LINEAR ALGEBRA I

VECTOR SPACE

A vector space $(V, +, .)$ is a set endowed by two operations :

$$+ \begin{cases} (V, V) \longrightarrow V \\ (x, y) \longmapsto x+y \end{cases}$$

that allows to add two vectors.

$$. \begin{cases} (\mathbb{R}, V) \longrightarrow V \\ (\alpha, x) \longrightarrow \alpha . x \end{cases}$$

that is the multiplication of a vector by a scalar

Formally a number of axioms need to be satisfied such that

$\rightarrow$ $(V, +)$ is an additive group

check what is a group after the class

$\rightarrow$ $\underset{\mathbb{R}}{\alpha} \cdot (\underset{V}{x} + \underset{V}{y}) = \alpha . x + \alpha . y$

$\rightarrow$ $\alpha_1 . (\alpha_2 . x) = (\alpha_1 \alpha_2) . x$

$\rightarrow$ $(\alpha_1 + \alpha_2) . x = \alpha_1 x + \alpha_2 x$

Elements of V are called vectors.

Q - Which vector space do you know ?

$(\mathbb{R}, +, .)$ is a vector space $\mathbb{R}$: is the set of real numbers

$(\mathbb{R}^n, +, .)$ is a vector space

x is a vector of $\mathbb{R}^n$ if $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $x_1 \in \mathbb{R}$, $x_n \in \mathbb{R}$

x_i are the components or entries of the vector

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} ; \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$x + y = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Example: Add the two vectors of $\mathbb{R}^2$:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Scalar in multiplication in $\mathbb{R}^n$:

$$\underset{\mathbb{R}}{\underset{\uparrow}{\alpha}} \cdot x = \alpha \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Home exercice: Simplify $3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

Can we substract vector of $\mathbb{R}^n$?

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ??$$

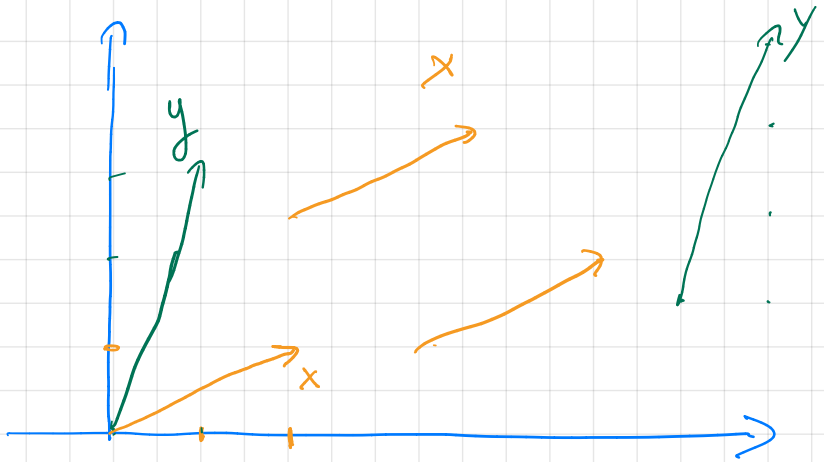
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Yes we can substract vectors because it is actually adding - the vector.

GEOMETRICAL INTERPRETATION :

In $\mathbb{R}^2$, we consider the vector

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



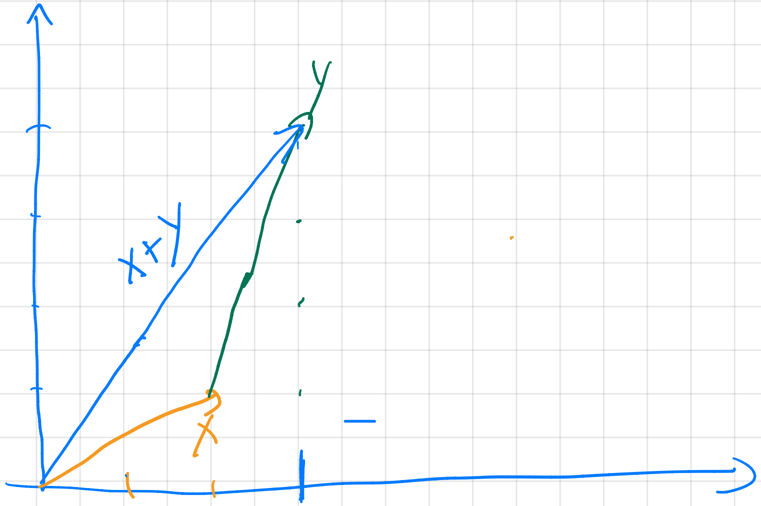
→ we do not care about the origin

$$y = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

→ a vector has a direction and an amplitude

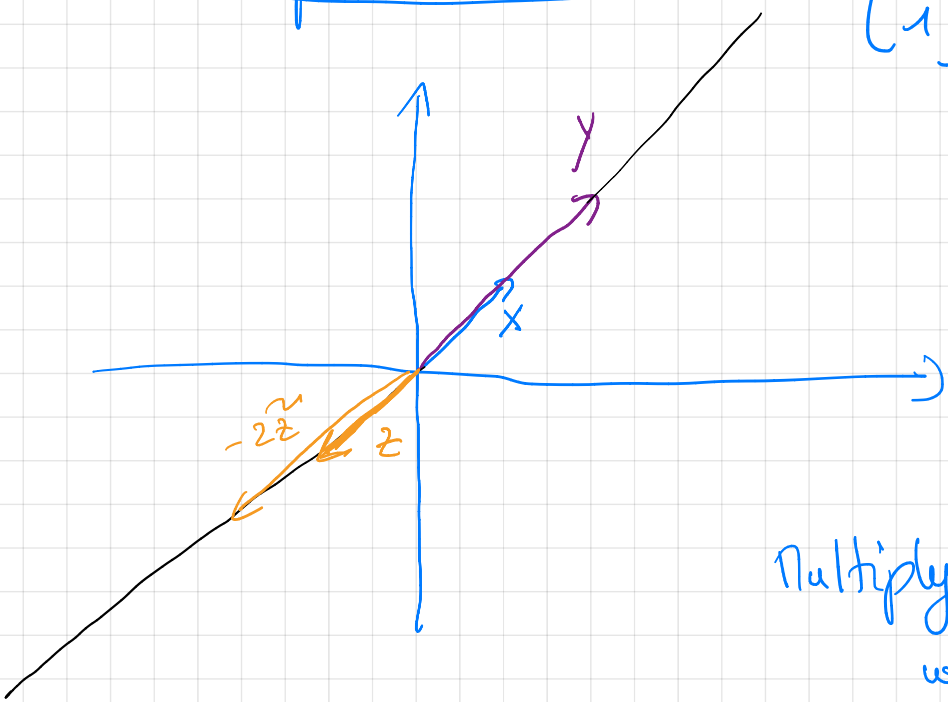
Adding two vectors

$$x + y = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Scalar multiplication:

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$y = 2 \cdot x = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$z = -1x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\tilde{z} = -2x$$

Multiplying by a positive scalar changes the amplitude but we keep the same direction.

LINEAR INDEPENDENCE, SPAN AND BASIS

$$\underset{\substack{\uparrow \\ \mathbb{R}}}{c_1} \cdot v_1 + \underset{\substack{\uparrow \\ \mathbb{R}}}{c_2} v_2 + \cdots + \underset{\substack{\uparrow \\ \mathbb{R}}}{c_k} v_k$$

$v_1, \ldots, v_k$ are vectors

$c_1, \ldots, c_k$ are scalars (reel numbers)

- This is a **linear combination** of the vectors $\{v_1, \ldots, v_k\}$
- The scalars $c_1, \ldots, c_k$ are called the weights.

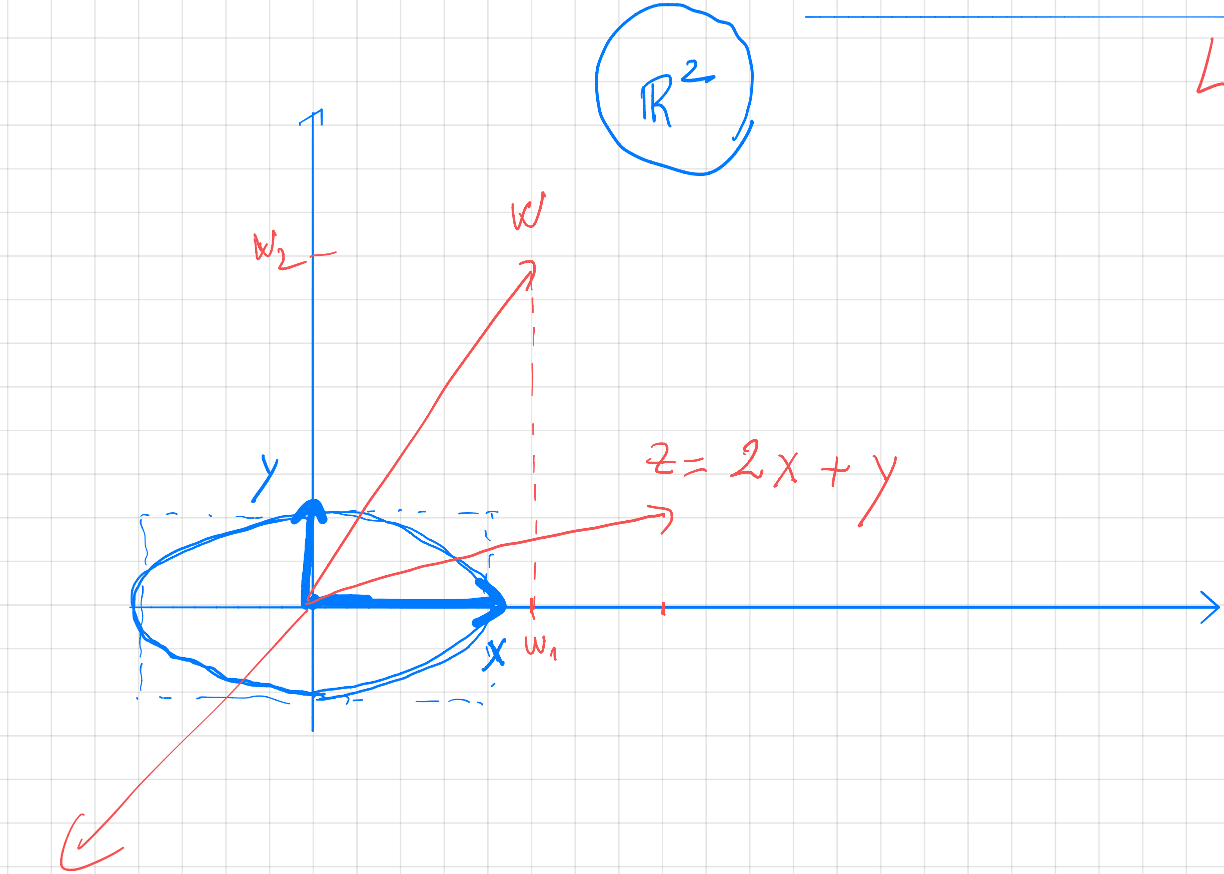
$$\underbrace{1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

this is a linear combination of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Definition: The **span** of a list of vectors is the set of all vectors which can be written as a linear combination of the vectors in the list.

Notation: $\text{span}(v_1, \ldots, v_k)$ [IN FRANCE $\text{Vect}(v_1, \ldots, v_k)$]

What is the span of $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?



Let w be a vector of $\mathbb{R}^2$

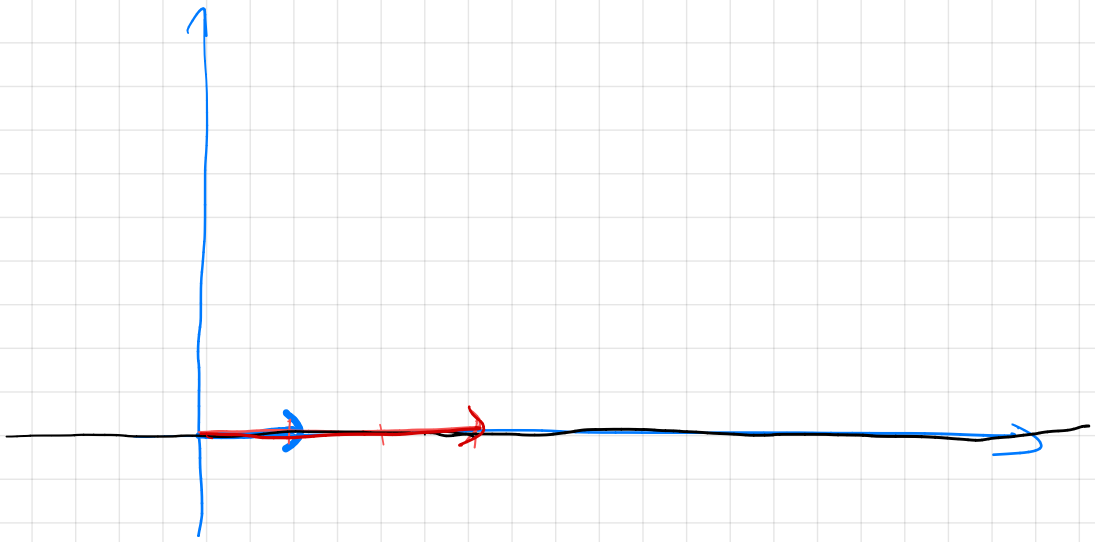
$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{w_1}{2} \times 2 \\ w_2 \end{bmatrix}$$

$$= \frac{w_1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

w is a linear combination of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with the weights $\frac{w_1}{2}, w_2$

Then $\operatorname{span}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = \mathbb{R}^2$

What is the span of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}$?



$$\operatorname{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\right) = \mathbb{R} \times \{0\}$$
$$= \left\{ \begin{pmatrix}x\\0\end{pmatrix}, x \in \mathbb{R} \right\}$$

$$\alpha \begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}\alpha + 3\beta\\ \alpha \cdot 0 + \beta \cdot 0\end{bmatrix}$$
$$= \begin{bmatrix}\alpha + 3\beta\\ 0\end{bmatrix}$$

Definition: A list of vectors is **linearly independent** if none of the vectors in the list can be written as a linear combination of the others.

Examples: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Are they linerly independent?

No, they are not lin. ind. because $v_2 = 3 v_1$

Same question with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Yes they are.

PROPOSITION: $(v_1, \dots, v_k)$ are linearly independent vectors

$$\iff$$

$$\lambda_1 v_1 + \dots + \lambda_k v_k = 0 \implies \lambda_1 = \lambda_2 = \dots = \lambda_k = 0$$

Use this proposition at home to show $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent

Definition: A **spanning list** of a vector space V is a list of vectors in V such that the span is equal to V.

Example: $\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a spanning list of $\mathbb{R}^2$

You can verify that $\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$ is a spanning list of $\mathbb{R}^2$

- We consider $V = \left\{\begin{pmatrix}x\\0\\0\end{pmatrix}, x \in \mathbb{R}\right\} \subset \mathbb{R}^3$

$\rightarrow$ V is a vector space (check this at home)

span $\begin{pmatrix}1\\0\\0\end{pmatrix} = V$ such that $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is a spanning list of V

Definition: A linearly independent spanning list of a vector space V is called **a basis**.

PROPOSITION: All basis of a vector space have the same length, which is called the dimension of the vector space.

Given a basis of V there is an **unique** way to write any vector of V as a linear combination of the elements of the basis.

If $\{v_1, \ldots, v_k\}$ is a basis of V.

Let $y \in V$, there exists an unique set of scalars $(\alpha_1, \ldots, \alpha_k)$ such that

$$y = \alpha_1 v_1 + \ldots + \alpha_k v_k$$

$(\alpha_1, \ldots, \alpha_k)$ are called the coordinates of y with respect to the basis $(v_1, \ldots, v_k)$

Let's go back to $\mathbb{R}^n$.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$. [HOME EXERCICE]

↳ the dimension $\mathbb{R}^n$ is n.

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, how do we decompose x into $\{e_1, e_2, \dots, e_n\}$?

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

So, the coordinates of a vector of $\mathbb{R}^n$ coincide with its coordinates with respect to the basis $\{e_1, \dots, e_n\}$

$$\boxed{\{e_1, \dots, e_n\} \text{ is called } \textbf{the standard basis} \text{ of } \mathbb{R}^n. \quad [\text{FRANCE: "base canonique"}]}$$

$\dim(\mathbb{R}^n) = n$

$\Rightarrow$ $\mathbb{R}^2$ is a 2-dimensional vector space.

$\mathbb{R}^3$ is a 3-dimensional vector space.

LINEAR TRANSFORMATIONS

Definition: $L : V \longrightarrow W$ is a **linear transformation**

vector spaces

if $\forall\ v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in \mathbb{R}$ the following is satisfied:

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$$

Example:

$$L : \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ x \longrightarrow 10x = \begin{pmatrix} 10x_1 \\ \vdots \\ 10x_n \end{pmatrix} \end{cases}$$

Verify (AT HOME) that L is a linear transformation.

Definition: The **rank** of a linear transformation $L : V \to W$ is the dimension of $L(V)$

where $L(V) = \{ y \in W \text{ such that } y = L(x), x \in V \}$

[FRANCE : $L(V)$ is the image of L]

$$\text{rank}(L) = \dim(L(V))$$

Image of L

Example:

Consider $L : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \end{cases}$

1) Show that L is a linear transformation

2) Find $\text{rank}(L)$.

Consider $L : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \end{cases}$

$$\underset{\underset{1}{\shortparallel}}{\operatorname{rank}(L)} \neq \underbrace{\dim\big(\underset{\underset{\mathbb{R}^2}{\shortparallel}}{\text{co-domain}}\big)}_{=2}$$

1) Show that L is a linear transformation

2) Find rank(L).

Answers: 1, ~~2~~, ~~$(1,2)$~~, ~~$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$~~

$$\operatorname{rank}(L) = \dim(L(\mathbb{R}^2))$$

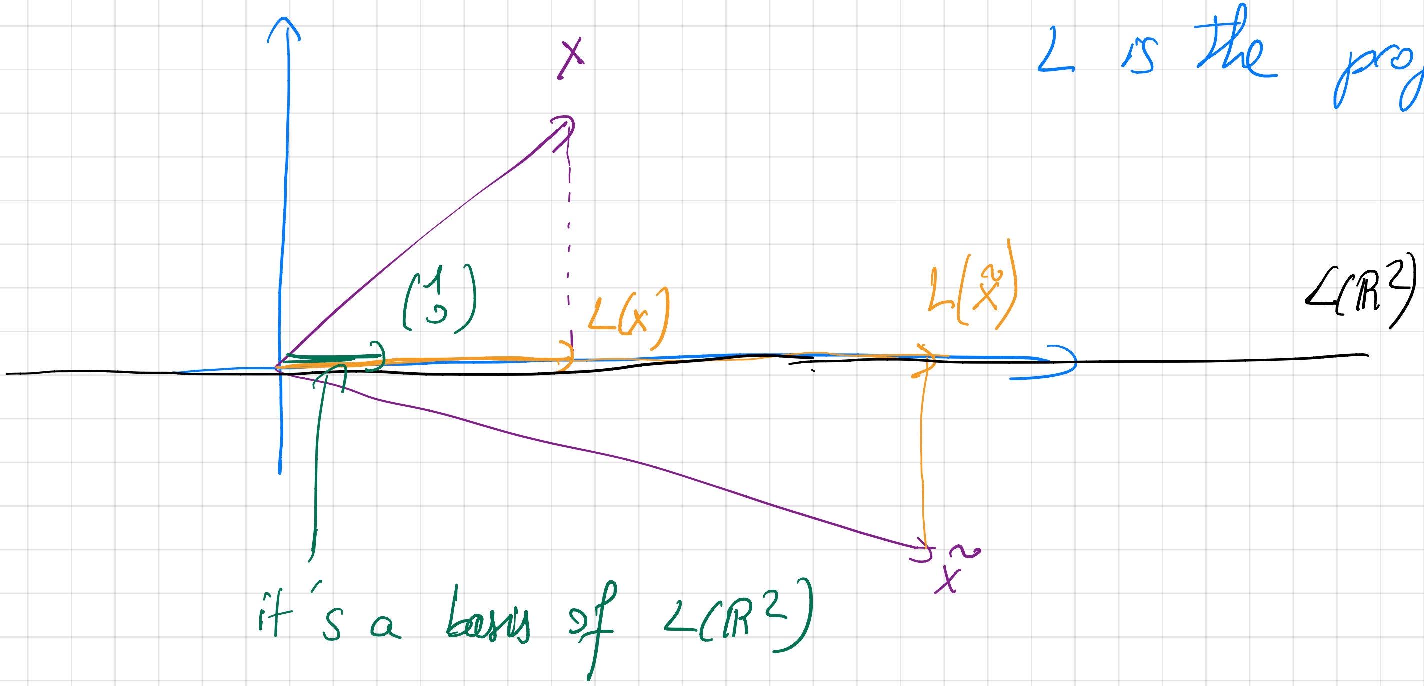
$$L(\mathbb{R}^2) = \left\{ y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix},\ y_1 \in \mathbb{R} \right\}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a basis of $L(\mathbb{R}^2)$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is independent and it is a spanning list of $L(\mathbb{R}^2)$

Let $y \in L(\mathbb{R}^2)$, $y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a spanning list of $L(\mathbb{R}^2)$

Then $\dim(L(\mathbb{R}^2)) = 1 = \operatorname{rank}(L)$



L is the projection of x onto the first axis.

Proposition of other example:

$$L: \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 5 \end{pmatrix} \end{cases}$$

$\operatorname{rank}(L) = 1$?

$L(x+y) = L(x) + L(y)$ if L is a linear transformation

$$\underbrace{L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)} = \begin{pmatrix} x_1 \\ 5 \end{pmatrix} + \begin{pmatrix} y_2 \\ 5 \end{pmatrix} = \begin{pmatrix} x_1 + y_2 \\ 10 \end{pmatrix}$$

$$L\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}\right)$$

$$\parallel$$

$$\begin{pmatrix} x_1 + y_1 \\ 5 \end{pmatrix} \neq \begin{pmatrix} x_1 + y_2 \\ 10 \end{pmatrix}$$

The example is not a linear transformation.

Consider $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z+y \\ z-y \\ 0 \end{bmatrix}$ $\qquad L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

HOME EXERCICE: show that L is linear

$\operatorname{rank}(L) = 2$

$L: \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 3x_2 \end{pmatrix} \end{cases}$ is linear transformation

$L: \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \end{cases}$

Definition:

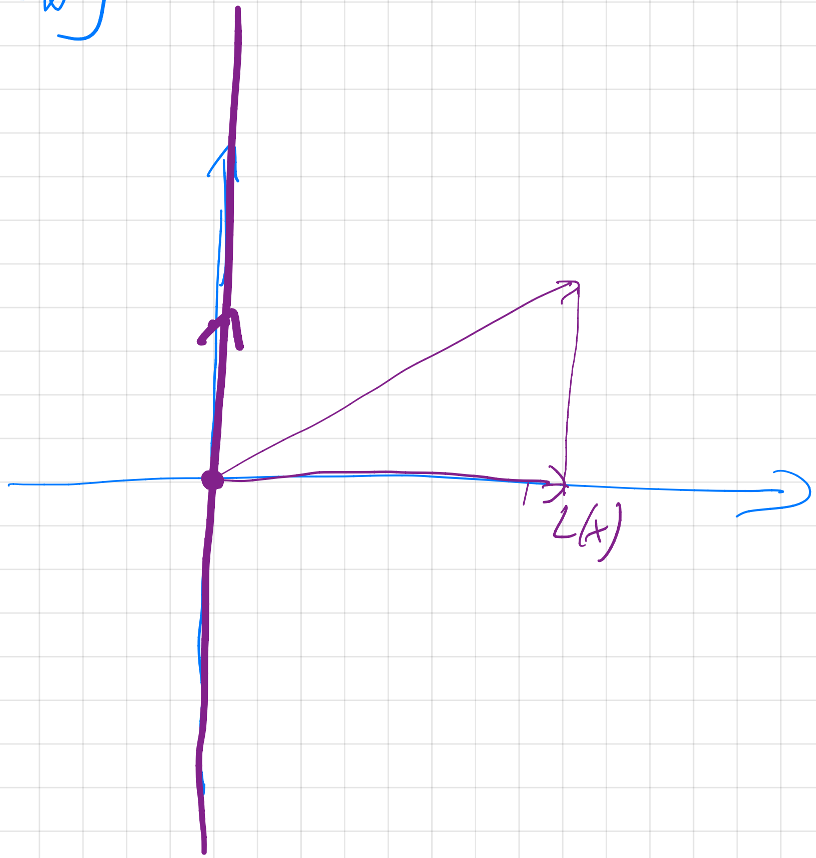
The **nullspace or kernel** of a linear transformation is the set of vectors mapped to 0.

Let $L : V \longrightarrow W$ be a linear transformation

$$\ker(L) = \{ v \in V,\ L(v) = 0_W \}$$

Example:

$$L : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \end{cases}$$



$$\ker(L) = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix},\ x_2 \in \mathbb{R} \right\}$$

$$\dim(\ker(L)) = 1$$

($\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis of the kernel)

Remark: $\dim(\ker(L)) + \dim(L(V)) = 1+1 = 2$

RANK NULLITY THEOREM

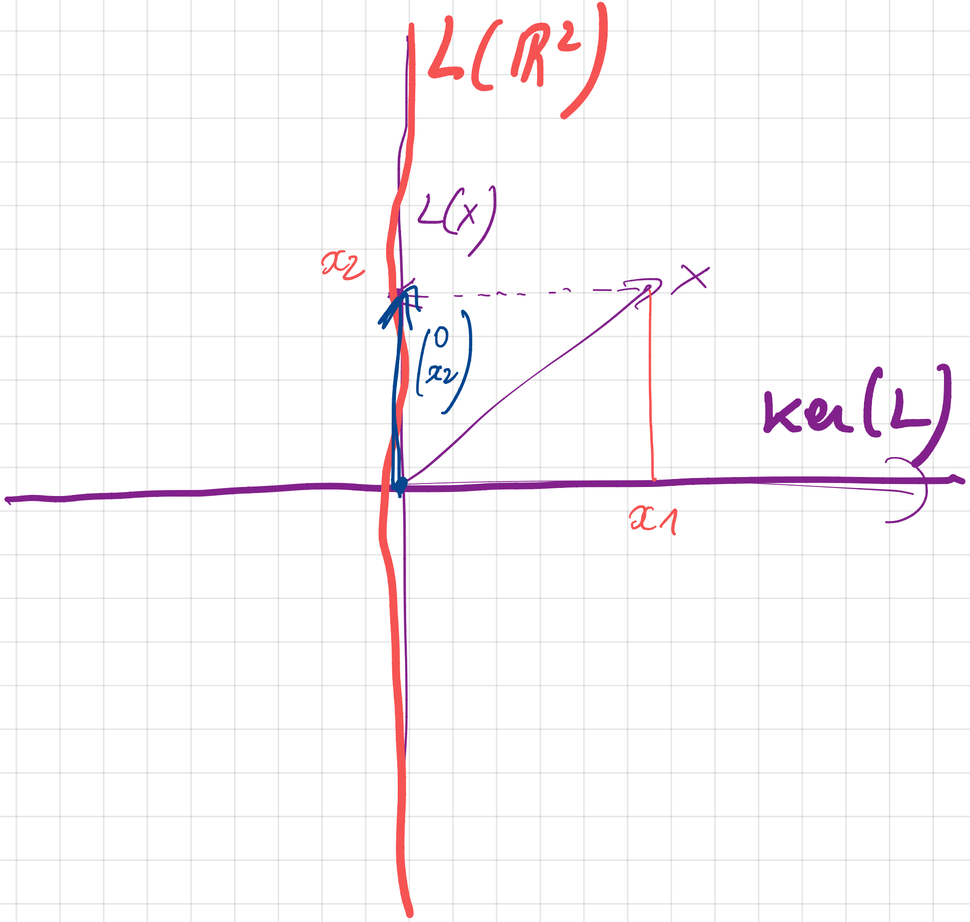
Let V, W be two vector spaces (V finite dimensional)
Let $L: V \longrightarrow W$ be a linear transformation.
Then
$$\operatorname{rank}(L) + \dim(\ker(L)) = \dim(V)$$

$$L: \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \end{cases}$$

$$L(\mathbb{R}^2) = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \ x_2 \in \mathbb{R} \right\}$$

$$\operatorname{rank}(L) = 1$$

$$\ker(L) = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \ x_1 \in \mathbb{R} \right.$$



$$\dim(\ker(L)) + \operatorname{rank}(L) = 1 + 1 = 2$$