CLASS 4 – EXERCICES

EXERCICE 1

Suppose A is a matrix with an eigenvector v associated to the eigenvalue 3 and an eigenvector w associated to the eigenvalue 2. Let $u = v + w$.

Prove that
$$\lim_{m \to +\infty} \frac{\|A^m u\|}{\|A^m v\|} = 1$$

EXERCICE 2

Let $\|.\|$ be a norm defined on vectors of $\mathbb{R}^n$. We define a norm on the set of $n \times n$ matrices as
$$|||A||| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

1/ Prove that $\forall x \quad \|Ax\| \leq |||A||| \, \|x\|$

2/ Deduce that $|||AB||| \leq |||A||| \, |||B|||$

3/ Prove that if $\|.\|$ is the Euclidean norm and A is symmetric, then
$$|||A||| = \max_{i=1,\dots,n} |\lambda_i| \qquad \lambda_i \text{ eigenvalues of } A.$$

EXERCICE 3

1/ Suppose that A is symmetric positive semi-definite, show that all eigenvalues of A are nonnegative (i.e ≥ 0)

2/ Suppose that A is symmetric positive definite. Show that all eigenvalues of A are strictly positive (i.e > 0)

EXERCICE 4

Let A be a $n \times m$ matrix.

Show that $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$

EXERCICE 5

Let $(\lambda_i, v_i)_{1 \leq i \leq n}$ be pairs of eigenvalues, eigenvectors of a matrix A where $\{v_1, \dots, v_n\}$ are linearly independent.

Identify eigenvalues and eigenvectors of $A^2, A^3, \dots, A^k$ $k \in \mathbb{N}$

Are those matrices diagonalizable ?