z_s, t) \partial_i \Phi(z_s, t) \int_{\partial B_s} x_i \frac{\partial \psi_j^{(s)}}{\partial \nu} \Big|_- d\sigma dt \\ &= -\varepsilon^d \sum_{s=1}^m (\gamma_0 - \gamma_s) \int_0^T \nabla U(z_s, t) \cdot M^{(s)} \nabla \Phi(z_s, t) dt, \end{aligned}$$

as desired.

## 5 Location search algorithms

In this section we apply the formula (2.8) in Theorem 2.1 (with an appropriate choice of test functions  $\Phi$  and background solutions  $U$ ) for the purpose of identifying the location of the inclusions  $D_s$ ,  $s = 1, \dots, m$ . We first design a real-time location search algorithm with good resolution and accuracy for detecting a single inclusion ( $m = 1$ ). This algorithm is related to the constant current projection algorithm developed in [20]. See also [3]. It makes use of constant heat flux and not surprisingly, it is limited in its ability to effectively locate multiple small inclusions.

Using many heat sources we then describe an efficient method to locate multiple inclusions ( $m > 1$ ) and illustrate its feasibility. Our approach is related to the linear sampling method [19] and MUSIC algorithm [13, 12]. For the sake of simplicity we only consider the two-dimensional case.

## 5.1 Detection of a single inclusion

For  $y \in \mathbb{R}^2 \setminus \overline{\Omega}$ , let

$$\Phi(x, t) = \Phi_y(x, t) := \frac{1}{4\pi\gamma_0(T-t)} e^{-\frac{|x-y|^2}{4\gamma_0(T-t)}}. \quad (5.1)$$

The function  $\Phi$  satisfies  $(\partial_t + \gamma_0\Delta)\Phi = 0$  in  $\Omega_T$  and the final condition  $\Phi|_{t=T} = 0$  in  $\Omega$ .

Suppose that there is only one inclusion  $D = z + \varepsilon B$  with conductivity  $\gamma$ . For simplicity assume that  $B$  is a disk. Then it follows from (2.7) that the polarization tensor associated with  $B$  and  $\gamma$  is given by  $M = -(2\gamma_0|B|/(\gamma + \gamma_0))I_2$ , where  $I_2$  is the  $2 \times 2$  identity matrix. Choose the background solution  $U(x, t) (= \theta(x))$  to be a harmonic (time-independent) function in  $\Omega_T$ . We compute

$$\begin{aligned} \nabla\Phi_y(z, t) &= \frac{y-z}{8\pi\gamma_0^2(T-t)^2} e^{-\frac{|z-y|^2}{4\gamma_0(T-t)}}, \\ M\nabla\Phi_y(z, t) &= \frac{|B|}{\gamma + \gamma_0} \frac{z-y}{4\pi\gamma_0(T-t)^2} e^{-\frac{|z-y|^2}{4\gamma_0(T-t)}}, \end{aligned}$$

and

$$\int_0^T M\nabla\Phi_y(z, t) dt = \frac{|B|}{\gamma_0 + \gamma} \frac{z-y}{4\pi\gamma_0} \int_0^T \frac{e^{-\frac{|z-y|^2}{4\gamma_0(T-t)}}}{(T-t)^2} dt.$$

But

$$\frac{d}{dt} e^{-\frac{|z-y|^2}{4\gamma_0(T-t)}} = \frac{-|z-y|^2}{4\gamma_0} \frac{e^{-\frac{|z-y|^2}{4\gamma_0(T-t)}}}{(T-t)^2}$$

and therefore

$$\int_0^T M\nabla\Phi_y(z, t) dt = \frac{|B|}{\gamma + \gamma_0} \frac{z-y}{\pi|z-y|^2} e^{-\frac{|z-y|^2}{4\gamma_0(T-t)}}.$$

Then the asymptotic expansion (2.8) in Theorem 2.1 yields

$$\begin{aligned} I_\Phi(T)(y) &= -\varepsilon^2(\gamma_0 - \gamma) \int_0^T \nabla\theta(z) \cdot M\nabla\Phi_y(z, t) dt + O(\varepsilon^3 |\log \varepsilon|) \\ &= \varepsilon^2 \frac{\gamma_0 - \gamma}{\gamma + \gamma_0} |B| \frac{\nabla\theta(z) \cdot (y-z)}{\pi|y-z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}} + 0(\varepsilon^3 |\log \varepsilon|). \end{aligned} \quad (5.2)$$

Now we are in a position to present our location search algorithms for detecting a single inclusion.

- **First algorithm** (with one measurement): We prescribe the initial condition  $\theta(x) = a \cdot x$  for some fixed unit constant vector  $a$  and choose  $g = \gamma_0 a \cdot \nu$  as an applied time-independent heat flux on  $\partial\Omega_T$ . Throughout this section,  $a$  is taken to be a coordinate unit vector. Take two observation lines  $\Sigma_1$  and  $\Sigma_2$  contained in  $\mathbb{R}^2 \setminus \overline{\Omega}$  such that

$$\Sigma_1 := \text{a line parallel to } a, \Sigma_2 := \text{a line normal to } a.$$

Next we find two points  $P_i \in \Sigma_i (i = 1, 2)$  so that  $I_\Phi(T)(P_1) = 0$  and

$$I_\Phi(T)(P_2) = \begin{cases} \min_{x \in \Sigma_2} I_\Phi(T)(x) & \text{if } \gamma_0 - \gamma < 0, \\ \max_{x \in \Sigma_2} I_\Phi(T)(x) & \text{if } \gamma_0 - \gamma > 0. \end{cases}$$

Finally, we draw the corresponding lines  $\Pi_1(P_1)$  and  $\Pi_2(P_2)$  given by

$$\Pi_1(P_1) := \{x|a \cdot (x - P_1) = 0\},$$

$$\Pi_2(P_2) := \{x|(x - P_2) \text{ is parallel to } a\}.$$

Then the intersecting point  $P$  of  $\Pi_1(P_1) \cap \Pi_2(P_2)$  is close to the inclusion  $D$ . The proof of following lemma follows from [20]. We sketch it for the reader's convenience.

**Lemma 5.1** *There exists a positive constant  $C$  independent of  $\varepsilon$  and  $z$  such that  $|P - z| \leq C\varepsilon|\log \varepsilon|$  for  $\varepsilon$  small enough.*

*Proof.* Fix for example  $\gamma_0 > \gamma$ . From (5.2) it follows that there exists a constant  $C$ , independent of  $y = (y_1, 0), z = (z_1, z_2)$ , and  $\varepsilon$  such that

$$I_\Phi(T)(y) \leq \varepsilon^2 \left( \frac{\gamma_0 - \gamma}{\gamma + \gamma_0} |B| \frac{(y_1 - z_1)}{\pi|y - z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}} + C\varepsilon|\log \varepsilon| \right),$$

and

$$I_\Phi(T)(y) \geq \varepsilon^2 \left( \frac{\gamma_0 - \gamma}{\gamma + \gamma_0} |B| \frac{(y_1 - z_1)}{\pi|y - z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}} - C\varepsilon|\log \varepsilon| \right),$$

for all  $y \in \Sigma_1$ . Therefore, for  $y = (y_1, 0) \in \Sigma_1$  satisfying

$$\frac{y_1 - z_1}{|y - z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}} \leq -\frac{C(\gamma + \gamma_0)\varepsilon|\log \varepsilon|}{(\gamma_0 - \gamma)|B|}$$

we have  $I_\Phi(T)(y) \leq 0$ . On the other hand, for  $y \in \Sigma_1$  satisfying

$$\frac{y_1 - z_1}{|y - z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}} \geq \frac{C(\gamma + \gamma_0)\varepsilon|\log \varepsilon|}{(\gamma_0 - \gamma)|B|}$$

we similarly have  $I_\Phi(T)(y) \geq 0$ . This implies that the quantity  $I_\Phi(T)(y)$  changes sign on  $\Sigma_1$  for  $y$  satisfying

$$-\frac{C(\gamma + \gamma_0)\varepsilon|\log \varepsilon|}{(\gamma_0 - \gamma)|B|} \leq \frac{y_1 - z_1}{|y - z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}} \leq \frac{C(\gamma + \gamma_0)\varepsilon|\log \varepsilon|}{(\gamma_0 - \gamma)|B|},$$

for some positive constant  $C$  independent of  $\varepsilon$ . Therefore, the zero point  $P_1$  satisfies  $|(P - z) \cdot a| \leq C'\varepsilon|\log \varepsilon|$  for some constant  $C'$  depending only on  $\Omega, B, T, \gamma_0$ , and  $\gamma$ . Let  $y = (0, y_2) \in \Sigma_2$ . The point  $y^* = (0, z_2) \in \Sigma_2$  is the unique point where

$$\varepsilon^2 \frac{\gamma_0 - \gamma}{\gamma_0 + \gamma} |B| \frac{\nabla\theta(z) \cdot (y - z)}{\pi|y - z|^2} e^{-\frac{|y-z|^2}{4\gamma_0 T}}$$

attains its maximum. Following [20], we can easily see that the point  $P_2 \in \Sigma_2$  where  $I_\Phi(T)(y)$  attains its maximum belongs to an  $\varepsilon$ -neighborhood of  $y^*$  and conclude that  $P = \Pi_1(P_1) \cap \Pi_2(P_2)$  satisfies  $|P - z| \leq C\varepsilon|\log \varepsilon|$  for some constant  $C$  depending only on  $\Omega, B, T, \gamma_0$ , and  $\gamma$ .  $\square$

- **Second algorithm** (with two measurements): Consider two measurements corresponding to the initial conditions  $\theta_1(x) = a \cdot x$  and  $\theta_2(x) = a^\perp \cdot x$  for some fixed constant vector  $a$ , where  $a^\perp$  is a unit vector orthogonal to  $a$  and choose  $g_1 = \gamma_0 a \cdot \nu$  and  $g_2 = \gamma_0 a^\perp \cdot \nu$  as applied time-independent heat flux on  $\partial\Omega_T$ . Here  $\Sigma_1$  and  $\Sigma_2$  are as in the first algorithm. Denote by  $I_\Phi^i(T)$  the function  $I_\Phi(T)$  corresponding to  $\theta = \theta_i$ ,  $i = 1, 2$ . Then we find two points  $P_i \in \Sigma_i$  ( $i = 1, 2$ ) so that  $I_\Phi^i(T)(P_i) = 0$ . We draw the corresponding lines  $\Pi_1(P_1)$  and  $\Pi_2(P_2)$  given by

$$\Pi_1(P_1) := \{x | a \cdot (x - P_1) = 0\},$$

$$\Pi_2(P_2) := \{x | (x - P_2) \text{ is parallel to } a\}.$$

From the proof Lemma 5.1 it immediately follows that the intersecting point  $P$  of  $\Pi_1(P_1) \cap \Pi_2(P_2)$  is close to the inclusion  $D$ : There exists a positive constant  $C$  independent of  $\varepsilon$  and  $z$  such that  $|P - z| \leq C\varepsilon |\log \varepsilon|$  for  $\varepsilon$  small enough.

## 5.2 Detection of multiple inclusions

Consider  $m$  inclusions  $D_s = \varepsilon B_s + z_s$ ,  $s = 1, \dots, m$ . Choose

$$U(x, t) = U_{y'}(x, t) := \frac{1}{4\pi\gamma_0 t} e^{-\frac{|x-y'|^2}{4\gamma_0 t}} \quad \text{for } y' \in \mathbb{R}^2 \setminus \overline{\Omega},$$

or equivalently  $g$  to be the heat flux corresponding to a heat source placed at the point source  $y'$  and the initial condition  $\theta(x) = 0$  in  $\Omega$ , to obtain that

$$\begin{aligned} I_\Phi(T) &= -\varepsilon^2 \sum_{s=1}^m \frac{(\gamma_0 - \gamma_s)}{64\pi^2\gamma_0^4} (y' - z_s) M^{(s)}(y - z_s) \int_0^T \frac{1}{t^2(T-t)^2} \exp\left(-\frac{|y - z_s|^2}{4\gamma_0(T-t)} - \frac{|y' - z_s|^2}{4\gamma_0 t}\right) dt \\ &\quad + O(\varepsilon^3 |\log \varepsilon|), \end{aligned}$$

where  $\Phi$  is given by (5.1).

Suppose for the sake of simplicity that all the domains  $B_s$  are disks. Then it follows from (2.7) that  $M^{(s)} = m^{(s)} I_2$ , where  $m^{(s)} = -2\gamma_0 |B_s| / (\gamma_s + \gamma_0)$ . Let the source points  $y_l \in \mathbb{R}^2 \setminus \overline{\Omega}$  for  $l \in \mathbb{N}$ . We assume that the countable set  $\{y_l\}_{l \in \mathbb{N}}$  has the property that any analytic function which vanishes in  $\{y_l\}_{l \in \mathbb{N}}$  vanishes identically.

Our location search algorithm for detecting multiple inclusions is as follows. For  $n \in \mathbb{N}$  sufficiently large, define the matrix  $A = [A_{ll'}]_{l, l'=1}^n$  by

$$A_{ll'} := -\varepsilon^2 \sum_{s=1}^m \frac{(\gamma_0 - \gamma_s)}{64\pi^2\gamma_0^4} m^{(s)} (y_{l'} - z_s) \cdot (y_l - z_s) \int_0^T \frac{1}{t^2(T-t)^2} \exp\left(-\frac{|y_l - z_s|^2}{4\gamma_0(T-t)} - \frac{|y_{l'} - z_s|^2}{4\gamma_0 t}\right) dt.$$

For  $z \in \Omega$ , we decompose the symmetric real matrix  $C$  defined by

$$C := \left[ \int_0^T \frac{1}{t^2(T-t)^2} \exp\left(-\frac{|y_l - z|^2}{4\gamma_0(T-t)} - \frac{|y_{l'} - z|^2}{4\gamma_0 t}\right) dt \right]_{l, l'=1, \dots, n}$$

as follows

$$C = \sum_{l=1}^p v_l(z) v_l(z)^*,$$

for some  $p \leq n$ , where  $v_l \in \mathbb{R}^n$  and  $v_l^*$  denotes the transpose of  $v_l$ . Define the vector  $g_z^{(l)} \in \mathbb{R}^{n \times 2}$  for  $z \in \Omega$  by

$$g_z^{(l)} = \left( (y_1 - z)v_{l1}(z), \dots, (y_n - z)v_{ln}(z) \right)^*, \quad l = 1, \dots, p.$$

Here  $v_{l1}, \dots, v_{ln}$  are the components of the vector  $v_l, l = 1, \dots, p$ . Let  $y_l = (y_{lx}, y_{ly})$  for  $l = 1, \dots, n$ ,  $z = (z_x, z_y)$ , and  $z_s = (z_{sx}, z_{sy})$ . We also introduce

$$g_{zx}^{(l)} = \left( (y_{1x} - z_x)v_{l1}(z), \dots, (y_{nx} - z_x)v_{ln}(z) \right)^*$$

and

$$g_{zy}^{(l)} = \left( (y_{1y} - z_y)v_{l1}(z), \dots, (y_{ny} - z_y)v_{ln}(z) \right)^*.$$

**Lemma 5.2** *The following characterization of the location of the inclusions in terms of the range of the matrix  $A$  holds:*

$$g_{zx}^{(l)} \text{ and } g_{zy}^{(l)} \in \text{Range}(A), \forall l \in \{1, \dots, p\} \text{ iff } z \in \{z_1, \dots, z_m\}. \quad (5.3)$$

*Proof.* Let for  $z \in \Omega$  suppose that  $g_{zx}^{(l)}$  and  $g_{zy}^{(l)} \in \text{Range}(A), \forall l \in \{1, \dots, p\}$ . Thus,

$$g_z^{(l)} = \sum_{s=1}^m \beta_s^{(l)} \left( (y_1 - z_s)v_{l1}(z_s), \dots, (y_n - z_s)v_{ln}(z_s) \right)^*,$$

for some constants  $\beta_s^{(l)}$ , which implies that

$$\int_0^T \frac{|y_l - z|^2}{t^2(T-t)^2} \exp\left(-\frac{|y_l - z|^2}{4\gamma_0(T-t)} - \frac{|y_l - z|^2}{4\gamma_0 t}\right) dt = \sum_{s=1}^m \alpha_s \int_0^T \frac{|y_l - z_s|^2}{t^2(T-t)^2} \exp\left(-\frac{|y_l - z_s|^2}{4\gamma_0(T-t)} - \frac{|y_l - z_s|^2}{4\gamma_0 t}\right) dt,$$

$\forall l = 1, \dots, n$ , for some constants  $\{\alpha_s\}_{s=1, \dots, m}$ . Therefore, provided that the countable set of sources  $\{y_l\}_{l \in \mathbb{N}}$  has the property that any analytic function which vanishes in  $\{y_l\}_{l \in \mathbb{N}}$  vanishes identically, it follows that

$$\int_0^T \frac{|y - z|^2}{t^2(T-t)^2} \exp\left(-\frac{|y - z|^2}{4\gamma_0(T-t)} - \frac{|y - z|^2}{4\gamma_0 t}\right) dt = \sum_{s=1}^m \alpha_s \int_0^T \frac{|y - z_s|^2}{t^2(T-t)^2} \exp\left(-\frac{|y - z_s|^2}{4\gamma_0(T-t)} - \frac{|y - z_s|^2}{4\gamma_0 t}\right) dt,$$

$\forall y \neq z, y \notin \{z_1, \dots, z_m\}$ , which proves that

$$\int_0^T \nabla U_y(z, t) \nabla \Phi_y(z, t) dt = \sum_{s=1}^m \alpha_s \int_0^T \nabla U_y(z_s, t) \nabla \Phi_y(z_s, t) dt, \quad \forall y \neq z, y \notin \{z_1, \dots, z_m\}.$$

Next, taking Laplacian in  $y$  yields

$$\Delta_y \int_0^T \nabla U_y(z, t) \nabla \Phi_y(z, t) dt = \sum_{s=1}^m \Delta_y \int_0^T \nabla U_y(z_s, t) \nabla \Phi_y(z_s, t) dt, \quad \forall y \neq z, y \notin \{z_1, \dots, z_m\}.$$

The singularity of  $\Delta_y \int_0^T \nabla U_y(z, t) \nabla \Phi_y(z, t) dt$  at the point  $z$ :

$$\Delta_y \int_0^T \nabla U_y(z, t) \nabla \Phi_y(z, t) dt = \nabla \delta_{y=z} + \text{smoother function},$$

gives that  $z \in \{z_1, \dots, z_m\}$ , as desired.  $\square$

Let the singular value decomposition (SVD) of the matrix  $A$  be defined by  $A = U\Sigma V^*$ . Suppose that the vectors  $\{g_{zx}^{(l)}, g_{zy}^{(l)}\}_{l=1, \dots, p}$  are linearly independent. The rank of  $A$  is then equal to  $2p$  and the first  $2p$  columns of  $U$ ,  $\{u_1, \dots, u_{2p}\}$ , provide a basis for column space of  $A$  which is denoted by  $U_S$ , and the rest of the matrix  $U$ ,  $\{u_{2p+1}, u_{2p+2}, \dots, u_n\}$ , provides a basis for the left null space of  $A$ , denoted by  $U_N$ .

From Lemma 5.2 we have that a test point  $z$  coincides with one of the positions  $z_s$  if and only if  $P(g_z^{(l)} \cdot c) = 0$ , for any vector  $c \in \mathbb{R}^2 \setminus \{0\}$ , where  $P = I - (U_S U_S^*)$  is the orthogonal projection onto the null space of  $A$ . Thus we can form an image of the locations  $\{z_s\}_{s=1}^m$  by plotting, at each point  $z \in \Omega$ , the quantity  $W_c^{(l)}(z) := \|g_z^{(l)} \cdot c\| / \|P(g_z^{(l)} \cdot c)\|$  for  $l = 1, \dots, p$ . The resulting plot will have large peaks at the locations of  $z_s$ ,  $s = 1, \dots, m$ .

Note that the smallest number  $n$  which is sufficient to efficiently recover the inclusions depends on the (unknown) number  $m$ . This is the main reason to take  $n$  sufficiently large.

## 5.3 Numerical experiments

### 5.3.1 Detection of a single inclusion

In this subsection we present numerical experiments for locating a single inclusion  $D$ . We consider  $\Omega$  as a unit disk centered at the origin in  $\mathbb{R}^2$ . The applied time-independent heat flux is given by  $g = e_1 \cdot \nu$  (and  $g_1 = e_1 \cdot \nu$ ,  $g_2 = e_2 \cdot \nu$  for the second algorithm) on  $\partial\Omega$ , and the inclusion  $D$  is taken as a homogeneous disk of diameter  $\varepsilon = 0.1$ , centered at  $z = (0.43, 0.27)$  with conductivity  $\gamma = 3$ . The thermal conductivity of the background  $\gamma_0 = 1$ . The observation lines are given by  $\Sigma_1 = \{(x, -2), x \in \mathbb{R}\}$  and  $\Sigma_2 = \{(2, x), x \in \mathbb{R}\}$ .

Figs. 5.3.1.1 and 5.3.1.2 show the result of our algorithms. Observing the function  $I_\Phi(T)(x)$  on the lines  $\Sigma_1$  and  $\Sigma_2$  we obtain the zero point  $P_1 = (0.43, -2)$  of  $I_\Phi(T)$  on  $\Sigma_1$  and the minimum point (or the zero point for the second algorithm)  $P_2 = (2, 0.27)$  of  $I_\Phi(T)$  on  $\Sigma_2$ , noting that  $\gamma_0 - \gamma$  is negative. Then the intersecting point  $P$  of  $\Pi_1(P_1) \cap \Pi_2(P_2)$  in the location search algorithm is given by  $P = (0.43, 0.27)$ .

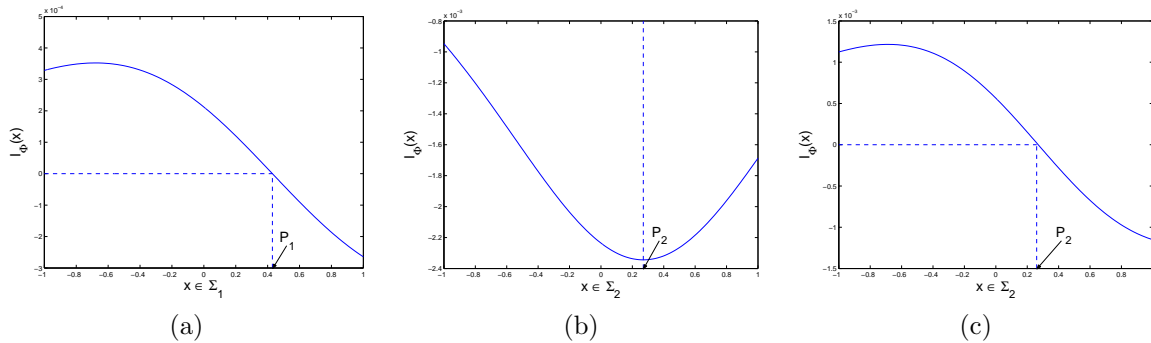


Figure 5.3.1.1: Patterns of  $I_\Phi(T)(x)$  calculated on the observation lines  $\Sigma_1 = \{(x, -2), x \in \mathbb{R}\}$  and  $\Sigma_2 = \{(2, x), x \in \mathbb{R}\}$ . (a) and (b) correspond to the first algorithm; (a) and (c) correspond to the second algorithm.

### 5.3.2 Detection of multiple inclusions

In this subsection we present numerical experiment for locating multiple inclusions. Let again  $\Omega$  be a unit disk in  $\mathbb{R}^2$  centered at the origin. We assume that we know the values  $A_{ll'}$  of the pattern  $I_\Phi(T)$ , for

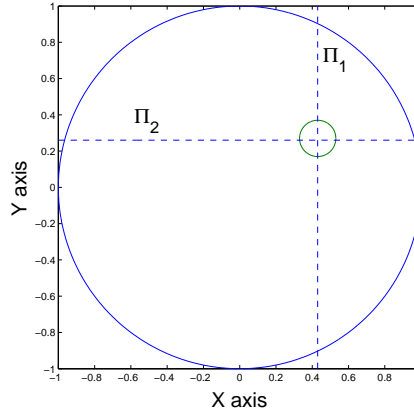


Figure 5.3.1.2: Location detection intersecting the lines  $\Pi_1(P_1)$  and  $\Pi_2(P_2)$ . The intersecting point is  $P = (P_1, P_2)$ .

some positive  $T$ , and for a small finite number of equidistantly distributed source points

$$y_l = \rho \left( \cos \frac{2\pi l}{n}, \sin \frac{2\pi l}{n} \right), \quad l = 1, \dots, n.$$

Here  $\rho$  denotes the (minimal) distance between the source points  $y_l$ ,  $l = 1, \dots, n$  and the origin. We have two cases of interest:  $\rho > T$  and  $\rho \leq T$ .

In the first numerical experiment we take two homogeneous circular disks of diameter  $\varepsilon = 0.1$  denoted as  $D_1$  and  $D_2$  and respectively centered at  $z_1 = (0.21, 0.32)$  and  $z_2 = (-0.43, -0.35)$ , to be retrieved using  $n = 10$  source points. Corresponding conductivities  $\gamma_s$ ,  $s = 1, 2$ , are equated to 3. The thermal conductivity of the background  $\gamma_0 = 1$ .

Within the above setting, the retrieval of the inclusions involves the calculation of the SVD  $A = U\Sigma V^*$  of the matrix  $A = [A_{ll'}] \in \mathbb{R}^{n \times n}$  and the decomposition of the matrix  $C$ . It was observed numerically that the rank of  $C$  is equal to one ( $p = 1$ ). Therefore, we expect to see four non-zero singular values of the matrix  $A$ . Denote by  $\{e_i\}_{i=1}^2$  the orthonormal basis in  $\mathbb{R}^2$ . Then, for each discrete location  $z \in \Omega$  (the sampling step henceforth is  $h = 0.03$ ), the identifier of interest is  $W_c(z)$  ( $:= W_c^{(1)}$ ),  $c = \{e_1, e_2, e_1 + e_2\}$ , calculated within  $\Omega$ . Plots of  $z \rightarrow W_c(z)$  illustrate the result achieved, sharp peaks being expected to occur at the locations of the inclusions,  $z_s$ ,  $s = 1, 2$ . Other accompanying results displayed consist of the singular values of  $A$ , using a standard log scale.

First we take  $\rho = 2.5$  and  $T = 1$  (first case of interest). The singular values are displayed in Fig. 5.3.2.3, the identifier  $W_c(z)$ , for  $c = e_1, e_2$  and  $e_1 + e_2$  is displayed in Fig. 5.3.2.4.

The results obtained are easy to interpret. Four singular values emerge from noise, noticing that two singular values are associated to one specific inclusion. The inclusions, be they observed via different  $W_c(z)$ , are clearly discriminated from the background, the visual aspect depending upon the choice of  $c$ .

In the second numerical example we take  $\rho = 2.5$  and  $T = 3$  (the second case of interest) and next we take  $\rho = 4$  and  $T = 3$  (the first case of interest).

Fig. 5.3.2.5 shows the distribution of the singular values of  $A$  for two cases. Fig. 5.3.2.6 shows the identifier  $W_c(z)$ ,  $c = e_1$ , for two cases, where the images of  $W_c(z)$  are obtained by using 5 and 4 largest singular vectors (associated to the 5 and 4 largest singular values of  $A$ , respectively) in the second case

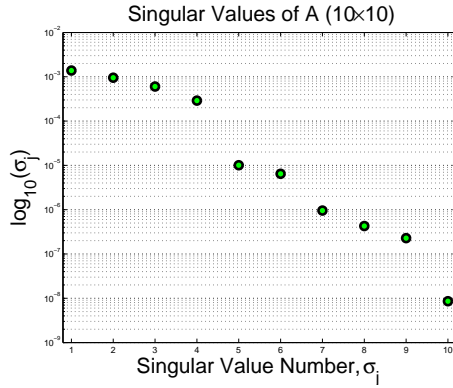


Figure 5.3.2.3:  $\rho > T$ : distribution of the singular values of  $A$  for  $n = 10$  sources.

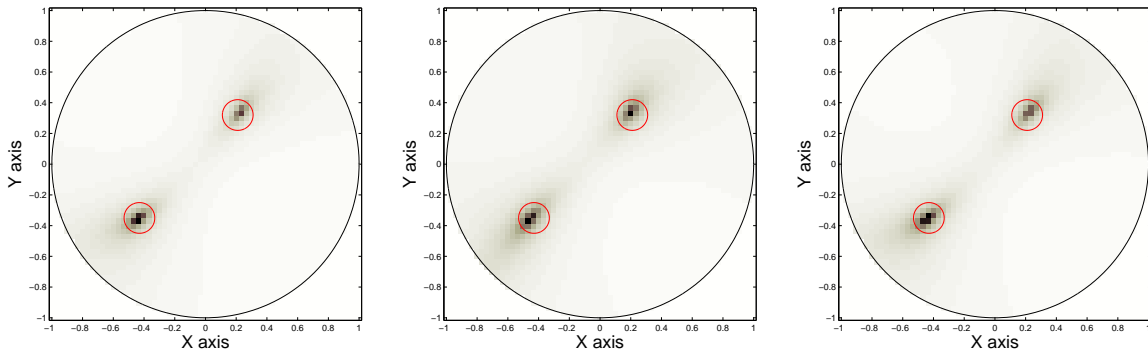


Figure 5.3.2.4:  $\rho > T$ : gray-level (or color) map of  $W_c(z)$ ,  $c = e_1, e_2$  and  $e_1 + e_2$ , ordered from left to right, for all points  $z$  in  $\Omega$ .

of interest and 4 largest singular vectors (associated to the 4 largest singular values of  $A$ ) in the first case of interest.

The results obtained are evidently less easy to interpret than before. Five singular values emerge from noise in the second case and four (as in the numerical example above) in the first case of interest. The inclusions, be they observed via  $W_c(z)$ ,  $c = e_1$ , are clearly discriminated from the background (Fig. 5.3.2.6 (b) and (c) ). Notice that in the second case the positions of the inclusions are moving away from the actual positions.

The second numerical example involves one more inclusion than the previous one. We add one more homogeneous circular disk of the same diameter  $\varepsilon = 0.1$  denoted as  $D_3$  and centered at  $z_3 = (0.4, -0.51)$ , to be retrieved using  $n = 14$  source points, with conductivity  $\gamma_3 = 3$ .

The result for  $\rho = 4, T = 1$  is shown in Fig. 5.3.2.7 (the distribution of the singular values of  $A$  for 14 source points and the identifier  $W_c(z)$ , for  $c = e_1$ ).

As in the first numerical example the results obtained are easy to interpret. Six singular values emerge from noise, noticing that two singular values are associated to one specific inclusion. The inclusions, be they observed via  $W_c(z)$ ,  $c = e_1$ , are clearly discriminated from the background.

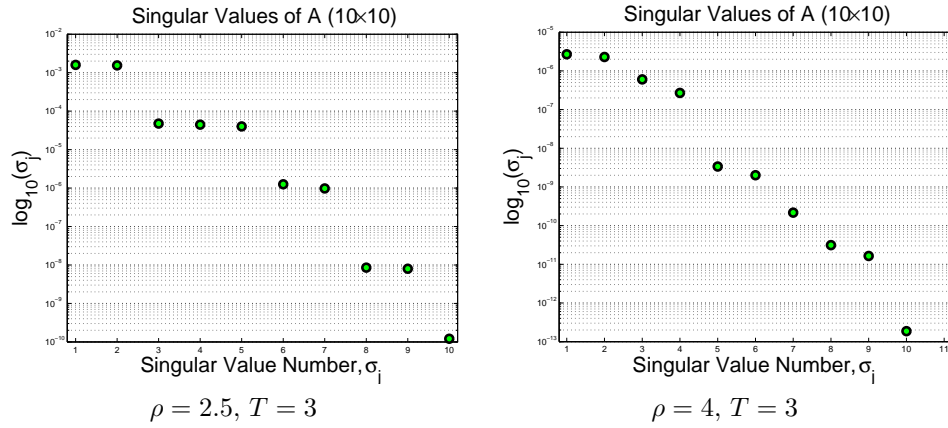


Figure 5.3.2.5: Distribution of the singular values of  $A$  for  $n = 10$  sources.

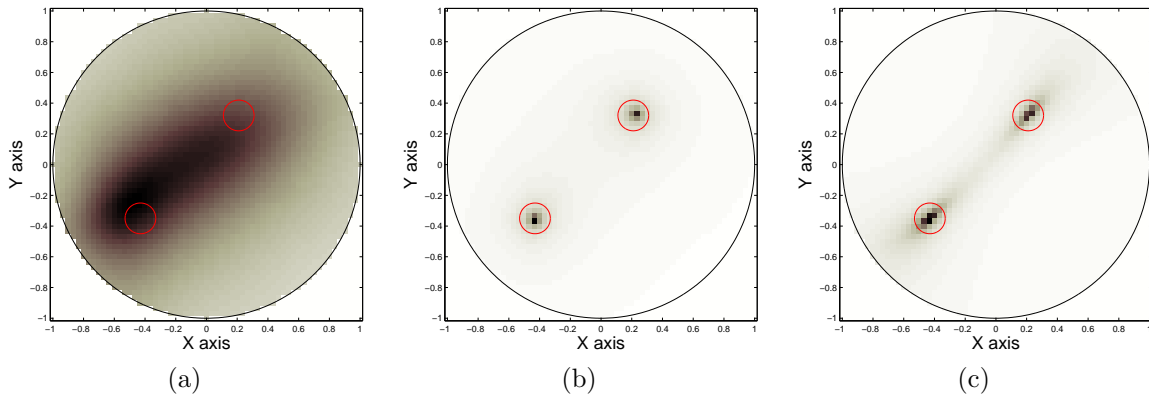


Figure 5.3.2.6: Gray-level (or color) map of  $W_c(z)$ ,  $c = e_1$  for all points  $z$  in  $\Omega$ . (a)  $\rho = 2.5$ ,  $T = 3$ : the map of  $W_c(z)$  is obtained by using 4 first singular vectors of  $A$ ; (b)  $\rho = 2.5$ ,  $T = 3$ : the map of  $W_c(z)$  is obtained by using 5 first singular vectors of  $A$ ; (c)  $\rho = 5$ ,  $T = 3$ : the map of  $W_c(z)$  is obtained by using 4 first singular vectors of  $A$ .

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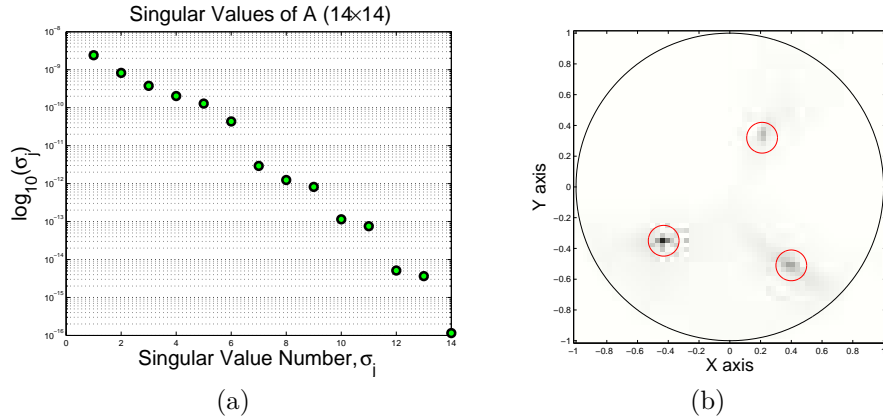


Figure 5.3.2.7:  $\rho > T$ : (a) distribution of the singular values of  $A$  for  $n = 14$  source points; (b) gray-level (or color) map of  $W_c(z)$ ,  $c = e_1$  for all points  $z$  in  $\Omega$ .

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