PRE 2 – CLASS 5

8.10.2020

EXERCICE 3 (Sheet for class 4)

A symmetric and which is positive semi-definite
[which means $\forall x, x^T A x \geq 0$]
We want to show that all eigenvalues of A are non-negative
which means ≥ 0.

→ Since A is symmetric, from the spectral theorem, we know that there exits P an orthogonal matrix such that: $A = PDP^T$ where

λ_i: eigenvalues of A. $D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$

$$A = P D P^T$$

Since A is positive semi-definite $x^T A x \geq 0 \quad \forall x$

i.e $x^T P D P^T x \geq 0$

$$(P^T x)^T D \underbrace{(P^T x)}_{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \geq 0 \qquad D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$$

If $P^T x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1$

$$\underbrace{\underset{1\times n}{(P^T x)^T} \; \underset{n \times n}{D} \; \underset{n \times 1}{(P^T x)}}_{1 \times 1} = (1, 0 \ldots, 0) \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (1, 0 \ldots, 0) \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1$$

If $x = (P^T)^{-1} e_1 = P e_1$, then $x^T A x = \lambda_1$

Since $\forall x$, $x^T A x \geq 0$, this implies that $\lambda_1 \geq 0$

I can take $x = P e_i$ with $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$, $x^T A x = \lambda_i$

since $x^T A x \geq 0$, this implies that $\lambda_i \geq 0$

$$\underbrace{(P^T x)}_{y}^{T} D \underbrace{(P^T x)}_{=y} = y^T D y \longrightarrow \sum_{i=1}^{m} \lambda_i y_i^2 \geq 0$$

If A is positive definite, we know $x^T A x = 0 \Rightarrow x = 0$

ie if $x \neq 0$, then $x^T A x > 0$.

Then if $x = Pe_i \neq 0$, therefore $x^T A x = \lambda_i > 0$.

EXERCICE 4 : A a $n \times m$ matrix, show that

$$\text{rank}(A^T A) = \text{rank}(A)$$

EXERCICE 4 : A a $n \times m$ matrix, show that

$$\text{rank}(\underbrace{A^T A}_{\substack{m\times n \times n \times m \\ m \times m}}) = \text{rank}(A)$$

From the rank-nullity theorem : $\text{rank}(A) + \dim \ker(A) = m$

$\text{rank}(A^TA) + \dim \ker(A^TA) = m$

Therefore if we show that $\dim(\ker A) = \dim \ker(A^TA)$, we will have

$\text{rank}(A) = \text{rank}(A^TA)$

We will prove that $\ker(A) = \ker(A^TA)$.

We want to show that

1) $\ker(A) \subset \ker(A^TA)$. Let $x \in \ker(A)$, $Ax = 0 \Rightarrow A^TAx = A^T0 = 0$

$\Rightarrow x \in \ker(A^TA)$

We want to show that

2) $\ker(A^TA) \subset \ker(A)$ Let $x \in \ker(A^TA)$, $A^TAx = 0$

I multiply by x^T, $\underbrace{x^TA^TAx}_{(Ax)^T(Ax)} = x^T0 = 0$

$\underset{1\times n}{(Ax)^T}\underset{n\times 1}{(Ax)} = 0$

$y = Ax$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $(Ax)^T(Ax) = (y_1, \dots, y_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{m} y_i^2 = 0 \Rightarrow y_i = 0 \ \forall i$

$\Rightarrow Ax = 0$ ie $x \in \ker(A)$

We have shown that $\ker(A^T A) = \ker(A)$

$\Rightarrow \dim(\ker(A^T A)) = \dim(\ker(A))$

$\Rightarrow \operatorname{rank}(A^T A) = \operatorname{rank}(A)$ (see above).

WRAP-UP :

Spectral theorem : A $n \times n$ symmetric can be diagonalized in an orthogonal basis, ie $\exists$ U orthogonal matrix, D diagonal

$$A = UDU^T \qquad D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix} \qquad \lambda_i : \text{eigenvalues of } A.$$

In addition if A is positive definite, $\lambda_i > 0$
— if A is positive semi-definite, $\lambda_i \geq 0$.

POLAR DECOMPOSITION : A $n \times n$ matrix, $\exists$ R orthogonal such that

$$A = R \sqrt{\underbrace{A^T A}_{}}$$

Gram matrix, positive semi-definite

SINGULAR VALUE DECOMPOSITION

Start (for the moment) from a square matrix A, $n \times n$

$$(1) \quad A = R \underbrace{\sqrt{A^T A}}$$ (polar decomposition)

symmetric positive semi-definite.

Then from the spectral theorem

$$(2) \quad \sqrt{A^T A} = V \Sigma V^T \qquad \Sigma = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$$

λ_i: eigenvalues of $\sqrt{A^T A}$ (= square root of eigenvalues of $A^T A$)

V: orthogonal

$\lambda_i \geq 0$ because $\sqrt{A^T A}$ is positive semi-definite.

I can put (1) and (2) together:

$$A = R V \Sigma V^T$$ with R and V orthogonal

RV is orthogonal since R & V are orthogonal, Let denote $U = RV$

Then: $A = U \Sigma V^T$ with U, V orthogonal
Σ diagonal with positive values.

$A = U \Sigma V^T$ is a singular value decomposition of A (SVD)

$\Sigma = \text{diag}(\lambda_i)$, λ_i are called the singular value of A.

$$\begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$$

Note: this is different from diagonalization where:

$$A = P \Sigma P^{-1}$$

Here U & V are different matrices.

Proof that if R and V are orthogonal, then RV is orthogonal.

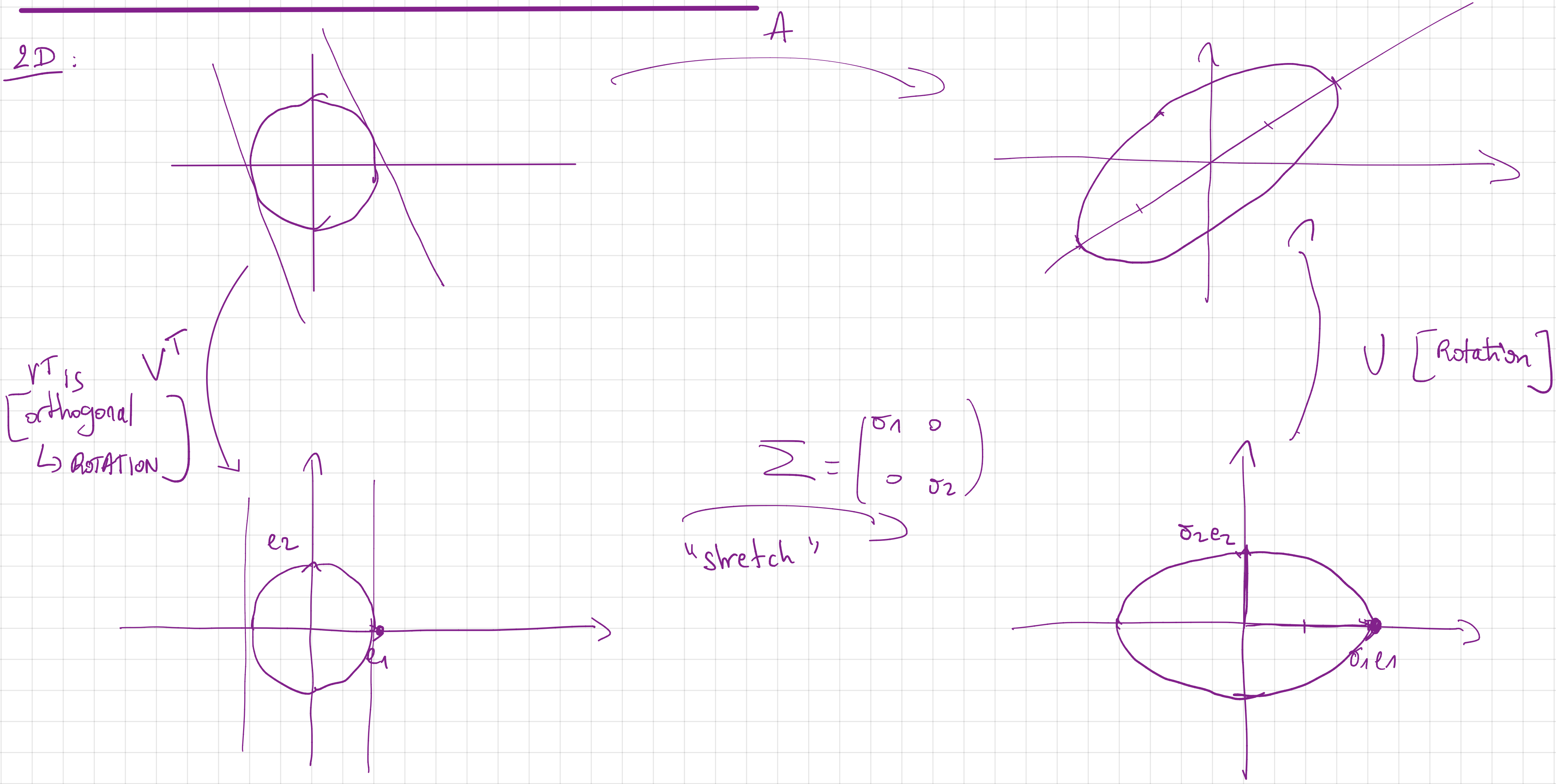
I need to show: $(RV)^T(RV) = I$ $\quad (RV)^T RV = V^T \underbrace{R^T R}_{I} V = V^T I V = V^T V = I$

R orthogonal

V orthogonal

GEOMETRICAL INTERPRETATION : $A = U \Sigma V^T$

2D :



SINGULAR VALUE DECOMPOSITION FOR RECTANGULAR MATRICES

Let A be $m \times n$ matrix, there exist U, V, Σ

U : $m \times m$ orthogonal matrix

V : $n \times n$ orthogonal matrix

Σ : $m \times n$

$$\Sigma = \begin{array}{c} \begin{pmatrix} \sigma_1 & & (0) \\ & \sigma_2 & \\ (0) & & \ddots \\ & & & \sigma_m \\ & (0) & & \end{pmatrix} \Big\updownarrow m \\ \longleftrightarrow \\ n \end{array}$$

$\sigma_i \geq 0$ singular value of A

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$

r = number of non zero singular value.

$r = \text{rank}(A) \left(= \text{rank}\, A^T A \right)$

$$A = U \Sigma V^T$$

Where do U, Σ, V come from?

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$$
$$= (V^T)^T\Sigma^T U^T U\Sigma V^T$$
$$= V\Sigma^T\underbrace{U^TU}_{I}\Sigma V^T$$
$$= V\underbrace{\underset{n\times m}{\Sigma^T}\,\underset{m\times n}{\Sigma}}_{n\times n}V^T \qquad \text{with } V \text{ orthogonal}$$

$\Sigma^T\Sigma$: diagonal matrix

$\Sigma^T\Sigma$ is the diagonal matrix that contains the eigenvalues of A^TA which are ≥ 0 because A^TA is positive semi-definite.

V contains the eigenvectors of A^TA.

→ Similary $AA^T = U\Sigma V^T(U\Sigma V^T)^T = U\Sigma V^TV\Sigma U^T = U\Sigma\Sigma^TU^T$

U contains eigenvectors from AA^T

Rank (A) = # non-zero singular values.

A square matrix is non-singular, ie invertible or full rank
$\iff$ all its singular value are non-zero.

- The ratio $\frac{\sigma_1}{\sigma_n}$ tells how close A is to being singular
(The larger the ratio, the closest the matrix is from being singular)

The ratio is called the condition number of A.

DETERMINANT

$\det(A) = |A|$ is a single number with lots of information.

$(A \text{ is invertible}) \iff (\det(A) \neq 0)$

$(A \text{ is singular}) \iff (\det(A) = 0)$

Three properties :

① $\det(I) = 1$

② If I exchange 2 rows of a matrix : reverse sign of determinant

Example

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

3a If I scale a row by t = determinant is multiplied by t:

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3b
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Linear funct° wrt 1st row, given 2nd fixed.

A and B are square matrices:

(i) $\det(A) = \det(A^T)$

(ii) $\det(AB) = \det(A)\det(B)$

If A is invertible: $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\left[\underbrace{\det(AA^{-1})}_{\det(I) = 1} = \det(A)\det(A^{-1}) \Rightarrow \det(A^{-1}) = \dfrac{1}{\det A} \right.$

$\det(A) = \pm 1$ if A is orthogonal

PROOF: A is orthogonal $\quad A^T A = I$

$$\det(A^T A) = \det(I) = 1$$

$$\Rightarrow \underset{\substack{\| \\ \det(A)}}{\det(A^T)} \times \det(A) = 1$$

$$\Rightarrow \det(A)^2 = 1 \quad \Rightarrow \det(A) = \pm 1$$

$$\det|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

BACK TO SVD:

For a square matrix $\quad \underset{n\times n}{A} = \underset{n\times n}{U} \underset{n\times n}{\Sigma} \underset{n\times n}{V^T}$

$$|\det(A)| = |\det(U \Sigma V^T)| = |\det(U)| |\det(\Sigma)| |\det(V^T)|$$

U is orthogonal, $|\det(U)| = 1$; $|\det(V^T)| = 1$

$$|\det(A)| = |\det(\Sigma)| = \underbrace{\sigma_1 \cdots \sigma_m}_{\text{singular values}}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & (0) \\ & \ddots & \\ (0) & & \sigma_m \end{pmatrix} \qquad \sigma_i \geq 0$$

$$\det(\Sigma) = \det\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{pmatrix} = \sigma_1 \det\begin{pmatrix} 1 & & & (0) \\ & \sigma_2 & & \\ & & \ddots & \\ (0) & & & \sigma_m \end{pmatrix} = \sigma_1\sigma_2 \det\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \sigma_3 & \\ & & & \ddots \\ & & & & \sigma_m \end{pmatrix} = \cdots = \sigma_1 - \sigma_m \underbrace{\det I}_{=1}$$

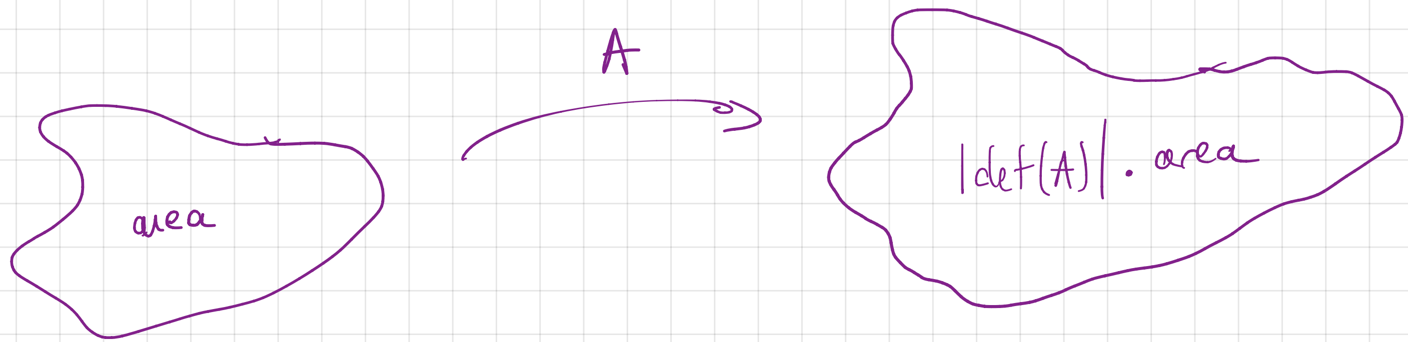
$$= \sigma_1 - \sigma_m$$

Given a matrix A, the absolute value of $\det(A)$

$=$ product of the singular values of A

GEOMETRICAL INTERPRETATION

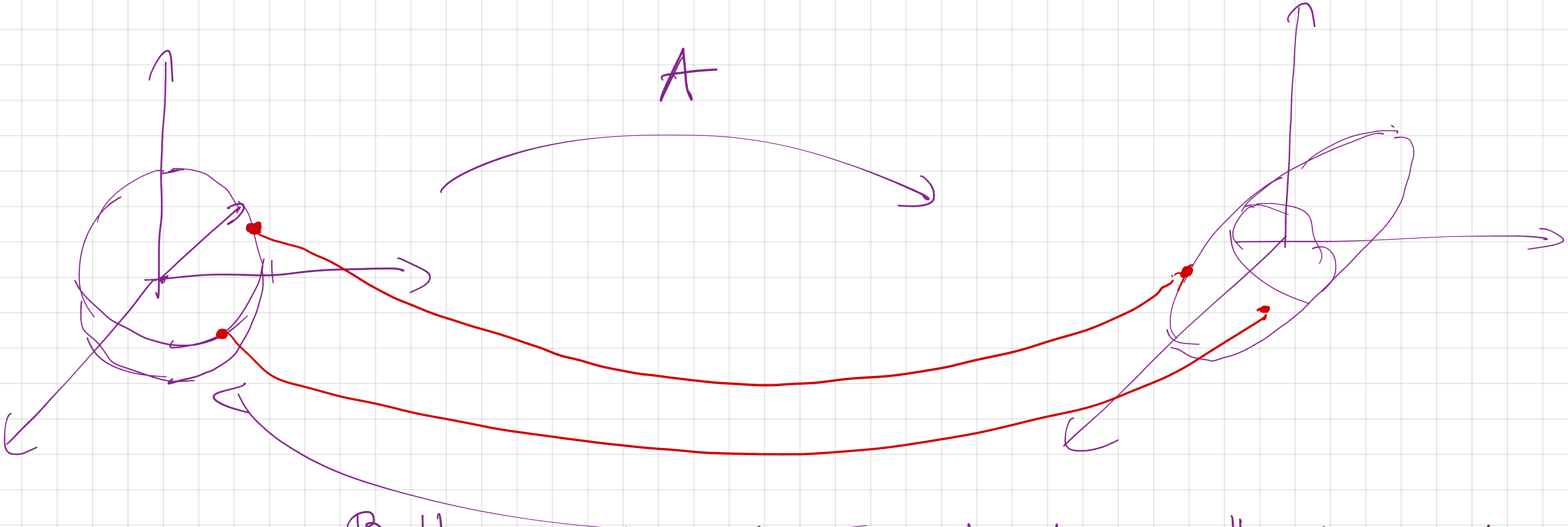
2D



The area is scaled via A by $|\det(A)|$

in higher dimension:

$$\text{Volume} \xrightarrow{A} \text{New volume} = |\det(A)| \cdot \text{Volume}.$$



Ball is transformed into hyper-ellipsoid via A

EXERCICE 5 (class 4)

A can be diagonalized $V = [v_1, \dots, v_n]$

$$A = VDV^{-1} \qquad D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$$

$$A^2 = (VDV^{-1})(VDV^{-1}) = VDV^{-1}VDV^{-1}$$
$$= VDIDV^{-1}$$
$$= VD^2V^{-1} \qquad D^2 = \begin{pmatrix} \lambda_1^2 & & (0) \\ & \ddots & \\ (0) & & \lambda_n^2 \end{pmatrix}$$

$$\vdots$$

$$A^k = V \underbrace{D^k}_{\text{Diagonal matrix}} V^{-1} \qquad D^k = \begin{pmatrix} \lambda_1^k & & (0) \\ & \ddots & \\ (0) & & \lambda_n^k \end{pmatrix}$$

A^k are diagonalizable, eigenvalues are λ_i^k, eigenvectors are the same as eigenvectors of A

Other solution :

v_i eigenvector of A associated to λ_i

$$A v_i = \lambda_i v_i$$

$$\Rightarrow \quad A^2 v_i = A(A v_i) = A(\lambda_i . v_i) = \lambda_i A v_i = \lambda_i^2 v_i$$

v_i eigenvector of A^2 associated to λ_i

EXERCICE 1 :

$$A v = 3v \qquad A w = 2w$$

$$A^n u = A^n (v + w) = ?$$

$$A v = 3v \Rightarrow A^2 v = 3 A v = 3^2 v \quad \dots \quad A^n v = 3^n v$$

$$A^n w = 2^n w$$

$$A^n u = A^n (v + w) = A^n v + A^n w = 3^n v + 2^n w$$

$$\frac{\| A^n (u) \|}{\| A^n v \|} = \frac{\| 3^n v + 2^n w \|}{\| 3^n v \|} = \frac{\| 3^n v + 2^n w \|}{3^n \|v\|} = \frac{\| v + \left(\frac{2}{3}\right)^n w \|}{\|v\|}$$

Since $\left(\frac{2}{3}\right)^n \xrightarrow[n \to +\infty]{} 0$ $\quad v + \left(\frac{2}{3}\right)^n w \xrightarrow[n \to +\infty]{} v$ $\quad \Rightarrow \lim\limits_{n \to +\infty} \frac{\| v + \left(\frac{2}{3}\right)^n w \|}{\|v\|} = \frac{\|v\|}{\|v\|} = 1$

Property: If $f : (E, \| \|) \longrightarrow \mathbb{R}$ is continuous in $a \in E$

then if $x_n \xrightarrow[n \to +\infty]{} a$ $\quad f(x_n) \xrightarrow[n \to +\infty]{} f(a)$

$$v + \left(\frac{2}{3}\right)^n w \xrightarrow[n \to +\infty]{} v$$

$f \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R} \\ x \longmapsto \| x \| \end{cases}$ is continuous in v, then $\underbrace{\left\| v + \left(\frac{2}{3}\right)^n w \right\|}_{f\left(v + \left(\frac{2}{3}\right)^n w\right)} \longrightarrow \| v \|$

$$f\left(v + \left(\frac{2}{3}\right)^n w\right) \longrightarrow f(v)$$