PRE2 - CLASS 4

We have seen last time that an orthogonal matrix is a square matrix U such that:

$$\boxed{U^T U = U U^T = I}$$

It means that the inverse of $U^{-1} = U^T$

<u>Example:</u>

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ is orthogonal $$U^{-1} = U^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U^{-1} = U^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

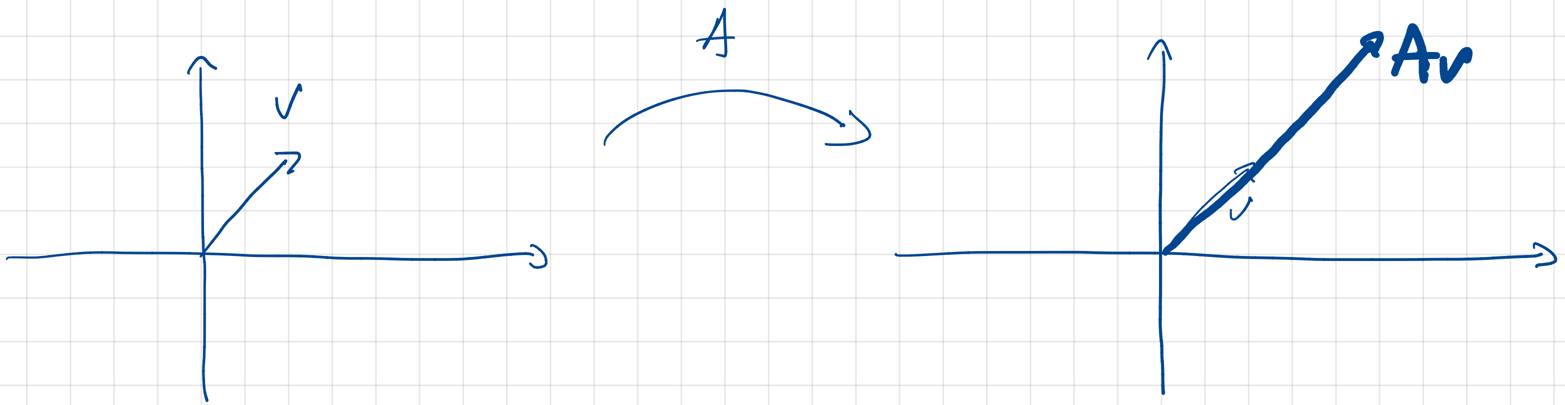
EIGENVALUES / EIGENVECTORS , MATRIX REDUCTION

DEFINITION: An eigenvector v of a $n \times n$ matrix is a non-zero vector such that, $\exists\, \lambda \in \mathbb{R}$ called eigenvalue such that

$$Av = \lambda v$$

GEOMETRICAL INTERPRETATION:

A a 2×2 matrix, v eigenvector of A; <u>ie</u> $Av = \lambda v$ say $\lambda > 1$



<u>Examples:</u> Find eigenvalues and eigenvectors of $I = \begin{pmatrix} 1 & & (0) \\ & \ddots & \\ (0) & & 1 \end{pmatrix}$, $D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$

If $I = \begin{pmatrix} 1 & & (0) \\ & \ddots & \\ (0) & & 1 \end{pmatrix}$, then any vector x satisfies $Ix = x$, so any vector is eigenvector associated to the eigenvalue 1.

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ are eigenvectors : CORRECT but actually any $x \in \mathbb{R}^n$ is eigenvector.

$D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$, $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow De_1 = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1 e_1$

e_1 is eigenvector associated to λ_1.

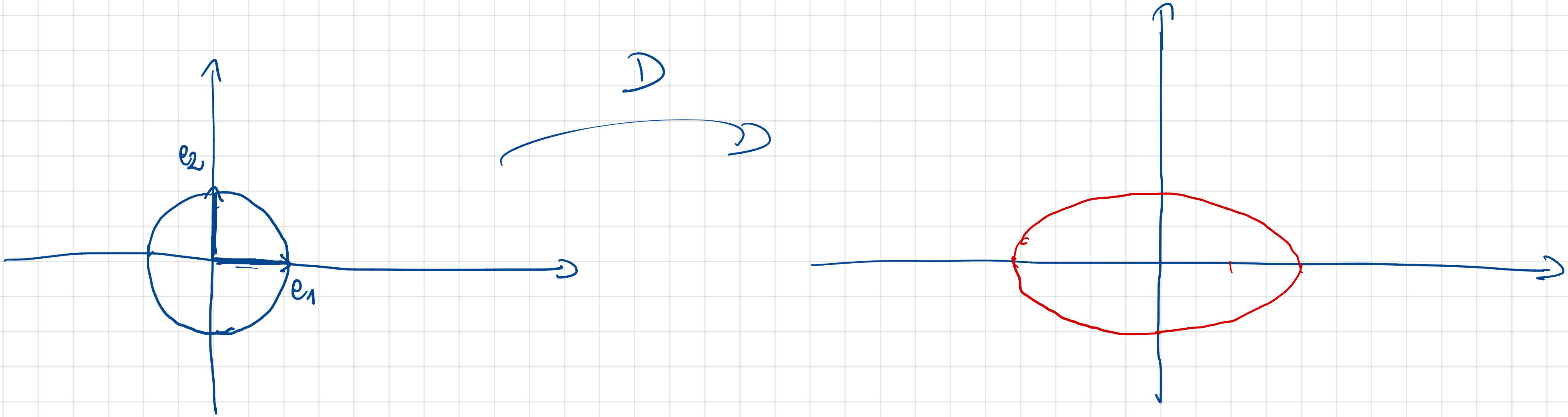
$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is eigenvector associated to λ_2

$\vdots$

$e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ ———————— to λ_n

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

In $\mathbb{R}^2$, how is the unit circle $\left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1^2 + x_2^2 = 1 \right\}$ transformed via D?



The circle is transformed into the red ellipsoid.

Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Ax$ with $x \in$ Unit circle (ie $x_1^2 + x_2^2 = 1$)

$$= \begin{pmatrix} 2x_1 \\ x_2 \end{pmatrix} \Rightarrow x_1 = \frac{y_1}{2}, \quad x_2 = y_2^2$$

$$x_1^2 + x_2^2 = 1$$

$$\Leftrightarrow \underbrace{\frac{y_1^2}{4} + y_2^2 = 1}_{\text{Equation of an ellipsoid.}}$$

Given $D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$, $D^k = \underbrace{DD\cdots D}_{k \text{ times}} = \begin{pmatrix} \lambda_1^k & & (0) \\ & \ddots & \\ (0) & & \lambda_n^k \end{pmatrix}$

$$D^2 = \begin{pmatrix} \lambda_1^2 & & (0) \\ & \ddots & \\ (0) & & \lambda_n^2 \end{pmatrix}$$

$A = \begin{pmatrix} a & d \\ c & b \end{pmatrix}$, $\cancel{A^2 = \begin{pmatrix} a^2 & d^2 \\ c^2 & b^2 \end{pmatrix}}$

If we have a $n \times n$ matrix A with n linearly independent eigenvectors $\{v_1, v_2, \dots, v_n\}$,

$$A v_i = \lambda_i v_i \qquad \lambda_i : \text{eigenvalue associated to } v_i$$

$\{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$.

$$V = \begin{bmatrix} v_1, & \dots, & v_n \end{bmatrix}, \text{ then we have } \boxed{AV = VD} \quad (*)$$

where $D = \begin{pmatrix} \lambda_1 & & (0) \\ & \lambda_2 & \\ (0) & & \ddots \\ & & & \lambda_n \end{pmatrix}$

$(*)$ is true : Indeed :

$$AV = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} v_1, & \dots, & v_n \end{pmatrix} = \begin{pmatrix} Av_1, Av_2, \dots, Av_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n \end{pmatrix} = \underbrace{\begin{pmatrix} v_1, \dots, v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}}_{VD}$$

$$VD \neq DV$$

multiplies the columns by d_i

multiplies the rows by d_i.

V is composed of n linearly independent vectors, $\Rightarrow \operatorname{span}(v_1, \dots, v_n) = \mathbb{R}^n$

$\Rightarrow \operatorname{rank}(V) = n$

$\Rightarrow V$ is an invertible matrix

$$AV = VD \iff AVV^{-1} = VDV^{-1}$$

$$\iff A = V \underbrace{D}_{\text{Diagonal}} V^{-1}$$

We say that the matrix A is diagonalizable.

EXERCICE : Let $(\lambda_i, v_i)_{1 \leq i \leq n}$ be pairs of eigenvalues, eigenvectors of a matrix A with $\{v_1, \dots, v_n\}$ are linearly independent.

Identify eigenvalues and eigenvectors of $A^2, \dots, A^k$ $k \in \mathbb{N}$

Are they diagonalizable?

POSITIVE - DEFINITE MATRICES

DEFINITION : A $n \times n$ matrix A is **positive definite** if A is symmetric and if:

1) $\forall x \in \mathbb{R}^n, \quad x^T A x \geq 0$ (positive)

2) $x^T A x = 0 \Rightarrow x = 0$ (definite)

Remark : we can have the same definition with negative definite.

A $n \times n$ matrix is positive **semi-definite** if it is symmetric and positive.

DEFINITION: (GRAM MATRIX) If A is a $m \times n$ matrix, then $A^T A$ is its Gram matrix.

The Gram matrix is always positive semi-definite.

PROOF: 1/ Let's prove that $A^T A$ is symmetric.

we need to prove that $(A^T A)^T = A^T A$

it is easy since $(AB)^T = B^T A^T$

$$\Rightarrow (A^T A)^T = A^T (A^T)^T = A^T A \Rightarrow A^T A \text{ is symmetric}$$

2/ Let $x \in \mathbb{R}^n$, we need to show that $x^T (A^T A) x \geq 0$

$$x^T A^T A x = (Ax)^T (Ax) \geq 0$$

Given y a vector $y^T y = \sum y_i^2 \geq 0$ $\quad y = \begin{pmatrix} y_1 \\ | \\ y_n \end{pmatrix}$

EXERCICE : $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$

THEOREM : (Spectral Theorem)

If A is a symmetric $n \times n$ matrix, then A is orthogonally diagonalizable, which means that A has n eigenvectors which are pairwise orthogonal and independent.

Hence, a symmetric matrix can be diagonalized as

$$A = V D \cdot V^T \quad \text{with } V \text{ orthogonal}.$$

and D is diagonal made of n eigenvalues of A

EXERCICE : If A is symmetric positive definite matrix, show that all eigenvalues of A are strictly positive.

POLAR DECOMPOSITION

Given a $n \times n$ matrix A, we consider the Gram matrix $A^T A$ which is symmetric and positive semi-definite.

According to the spectral theorem, $\exists$ V orthogonal and D diagonal (composed of eigenvalues of $A^T A$) such that:

$$A^T A = V D V^T \qquad D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$$

Since $A^T A$ is positive semi-definite, $\lambda_i \geq 0$

We define $D^{1/2} := \begin{pmatrix} \sqrt{\lambda_1} & & (0) \\ & \ddots & \\ (0) & & \sqrt{\lambda_n} \end{pmatrix}$ $\quad \left[D^{1/2} D^{1/2} = D \right]$

I define $\sqrt{A^T A} := V D^{1/2} V^T$

$$\left[\sqrt{A^T A} \sqrt{A^T A} = V D^{1/2} \underbrace{V^T V}_{I} D^{1/2} V^T = V D^{1/2} D^{1/2} V^T = V D V^T = A^T A \right]$$

The matrix $\sqrt{A^TA}$ is simpler to understand than A because it is symmetric and positive semi-definite, yet it transforms the space nearly in the same way.

<u>INDEED</u>: Let $x \in \mathbb{R}^n$, then

$$\begin{aligned} \|Ax\|^2 = (Ax)^T Ax &= x^T A^T A x \\ &= x^T \left(\sqrt{A^TA}\right)^T \sqrt{A^TA}\, x \\ &= \|\sqrt{A^TA}\, x\|^2 \end{aligned}$$

$$\begin{tikzcd} & Ax \\ x \arrow[ur] \arrow[dr] & \\ & \sqrt{A^TA}\, x \end{tikzcd}$$

have the same norm

Hence there is a norm preserving transformation (ie orthogonal matrix) to go from Ax to $\sqrt{A^TA}\,x$

$$Ax = R\sqrt{A^TA}\,x \qquad R \text{ is orthogonal}$$

It turns out that R does not depend on x. (From the POLAR DECOMPOSITION THEOREM)

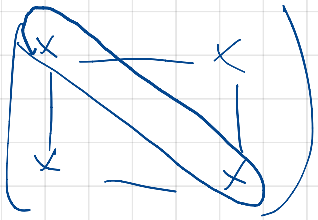
THEOREM: (POLAR DECOMPOSITION)

For any $n \times n$ matrix A, there exists an orthogonal matrix R such that

$$A = R\sqrt{A^T A}$$

TRACE and MATRIX NORMS

Given a $n \times n$ matrix A, $\quad Tr(A) = \sum_{i=1}^{n} a_{ii} \qquad A = (a_{ij})_{i,j}.$

$A =$ 

- $Tr(AB) = Tr(BA)$
- $Tr(A+B) = Tr(A) + Tr(B)$

$$\left[\, Tr(I) = n \,\right]$$

<u>Consequence</u>: Let A is a $n \times n$ diagonalizable matrix

$$\boxed{Tr(A) = \sum_{i=1}^{n} \lambda_i}$$ with λ_i eigenvalues of A.

<u>PROOF</u>:

$$A = VDV^{-1}$$

$$Tr(A) = Tr((VD)V^{-1}) = Tr(V^{-1}(VD)) = Tr(ID) = Tr(D)$$

$$D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$$ λ_i eigenvalues of A

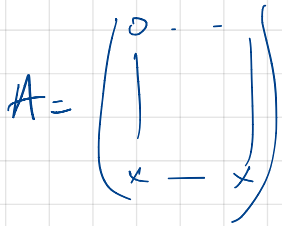
$$Tr(D) = \sum_{i=1}^{n} \lambda_i$$

<u>Example of application</u>: $A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$ is symmetric

$Tr(A) = 3 = \lambda_1 + \lambda_2$ where λ_1 and λ_2 are eigenvalues of A.

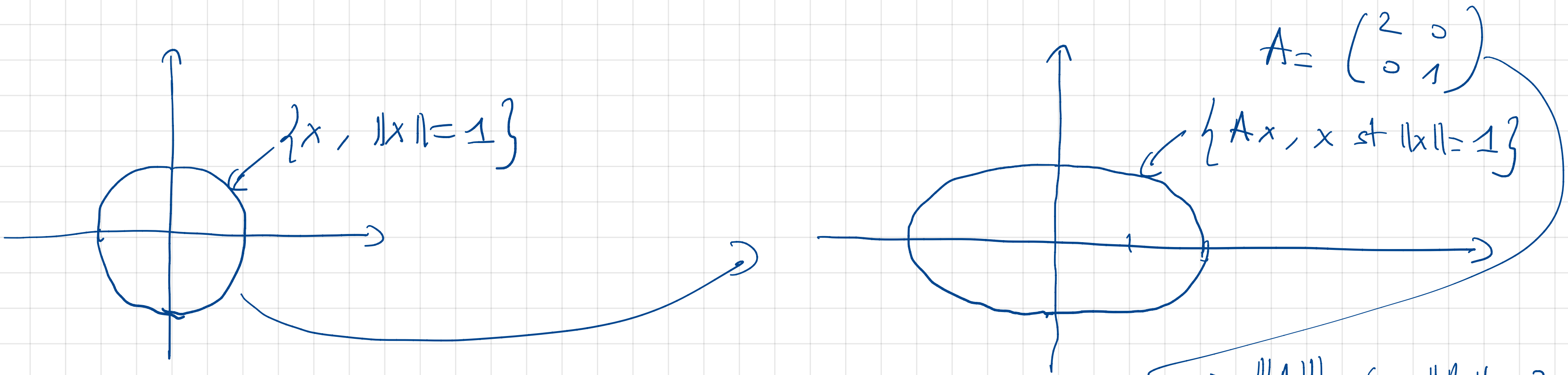
Matrix norms: We can define norms on the set of matrices.

$$\|A\|_\infty = \max_{i,j} |a_{ij}|$$



Given a norm $\|\cdot\|$ defined on vectors of $\mathbb{R}^n$ (any norm), we can define a norm on matrices as:

$$|||A||| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$



$$|||A||| = \sup_{\|x\|=1} \|Ax\| = 2$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\ker(A) = \mathbb{R}^3}$ $\quad x \in \mathbb{R}^3$, $\quad Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, all $x \in \mathbb{R}^3$ belong $\ker(A)$.

A 3×3 invertible matrix, $\ker(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$