PRE 2 - CLASS 3

Correction from :

$$\Big(\underset{(i)}{x \longmapsto Ax \text{ is injective}}\Big) \Longleftrightarrow \Big(\underset{(ii)}{\mathrm{Ker}(A) = \{0\}}\Big)$$

$(i) \Rightarrow (ii)$ Let $z \in \mathrm{Ker}(A)$, then $Az = 0 = A0$

by injectivity $z = 0$

$(ii) \Rightarrow (i)$ Let $x, \tilde{x}$ be two vectors such that $Ax = A\tilde{x}$

by linearity $\Rightarrow A(x - \tilde{x}) = 0$

then $x - \tilde{x} \in \mathrm{Ker}(A)$ and thus $x - \tilde{x} = 0$

$\Rightarrow x = \tilde{x}$

Elements for the correction of the proof of the last THEOREM from the last class.

Since bijective means injective and surjective we only need to prove that

$(ii) \Leftrightarrow (iii)$

Let us prove that (ii) $\Rightarrow$ (iii)

We know that $\{0\} \subset \operatorname{Ker}(A)$ [because $A0 = 0$]

With the rank nullity theorem :

$$\dim(\operatorname{Ker}(A)) + \underbrace{\operatorname{rank}(A)}_{\overset{\shortparallel}{n}} = n$$

[because $x \mapsto Ax$ is surjective]

By assumption

$\operatorname{range}(A) = \mathbb{R}^n$ $\quad \operatorname{rank}(A) = \dim(\operatorname{range} A)$

$= \dim(\mathbb{R}^n)$

$= n$

$$\Rightarrow \dim(\operatorname{Ker}(A)) = 0$$

$$\Rightarrow \operatorname{Ker}(A) = \{0\}$$

(iii) $\Rightarrow$ (ii) If $\operatorname{Ker}(A) = \{0\} \Rightarrow \dim(\operatorname{Ker}(A)) = 0$

By the rank nullity theorem $\Rightarrow \operatorname{rank}(A) = n$

$$\underbrace{\dim(\operatorname{Ker}(A))}_{\overset{\shortparallel}{0}} + \operatorname{rank}(A) = n$$

$$\Rightarrow \operatorname{rank}(A) = n$$

$\Leftrightarrow x \mapsto Ax$ is surjective.

For a square matrix.
When $x \longmapsto Ax$ is bijective, we can talk about the inverse of the matrix A, we denote it A^{-1}.

It satisfies $A^{-1}A = AA^{-1} = I = \begin{pmatrix} 1 & & & (0) \\ & 1 & & \\ & & \ddots & \\ (0) & & & 1 \end{pmatrix}$

Identity matrix, sometimes we denote it I_n or Id

- The previous theorem tells us that the inverse of a matrix exists $\Longleftrightarrow \left(\operatorname{Ker}(A) = \{0\} \right) \Longleftrightarrow \left(\operatorname{rank}(A) = n \right)$

Properties of identity matrix: Let A a $n \times n$ matrix

- $AI_n = I_nA = A$
- $I_n I_n = I_n \Rightarrow I_n^{-1} = I_n$
- $I_n \underset{\mathbb{R}^n}{\underset{\cap}{x}} = x$

Inverse of product of matrices :

Let A and B be $n\times n$ matrices, invertible :

$$(AB)^{-1} = B^{-1}A^{-1}$$

PROOF : if $(AB)(B^{-1}A^{-1}) = I_n$ then $(AB)^{-1} = B^{-1}A^{-1}$

$$\hookrightarrow (AB)B^{-1}A^{-1} = A\underbrace{BB^{-1}}_{I_n}A^{-1} = \underbrace{AI_n}A^{-1}$$
$$= AA^{-1} = I_n$$

EXERCICE : Are the following matrices invertible ? If so compute the inverse :

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} ; \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in Ker(A)$$

A is invertible (rank (A) = 2) , B is not invertible (rank(B) = 2 or $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in Ker(B)$)

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

A is a specific diagonal matrix.

<u>Definition</u>: A diagonal matrix, is a matrix with zeros on off-diagonal elements

$$D = \begin{pmatrix} d_1 & & & (0) \\ & d_2 & & \\ & & \ddots & \\ (0) & & & d_n \end{pmatrix}$$

A diagonal matrix is invertible if $d_1 \neq 0, d_2 \neq 0, \dots, d_n \neq 0$

If $d_i \neq 0 \quad \forall i = 1, \dots, n$ then $D^{-1} = \begin{pmatrix} \frac{1}{d_1} & & & (0) \\ & \frac{1}{d_2} & & \\ & & \ddots & \\ (0) & & & \frac{1}{d_n} \end{pmatrix}$

<u>PROOF</u>: $\begin{pmatrix} d_1 & & (0) \\ & \ddots & \\ (0) & & d_n \end{pmatrix} \begin{pmatrix} \frac{1}{d_1} & & (0) \\ & \ddots & \\ (0) & & \frac{1}{d_n} \end{pmatrix} = I$

Why do we want to inverse a matrix?
for instance to solve a linear system.

Consider the system of equations:

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 2 \\ \phantom{x_1 + {}} 2x_2 + x_3 = 1 \\ x_1 + \phantom{3x_2 + {}} 3x_3 = 2 \end{cases} \qquad (S)$$

Show that (S) is equivalent to find $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that

$$\underbrace{\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{=A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}_{=b}$$

i.e. to solve (S) we need to find x such that $Ax = b$

If we know that A is invertible, then

$$A x = b$$
$$\Leftrightarrow \quad \underbrace{A^{-1} A}_{} x = A^{-1} b$$
$$\Leftrightarrow \quad I x = A^{-1} b$$
$$\Leftrightarrow \quad x = A^{-1} b$$

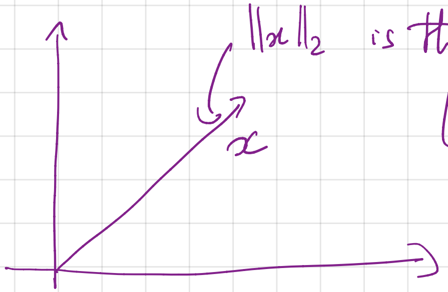
NORMS

Given a vector space E. A norm on E is a function : $p : E \longrightarrow \mathbb{R}_+ = [0, +\infty[$ and that satisfies :

$\forall \lambda \in \mathbb{R}, \ \forall u, v \in E$

- $p(u+v) \leq p(u) + p(v)$ TRIANGE INEQUALITY
- $p(\lambda u) = |\lambda| \, p(u)$
- $p(v) = 0 \Rightarrow v = 0$

<u>Example in $\mathbb{R}^n$</u> : $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$



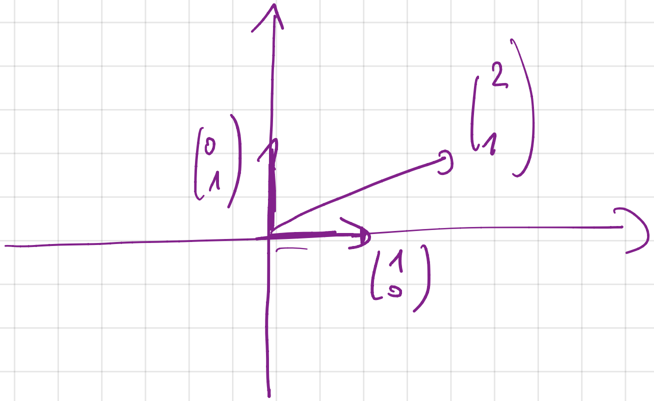
$\|x\|_2$ is the length of x (usual length)

Given $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\qquad \|x\|_2 = \sqrt{\left(x_1^2 + \dots + x_n^2\right)}$

example : $\left\|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\|_2 = \sqrt{4+1} = \sqrt{5}$

$$\left\|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|_2 = \sqrt{1+0} = 1$$

$$\left\|\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\|_2 = \sqrt{0+1} = 1$$



Other norms :

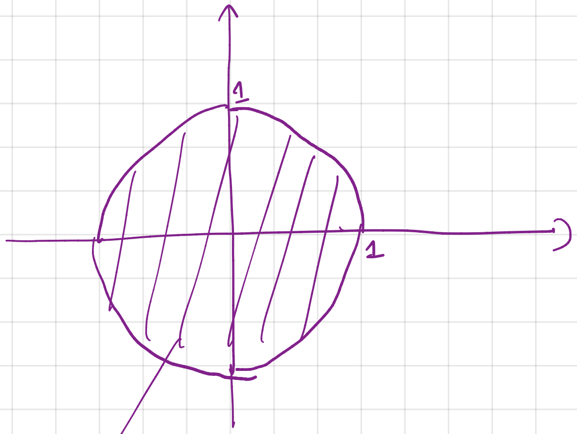
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$\|x\|_\infty = \max |x_i|$$

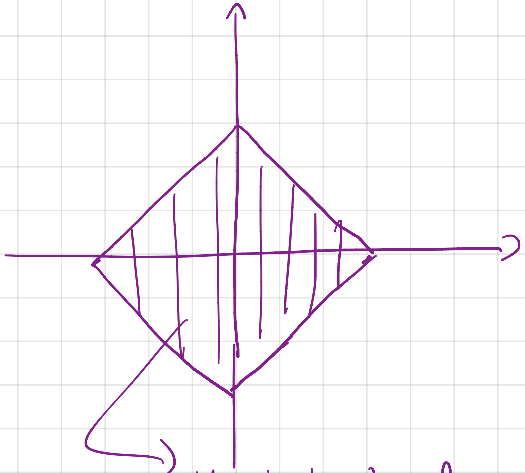
Display in $\mathbb{R}^2$, the unit balls with respect to the different norms

$$B(0, 1) = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$$

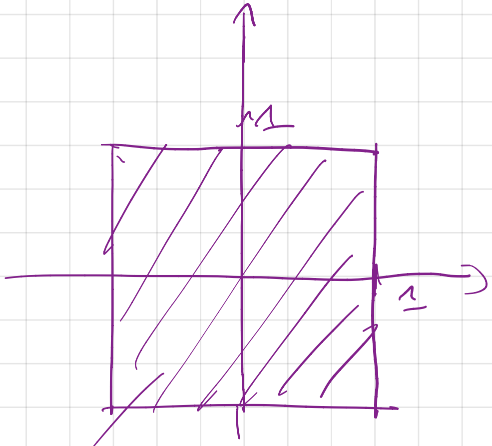


$\{x \mid \|x\|_2 \leq 1\}$

$\sqrt{x_1^2+x_2^2} \leq 1$



Unit ball for $\|x\|_1$



Unit ball for $\|\cdot\|_\infty$

Norms are useful to measure distances:

$$d(x,y) = \|x-y\| \left(= \|y-x\| \right)$$

Two different norms will give two different "measure" of distances.

DOTS PRODUCTS AND ORTHOGONALITY

Examples:

$$p = \begin{bmatrix} 4.90 \\ 2.20 \\ 1.50 \end{bmatrix} \qquad q = \begin{bmatrix} 20 \\ 100 \\ 200 \end{bmatrix}$$

PRICES OF PRODUCTS

quantity of product sold.

Value of what you have sold: $\underbrace{4.90 \times 20 + 2.20 \times 100 + 1.50 \times 200}_{} =$

$= p \cdot q$

dot product

DEFINITION: The **dot product** or **scalar product** of two vectors in $\mathbb{R}^n$ is defined as

$$x.y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

$$\|$$

$$\langle x, y \rangle$$ (other standard notation)

Connection to Euclidean norm: $x.x = \sum_{i=1}^{n} x_i^2 = \|x\|_2^2$

PROPERTIES :

$$x \cdot y = y \cdot x \quad \forall x, y \quad [\text{symmetry}]$$

$$\underset{\underset{\mathbb{R}}{\uparrow}}{\lambda} x \cdot y = x \cdot (\lambda y) = \lambda (x \cdot y) \quad \forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$$

$$x \cdot (\lambda y + z) = \lambda x \cdot y + x \cdot z \quad \lambda \in \mathbb{R}, x, y, z \in \mathbb{R}^n$$

EXERCICE : Show that

$$\|x+y\|^2 = (x+y) \cdot (x+y) = \|x\|^2 + 2x \cdot y + \|y\|^2 \quad \forall x, y \in \mathbb{R}^n$$

where $\| \cdot \|$ is the Euclidean norm.

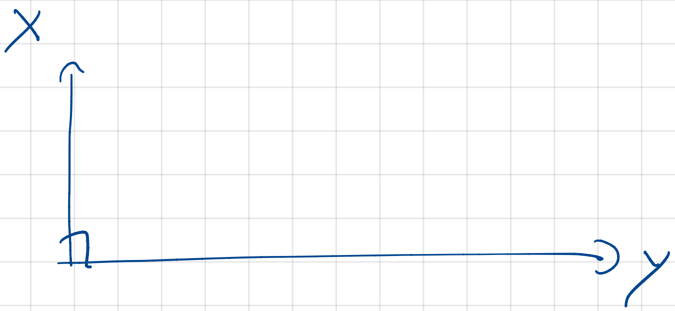
The dot product $x \cdot y$ has a geometric connection with the angle θ between two vectors x and y :

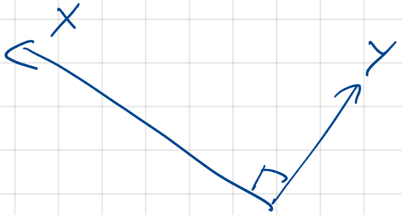
$$x \cdot y = \|x\| \|y\| \cos(\theta)$$

When $x \cdot y = 0$ we say that x and y are **orthogonal**.

GEOMETRIC INTERPRETATION : If $x \cdot y = 0$ then $\cos(\theta) = 0$, $\theta = 90°$

Examples of orthogonal vectors :





EXERCICE : Let $v_1, \dots, v_n$ be a list of orthogonal non zero vectors, i.e $v_i \cdot v_j = 0$ $\forall i \neq j$. Prove that they are linearly independent.

TRANSPOSE OF A MATRIX

DEFINITION : Given a $m \times n$ matrix A, its transpose A' or A^T is a $n \times m$ where ith row is equal to the ith column of A.

If $A = (a_{ij})$, $A^T = (\bar{a}_{ij})$ where $\bar{a}_{ij} = a_{ji}$

Example:

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 4 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 1 & 4 \\ 3 & 2 & 0 \end{pmatrix}$$

For a square matrix :

$$A = \begin{pmatrix} a_{11} & \bigcirc & \bigcirc & - & \\ \boxed{a_{21}} & \ddots & & & \\ \bigcirc & & \ddots & & \\ \vdots & & & \ddots & \\ a_{n1} & & & & a_{nn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & - & - & - & a_{n1} \\ a_{12} & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{1n} & & & & a_{nn} \end{pmatrix}$$

$$\longleftrightarrow$$

Swap symmetric elements wrt diagonal.

PROPERTIES

- $(\underset{\overset{\uparrow}{\in \mathbb{R}}}{\lambda} A + B)^T = \lambda A^T + B^T$
- .. $(AB)^T = B^T A^T$
- $(A^T)^T = A$

DEFINITION ; A $n \times n$ matrix A satisfying $A^T = A$ is called **Symmetric.**

Example:

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \text{ is symmetric}$$

$$\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 3 & \gamma \\ \beta & \gamma & 6 \end{pmatrix} \text{ is symmetric}$$

WRITING THE DOT PRODUCT WITH TRANSPOSE

Let $x, y \in \mathbb{R}^n$ two vectors (or $n \times 1$ matrices)

$$\underset{\substack{\uparrow \\ \text{dot product}}}{x \cdot y} = \underbrace{x^T y}_{\substack{\uparrow \\ \text{product} \\ \text{between 2 matrices}}}$$

$$x^T y = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n$$

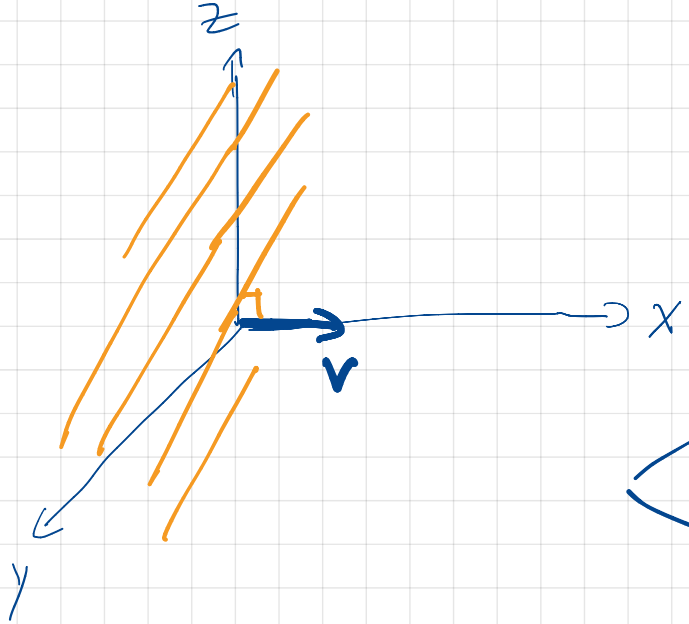
EXERCICE: Show that $\underline{u \cdot (Av) = A^T u \cdot v}$ $\forall u \in \mathbb{R}^m, v \in \mathbb{R}^n$,
A $m \times n$ matrix

ORTHOGONAL COMPLEMENT

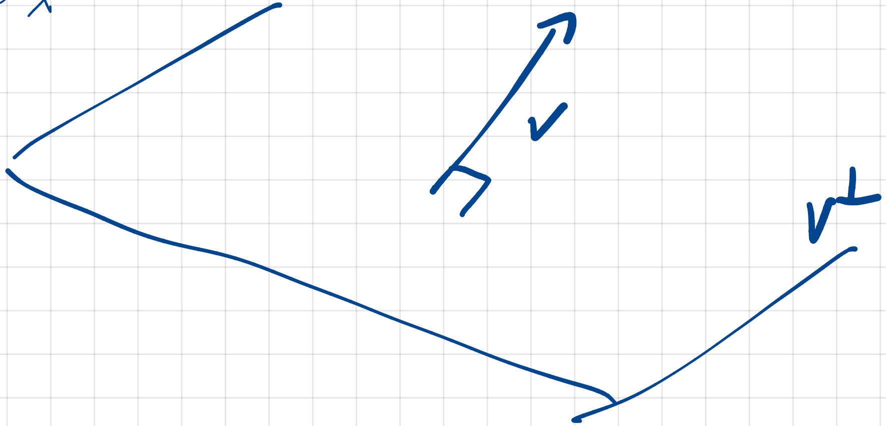
The orthogonal complement $V^\perp$ of a vector space $V \subset \mathbb{R}^n$ is the set of vectors which are orthogonal to every vector in V.

Example: In $\mathbb{R}^3$. $v = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $V = \text{span}(v)$

What is $V^\perp$?



$$V^\perp = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$



An orthogonal basis is a basis $\{e_1, -, e_n\}$ such that

$$e_i \cdot e_j = 0 \quad i \neq j$$

Which orthogonal basis of $\mathbb{R}^n$ do you know?

standard basis, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\dots$, $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

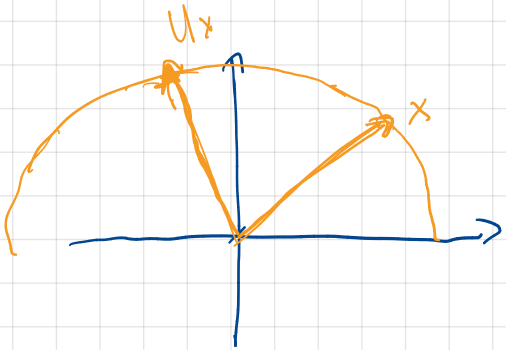
THEOREM: Every vector space $V \subset \mathbb{R}^n$ has an orthogonal basis.

ORTHOGONAL MATRICES

DEFINITION: A square matrix U is **orthogonal** if $U^T U = U U^T = I$

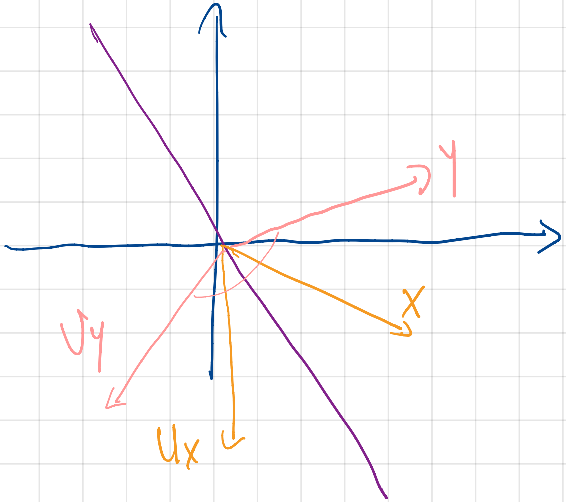
Orthogonal matrices are norm preserving:

U $n \times n$ orthogonal matrix, $\forall\ x \in \mathbb{R}^n$ $\|U x\| = \|x\|$



↳ How do you prove this?

In $\mathbb{R}^2$, orthogonal matrices are either rotations or reflexion along axis



Let us prove that $\|Ux\| = \|x\|$ if U is orthogonal

This is equivalent to show that $\|Ux\|^2 = \|x\|^2$

i.e $Ux \cdot Ux = x \cdot x$

$$Ux \cdot Ux = \underbrace{U^T U}_{I} x \cdot x = x \cdot x$$

Remark: $U^T U = I \iff$ The columns of U are orthogonal and of norm 1

$U = [U_1, \ldots, U_n]$

$$\boxed{U^T U = I \iff \|U_i\| = 1 \ \forall i \quad U_i \cdot U_j = 0 \ \forall i \neq j}$$

Find some 3 x 3 orthogonal matrices.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A \; ; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \; ; \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \; ; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

permutation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

↳ orthogonal - Rotation matrix

rotation in the plane (x,y)

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad Ax = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

A is a permutation, it permutes the coordinate of x.

$$A = \begin{array}{ccc} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \shortparallel & \shortparallel & \\ A_1 & A_2 & A_3 \end{array}$$

$$\|A_1\| = \sqrt{0+0+1} = 1$$

$$\|A_2\| = 1$$

$$\|A_3\| = 1$$

$$A_1 . A_2 = 0 \times 0 + 0 \times 1 + 1 \times 0 = 0$$

$$A_1 . A_3 = \dots = 0$$

$$A_2 . A_3 = \dots = 0$$

U is an orthogonal matrix $U^T U = I$

$\iff$ $U = \left[U_1, \dots, U_n \right]$ $\|U_i\| = 1$, $U_i \cdot U_j = 0$ $i \neq j$

But it is not true that $x \in \mathbb{R}^n$ and Ux are orthogonal for U orthogonal.

Take : $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ U is orthogonal

$$\underset{U_1}{\shortparallel} \qquad \underset{U_2}{\shortparallel}$$

$\|U_1\|^2 = \cos^2\theta + \sin^2\theta = 1$; $\|U_2\|^2 = \sin^2\theta + \cos^2\theta = 1$

$U_1 \cdot U_2 = \cos\theta(-\sin\theta) + \sin\theta\cos\theta = 0$

x , Ux