CLASS 2 - LINEAR ALGEBRA 2

PRE2- MATH FOR DATASCIENCE

We come back on the example from last time

$$L:\begin{cases}\mathbb{R}^2 \longrightarrow \mathbb{R}^2\\ \begin{pmatrix}x_1\\x_2\end{pmatrix}\longrightarrow\begin{pmatrix}x_1\\5\end{pmatrix}\end{cases}$$

This is not a linear transformation because

$$L\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\5\end{pmatrix}\neq 0$$

Proposition: If L is a linear transformation then

$$L(0_V) = 0_W$$

$L: V \longrightarrow W$

Since L is linear, $L(x+y)=L(x)+L(y)$

Hence $L(\underbrace{0_V+0_V}_{=0_V}) = L(0_V)+L(0_V)$

$\Rightarrow L(0_V) = L(0_V)+L(0_V) \Rightarrow L(0_V) = 0_W$

MATRIX

A matrix is a rectangular array of numbers.

$$A = \begin{array}{r} \\ \\ \\ \\ i \longrightarrow \\ \\ \end{array} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & & a_{ij} & \vdots \\ a_{m,1} & & & a_{m,n} \end{pmatrix} \updownarrow \ m \text{ rows}$$

$$\underset{n \text{ columns}}{\overset{\overset{\uparrow}{j}}{\longleftrightarrow}}$$

A is a $m \times n$ matrix

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

Example:

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 0 \end{pmatrix}$$ is a 2×2 matrix

If $m = n$, we talk about a square matrix.

Adding two matrices

Let A be a $m \times n$ matrix

— B ——— $m \times n$ matrix

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{n,1} \\ \vdots & & \vdots \\ a_{m,1} & & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & & \vdots \\ b_{m,1} & & b_{m,n} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11}, & \cdots, & a_{n1}+b_{n1} \\ a_{21}+b_{21}, & & \vdots \\ \vdots & & \vdots \\ a_{m1}+b_{m1}, & & a_{m,n}+b_{m,n} \end{bmatrix}$$

$$A+B = \left(a_{ij} + b_{ij} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq n}}$$

Examples:

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

Impossible to add

MULTIPLICATION BY A SCALAR

$$tA = \begin{bmatrix} ta_{11} & \cdots & ta_{n1} \\ \vdots & & \vdots \\ ta_{m1} & & ta_{m,n} \end{bmatrix}$$

ex : $3 \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 3 & 3 \\ 0 & 3 \end{bmatrix}$

MATRIX-VECTOR MULTIPLICATION

Let A a $m \times n$ matrix ; x a n-dim vector (that we can also see as a $n \times 1$ matrix)

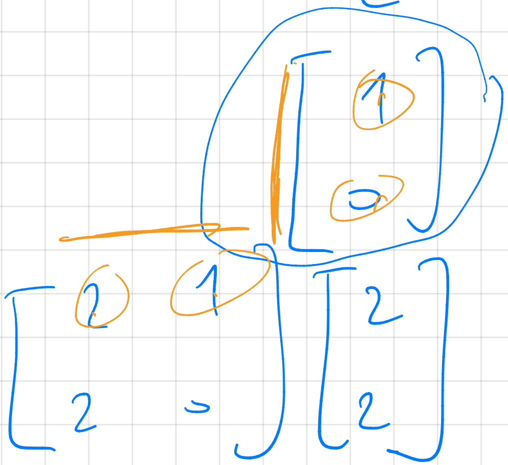
$$x \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$\underset{(m \times n)}{A} \; \underset{(n \times 1)}{x}$$

Example:

$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$; $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute $Ax = ?$



$$Ax = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

2x 1 matrix or 2-dim vector

Another way to see the matrix-vector multiplication:

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & - - & a_{1n} \\ | & & | \\ | & & | \\ a_{m1} & & a_{mn} \end{bmatrix} \\ \begin{array}{ccc} \parallel & \qquad & \\ A_1 & & A_n \end{array} \end{array} \qquad x = \begin{bmatrix} x_1 \\ | \\ | \\ x_n \end{bmatrix}$$

$$Ax = \underbrace{\overset{\in \mathbb{R}}{x_1} A_1 + \dots + x_n A_n}$$

linear combination of the columns of A (which are vectors) with weights $x_1, \dots, x_n$ (coordinates of vector x)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$Ax = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Let A be a $m \times n$ matrix, $A = \underbrace{\left.\begin{bmatrix} A_1, \dots, A_n \end{bmatrix}\right\updownarrow m}_{n}$

$$L \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ x \longrightarrow Ax \left(= x_1 A_1 + \cdots + x_n A_n\right) \end{cases}$$

PROPOSITION: $x \xmapsto{L} Ax$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

PROOF: Verify: $L(x+y) = L(x) + L(y)$

$L(\lambda x) = \lambda L(x)$

We have seen that we can associate to a matrix a linear transformation. The converse is true, we can associate to a linear transformation from two vector spaces of finite dimension a matrix.

Let $L : E \longrightarrow V$ be a linear transformation. E : is a n dim - vector space

V is a m dim - vector space

Let $\{e_1, --, e_n\}$ be a basis of vectors of E

Let $\{b_1, -, b_m\}$ ______ of V

$L(e_1) \in V$, I can decomposite it in a unique way on $\{b_1, -, b_m\}$

$$L(e_1) = \alpha_{11} b_1 + --- + \alpha_{m1} b_m$$
$$L(e_2) = \alpha_{12} b_1 + — - + \alpha_{m2} b_m$$
$$\vdots$$

$$B = \begin{array}{c} L(e_1) \\ \left[\begin{array}{ccc} \alpha_{11}, & \alpha_{12} & \alpha_{1n} \\ \vdots & \vdots & \vdots \\ \alpha_{m1}, & \alpha_{m2} & \alpha_{mn} \end{array}\right] \\ \xleftrightarrow[n]{\qquad\qquad} \end{array} \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} \Big\updownarrow m$$

We say that B is the representation of L in the basis $\{e_1, -, e_n\}$; $\{b_1, -, b_m\}$

B is a $m \times n$ matrix

EXERCICE : Find the matrix corresponding to the linear transformation

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix} \qquad T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

in the standard basis.

Standard basis in $\mathbb{R}^3$, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$T(e_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad T(e_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$= 0e_1 + 1e_2 + 0e_3$ $\quad$ $= 0e_1 + 0e_2 + 1e_3$ $\quad$ $= 1e_1 + 0e_2 + 0e_3$

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} T(e_1) & T(e_2) & T(e_3) & \\ 0 & 0 & 1 & e_1 \\ 1 & 0 & 0 & e_2 \\ 0 & 1 & 0 & e_3 \end{array}$$

A is the representation of T in the standard basis.

In $\mathbb{R}^n$ the standard basis is $\{e_1, -, e_n\}$

$$e_i = \begin{pmatrix} 0 \\ | \\ 0 \\ 1 \\ 0 \\ | \\ 0 \end{pmatrix} \leftarrow i$$

⚠ There is not an unique way to represent a linear transformation as a matrix. If you change the basis, this will change the matrix representation.

If you consider the basis $\tilde{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\tilde{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \tilde{e}_3$ will be different.

If $L : E \to V$ is a linear transformation, B its matrix representation in $\{e_1, -, e_n\}$ and $\{b_1, -, b_m\}$

$x \in E$ represented $x = \begin{pmatrix} x_1 \\ | \\ x_n \end{pmatrix}$ in $\{e_1, -, e_n\}$

<u>Then</u> $L(x)$ is represented as $\underbrace{Bx}_{\text{Product. matrix, vector.}}$ in $\{b_1, -, b_m\}$

MATRIX MULTIPLICATIONS :

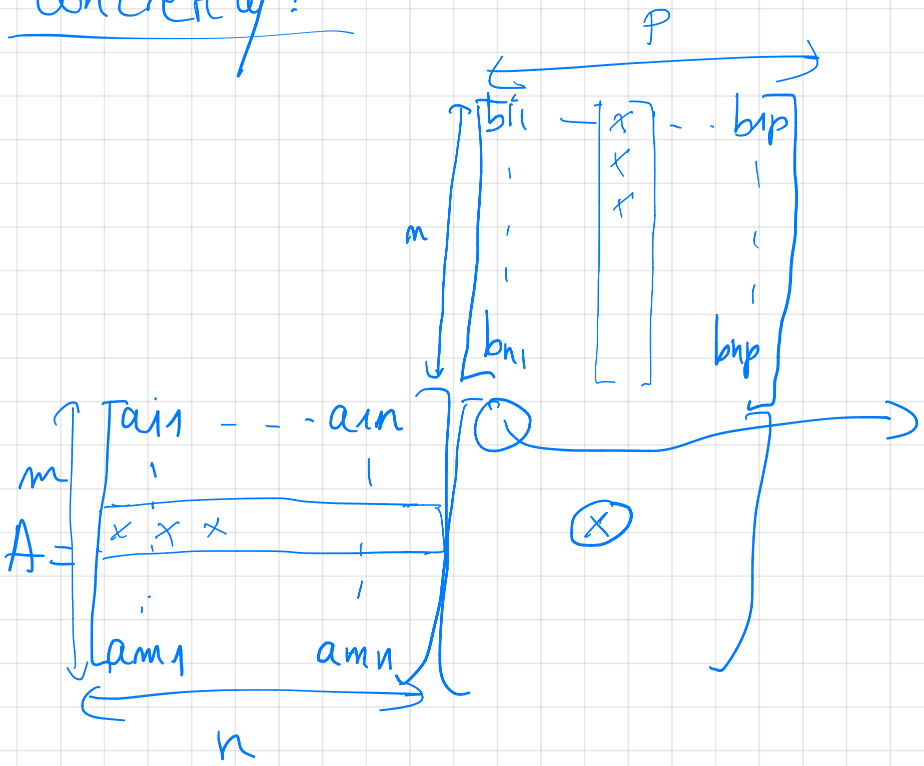
Let A be a $m \times n$ matrix

Let B be a $n \times p$ matrix

Then $\underset{(m,p)}{\underset{(m \times n)(n \times p)}{AB}}$ is a $m \times p$ matrix such that

$$\forall x \in \mathbb{R}^p \quad (AB)x = A(Bx)$$

Concretely :



$a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Multiply ;

$$\begin{array}{c|c} & \begin{bmatrix}2&0&4&1\\1&0&0&1\end{bmatrix} \\ \hline \begin{bmatrix}3&0\\0&1\\1&0\end{bmatrix} & \begin{bmatrix}6&0&12&3\\1&0&0&1\\2&0&4&1\end{bmatrix}\end{array}$$

⚠ Be careful about sizes, you can't multiply a (2x2) & (3x4) matrix

$$\xcancel{\begin{array}{cc} & \begin{bmatrix}x&x\\x&x\\x&x\end{bmatrix} \\ \begin{bmatrix}x&x\\x&x\end{bmatrix} & \end{array}}$$

impossible

Remember ;

$$(m \times \underbrace{n)\;(n}_{\text{same}} \times p)$$

$$\rightarrow (m \times p) \text{ matrix}$$

RANK, KERNEL (NULLSPACE), INVERSE OF A MATRIX

DEFINITION : The **rank** of a matrix A is defined as the rank of the linear transformation

$$A \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ x & \longrightarrow & Ax \end{array} \right.$$

$$\mathrm{rank}(A) = \dim\Big(\underbrace{A(\mathbb{R}^n)}_{\text{range of } A \text{ or Image of } A} \Big)$$

Since $Ax = \underbrace{x_1 A_1 + x_2 A_2 + \cdots + x_n A_n}_{\text{linear combinations of columns of } A}$

Then $\mathrm{rank}(A)$ is the dimension of the span $(A_1, \_, A_n)$

Examples:

$$\mathrm{rank}\left(\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right) = 2 \qquad \mathrm{rank}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1 \qquad \mathrm{rank}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = 2$$

DEFINITION: The **kernel or nullspace** of a $m \times n$ matrix A is the kernel of $x \in \mathbb{R}^n \longmapsto Ax \in \mathbb{R}^n$ that is

$$\mathrm{Ker}(A) = \{ x \in \mathbb{R}^n , \ Ax = 0_{\mathbb{R}^m} \}$$

Examples: Rank and Kernel of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

rank ?

$$\mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} = \mathbb{R}^2 \qquad \boxed{\mathrm{rank}(A) = 2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} \mathrm{Ker}(A) &= \{ x \in \mathbb{R}^3 \mid Ax = 0 \} \\ &= \left\{ x \in \mathbb{R}^3 \,\middle|\, \begin{bmatrix} x_2 \\ 2x_3 \end{bmatrix} = 0 \right\} \\ &= \{ x \in \mathbb{R}^3 \mid x_2 = x_3 = 0 \} \\ &= \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, x_1 \in \mathbb{R} \right\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

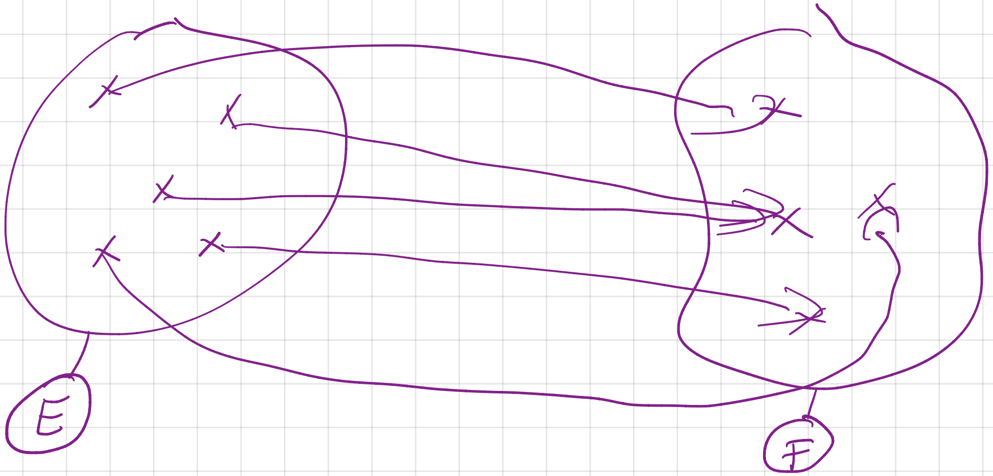
$$\boxed{\dim(\mathrm{Ker}\, A) = 1}$$

The rank(A) and dim Ker(A) are consistent with the RANK NULLITY Theorem

$$\underset{\substack{\shortparallel \\ 2}}{\mathrm{rank}(A)} + \underset{\substack{\shortparallel \\ 1}}{\dim(\mathrm{ker}(A))} = \underset{\substack{\shortparallel \\ 3}}{\dim(\mathbb{R}^3)}$$

We forget about matrix, linear transformation and we want to talk for a mapping $f: \underset{\substack{\nearrow \\ \text{set}}}{E} \longrightarrow \underset{\substack{\uparrow \\ \text{set}}}{F}$ bijection, surjection, injection.

f can be $f: \exp \begin{cases} \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0} \\ x \longrightarrow \exp(x) \end{cases}$



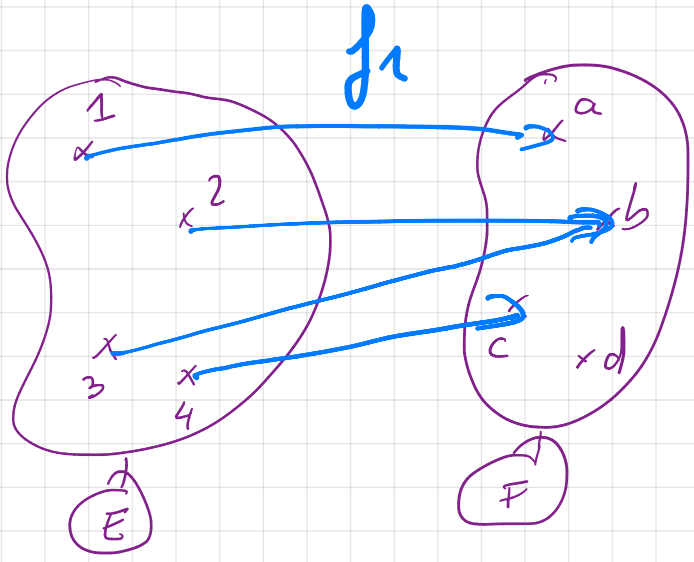
DEFINITION: We say that f is **surjective** if all elements of F have a pre-image by f, i.e. $\forall y \in F, \exists x \in E, f(x) = y$

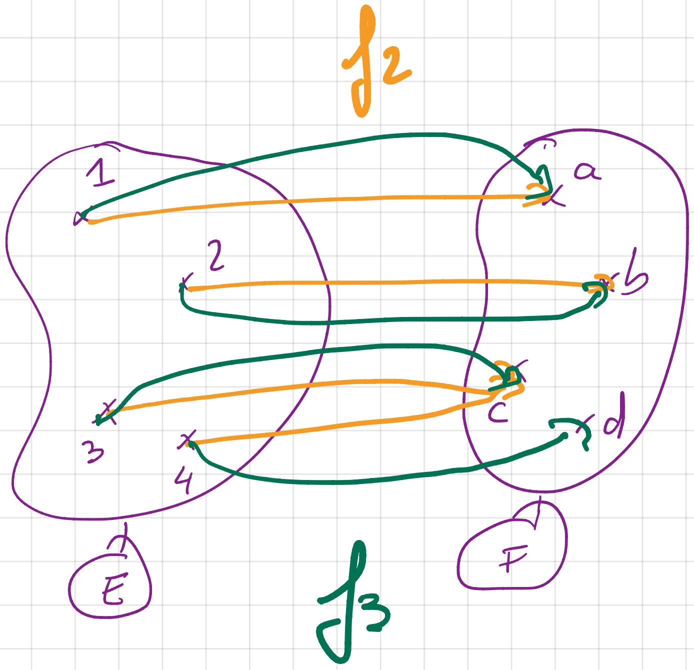
"antecedent" IN FRENCH

DEFINITION: We say that f is **injective** if two elements of E are sent on different elements of F

I.E. $\forall x_1, x_2 \in E$, $f(x_1)=f(x_2) \Rightarrow x_1=x_2$

EXAMPLES





f_1, injective, surjective ?

f_2 ______ ?

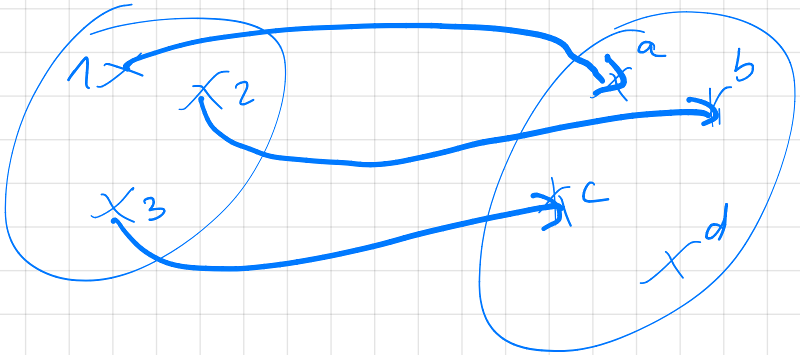
f_3 ______ ?

→ f_1 is not surjective because d has no pre-image.

f_1 ⟶ injective because $f_1(2)=f_1(3)$

Same for f_2: not injective and not surjective.

f_3 is both injective and surjective



This is injective but not surjective

DEFINITION : If a mapping $f : E \longrightarrow F$ is both injective and surjective, we say that f is **bijective**.

If f is bijective then $\forall\, y \in F$ there exist an unique $x \in E$ such that $y = f(x)$.

$\exists\, !$

exists unique

If f is bijective there exists an inverse mapping denoted f^{-1} such that

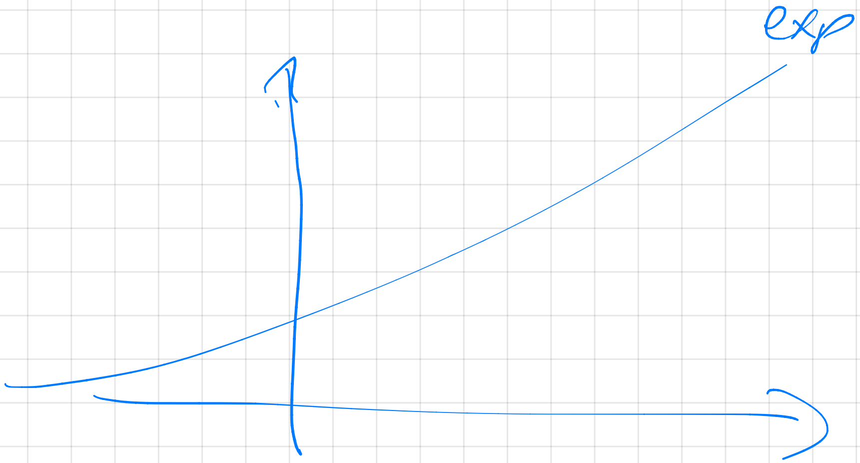
$$f \circ f^{-1} = \underbrace{f^{-1} \circ f}_{f^{-1}(f(x)) = x} = id$$

$$\begin{tikzcd} E \arrow[r, "f"] & F \arrow[r, "f^{-1}"] & E \\ x \arrow[r] & f(x) \arrow[r] & f^{-1}(f(x)) = x \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[rr, bend right, "f^{-1}\circ f"'] & F \arrow[r, "f^{-1}"] & E \end{tikzcd}$$

Example :

$$\exp : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R}_{>0} \\ x \longrightarrow \exp(x) \end{cases}$$



$\exp^{-1} = \ln$ $\qquad$ $\exp(\ln(x)) = x$

We go back to matrices.

Proposition: Let A be a $m \times n$ matrix, then

$$\left(x \in \mathbb{R}^n \longmapsto Ax \quad \text{is injective} \right) \iff \left(\operatorname{Ker}(A) = \{0\} \right)$$

PROOF: Look for the proof for the next class [EXERCICE]

$$\left(x \overset{A}{\longmapsto} Ax \quad \text{is surjective} \right) \iff \underset{\overset{\|}{A(\mathbb{R}^n)}}{\operatorname{range}(A)} = \mathbb{R}^m$$

THEOREM: If A is a $n \times n$ matrix (square). Then the following are equivalent:

(i) the transformation $x \mapsto Ax$ is bijective

(ii) $\operatorname{range}(A) = \mathbb{R}^n$ ($x \longmapsto Ax$ is surjective)

(iii) $\operatorname{Ker} A = \{0\}$ ($x \longmapsto Ax$ is injective)

PROOF: EXERICE

ELEMENTS FOR THE

CORRECTION OF EXERCICES

$$L_1 : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

L_1 the symmetry with respect to "$x = y$"