The effect of competition on the height and length of the forest of genealogical trees of a large population

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Consider a Galton–Watson branching process $X^m$ in continuous time with $m$ ancestors at time $t = 0$: for $k \geq 1$, the process $X^m$ jumps

$$
\begin{align*}
 k &\rightarrow k + 1, \quad \text{at rate } \mu k; \\
 k &\rightarrow k - 1, \quad \text{at rate } \lambda k.
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In order to model competition within the population described by $X^m$, we superimpose to each individual a death rate due to competition, equal to $\gamma$ times the number of presently alive individuals in the population, which amounts to add a global death rate equal to $\gamma (X^m_t)^2$ at time $t$.

Set $m = \lfloor Nx \rfloor$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma / N$, and $Z^N, x = \frac{X^m_t}{N}$. The “total population mass process” $Z^N$ converges weakly to the solution of the Feller SDE with logistic drift

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\begin{align*}
 dZ^x_t &= \left[ \theta Z^x_t - \gamma (Z^x_t)^2 \right] dt + 2 \sqrt{Z^x_t} dW_t, \quad Z^x_0 = x.
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This equation has been studied by Lambert, Pardoux and Wakolbinger. Clearly the forest of those $m$ trees is finite a.s.

One can define the height and the length of the discrete forest of genealogical trees

$$H^m = \inf\{t > 0, X^m_t = 0\}, \quad L^m = \int_0^{H^m} X^m_t dt, \quad \text{for } m \geq 1,$$

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We generalize the above models, replacing the death rate \( \gamma(X_t^m)^2 \) by \( \gamma(X_t^m)^\alpha \).

Set \( m = [Nx] \), \( \mu_N = 2N + \theta \), \( \lambda_N = 2N \) and \( \gamma_N = \gamma/N^{\alpha-1} \), and \( Z^{N,x} := \frac{X^{[Nx]}}{N} \). We show that the process \( Z^N \) converges weakly to to a Feller SDE with a negative polynomial drift.

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dZ^x_t = [\theta Z^x_t - \gamma(Z^x_t)^\alpha] \, dt + 2\sqrt{Z^x_t} \, dW_t, \quad Z^x_0 = x.
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Results

- Both $\mathbb{E}[\sup_m H^m] < \infty$ and $\mathbb{E}[\sup_x T^x] < \infty$ if $\alpha > 1$, while $H^m \to \infty$ as $m \to \infty$ and $T^x \to \infty$ as $x \to \infty$ a.s. if $\alpha \leq 1$.
- Both $\mathbb{E}[\sup_m L^m] < \infty$ and $\mathbb{E}[\sup_x S^x] < \infty$ if $\alpha > 2$, while $L^m \to \infty$ as $m \to \infty$ and $S^x \to \infty$ as $x \to \infty$ a.s. if $\alpha \leq 2$.

This necessitates to define in a consistent way the population processes jointly for all initial population sizes, i.e. we will need to define the two-parameter processes $\{X^m_t, t \geq 0, m \geq 1\}$ and $\{Z^x_t, t \geq 0, x > 0\}$. 
The model with asymmetric effect of the competition

- The description of the process \((X^m_t, t \geq 0)\) is valid for one initial condition \(m\), but it is not sufficiently precise to describe the joint evolution of \(\{(X^m_t, X^n_t), \ t \geq 0\}\), \(1 \leq m < n\).
- Modelize the effect of the competition in a asymmetric way. The idea is that the progeny \(X^m_t\) of the \(m\) “first” ancestors should not feel the competition due to the progeny \(X^n_t - X^m_t\) of the \(n - m\) “additional” ancestors which is present in the population \(X^n_t\).
- We order the ancestors from left to right, this order being passed to their progeny.
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- We order the ancestors from left to right, this order being passed to their progeny.
The individual placed at the position $i$ at time $t$ dies because of competition at rate $\gamma [L_i(t)^\alpha - (L_i(t) - 1)^\alpha]$, $L_i(t)$ is the number of alive individuals at time $t$, who are located at his left on the planar tree.
Discret model in the asymmetric competition picture

- \( \{X_t^m, \ t \geq 0\} \) is a continuous time \( \mathbb{Z}_+ \)-valued Markov process, which evolves as follows. \( X_t^m \) jumps to \( k + 1 \), at rate \( \mu k \);
  \[ \begin{align*}
  k - 1, & \quad \text{at rate } \lambda k + \gamma(k - 1)\alpha.
  \end{align*} \]

- The above description specifies well the evolution of the two parameters process \( \{X_t^m, \ t \geq 0, m \geq 0\} \).

- If \( \alpha \neq 1 \), \( \{X_t^m, \ m \geq 1\} \) is not a Markov chain for fixed \( t \). The conditional law of \( X_t^{n+1} \) given \( X_t^n \) differs from its conditional law given \( (X_t^1, X_t^2, \ldots, X_t^n) \).

- However, \( \{X_t^m, \ m \geq 0\} \) is a Markov chain with values in the space \( D([0, \infty); \mathbb{Z}_+) \), which starts from 0 at \( m = 0 \).
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If \( \alpha \neq 1 \), \{X_t^m, \ m \geq 1\} is not a Markov chain for fixed \( t \). The conditional law of \( X_{t}^{n+1} \) given \( X_{t}^{n} \) differs from its conditional law given \((X_t^1, X_t^2, \ldots, X_t^n)\).

However, \{X_t^m, \ m \geq 0\} is a Markov chain with values in the space \( D([0, \infty); \mathbb{Z}_+) \), which starts from 0 at \( m = 0 \).
For arbitrary $0 \leq m < n$, let $V_{t}^{m,n} := X^n_t - X^m_t$, $t \geq 0$. Conditionally upon \{X^{\ell}, \ell \leq m\}, and given that $X^m_t = x(t)$, $t \geq 0$, \{\{V_{t}^{m,n}, t \geq 0\} is a $\mathbb{Z}_+-$valued time inhomogeneous Markov process starting from $V_0^{m,n} = n - m$, whose time–dependent infinitesimal generator \{\{Q_{k,\ell}(t), k, \ell \in \mathbb{Z}_+\} is such that its off–diagonal terms are given by

\[
Q_{0,\ell}(t) = 0, \quad \forall \ell \geq 1,
\]

\[
Q_{k,k+1}(t) = \mu_k,
\]

\[
Q_{k,k-1}(t) = \lambda_k + \gamma(x(t) + k - 1)^\alpha,
\]

\[
Q_{k,\ell}(t) = 0, \quad \forall \ell \notin \{k - 1, k, k + 1\}.
\]
The continuous Model

- \( \{Z^x_t, \ t \geq 0, \ x \geq 0\} \) which such that for each fixed \( x > 0, \ \{Z^x_t, t \geq 0\} \) is continuous process, solution of the SDE (1).

- For any \( 0 < x < y \), \( \{V^{x,y}_t := Z^y_t - Z^x_t, \ t \geq 0\} \) solves the SDE

\[
dV^{x,y}_t = \left[ \theta V^{x,y}_t - \gamma \left( (Z^x_t + V^{x,y}_t)^\alpha - (Z^x_t)^\alpha \right) \right] dt + 2\sqrt{V^{x,y}_t} dW'_t,
\]

\( V^{x,y}_0 = y - x, \)

\( \{W'_t, \ t \geq 0\} \) is independent from the Brownian motion \( W \) which drives the SDE (1) for \( Z^x_t \).

- \( \{Z^x, \ x \geq 0\} \) is a Markov process with values in \( C([0, \infty), \mathbb{R}_+) \), starting from 0 at \( x = 0 \).
Convergence result

Theorem

As \( N \to \infty \),

\[
\{ \tilde{Z}_t^N, \ t \geq 0, x \geq 0 \} \Rightarrow \{ Z_t^x, \ t \geq 0, x \geq 0 \}
\]

in \( D([0, \infty); C([0, \infty); \mathbb{R}_+)) \), equipped with the Skohorod topology of the space of càdlàg functions of \( x \), with values in the space \( C([0, \infty); \mathbb{R}_+) \) equipped with the topology of locally uniform convergence.
Height of the discrete tree

Theorem

• If $0 < \alpha \leq 1$, then
  \[ \sup_{m \geq 1} H^m = +\infty \quad \text{a. s.} \]

• If $\alpha > 1$, then
  \[ \mathbb{E} \left[ \sup_{m \geq 1} H^m \right] < \infty. \]
Proof of the Theorem[Height of the discrete tree] for $\alpha > 1$

Let $H_1^m = \inf \{ s \geq 0; X_s^m = 1 \}$.

**Proposition**

For $\alpha > 1$, $\lambda = 0$, $\forall \ m \geq 1$, $\mathbb{E}(H_1^m)$ is given by

$$\mathbb{E}(H_1^m) = \sum_{k=2}^{m} \frac{1}{\gamma(k-1)^\alpha} \sum_{n=0}^{\infty} \frac{\mu^n}{\gamma^n} \frac{1}{[k(k+1)\cdots(k+n-1)]^{\alpha-1}} < \infty.$$ 

Moreover we have

$$H^m \leq H_1^m + G H_1^2 + \sum_{i=1}^{G} T_i,$$

where $G$ is geometric variable with parameter $\frac{\lambda}{\lambda+\mu}$ and $T_i$ is exponential with mean $1/(\lambda + \mu)$. 
We have $X_t^{\alpha,m} \geq X_t^{1,m}$, for all $m \geq 1$, $t \geq 0$, a. s.. \{X_t^m, t \geq 0\} is the sum of $m$ mutually independent copies of \{X_t^1, t \geq 0\}. The result follows from the fact that $\mathbb{P}(H^1 > t) > 0$, for all $t > 0$. 

Length of the discrete tree

- Time change of $X^m$:

\[ A_t^m := \int_0^t X_r^m \, dr, \quad \eta_t^m = \inf \{ s > 0; \ A_s^m > t \} . \]

\[ U^m := X^m \circ \eta^m, \ \text{and} \ S^m = \inf \{ r > 0; U_r^m = 0 \} . \]

- We have $L^m = S^m$ since $S^m = \int_0^{H^m} X_r^m \, dr$. 
Length of the discrete tree

\[ X_t^m = m + P_1 \left( \int_0^t \mu X_r^m \, dr \right) - P_2 \left( \int_0^t \left[ \lambda X_r^m + \gamma (X_r^m - 1)^\alpha \right] \, dr \right), \]

\[ U_t^m = m + P_1(\mu t) - P_2 \left( \int_0^t \left[ \lambda + \gamma (U_r^m)^{-1} (U_r^m - 1)^\alpha \right] \, dr \right). \]

On the interval \([0, S^m]\), \(U_t^m \geq 1\), and consequently we have

\[ m - P_2 \left( \int_0^t \left[ \lambda U_r^m + \gamma (U_r^m - 1)^{\alpha - 1} \right] \, dr \right) \leq U_t^m \]

\[ \leq m + P_1 \left( \int_0^t \mu U_r^m \, dr \right) - P_2 \left( \int_0^t \left[ \frac{\gamma}{2} (U_r^m - 1)^{\alpha - 1} \right] \, dr \right). \]
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On the interval \([0, S^m]\), \( U_t^m \geq 1 \), and consequently we have

\[ m - P_2 \left( \int_0^t \left[ \lambda U_r^m + \gamma (U_r^m - 1)^{\alpha - 1} \right] \, dr \right) \leq U_t^m \leq m + P_1 \left( \int_0^t \mu U_r^m \, dr \right) - P_2 \left( \int_0^t \left[ \frac{\gamma}{2} (U_r^m - 1)^{\alpha - 1} \right] \, dr \right). \]
Length of the discrete tree

**Theorem**

*If* \( \alpha \leq 2 \), *then*

\[
\sup_{m \geq 0} L^m = \infty \quad \text{a. s.}
\]

*If* \( \alpha > 2 \), *then*

\[
\mathbb{E} \left[ \sup_{m \geq 0} L^m \right] < \infty.
\]
Consider again \( \{Z^x_t, \ t \geq 0\} \) solution of (1). We have

**Theorem**

- *If* \( 0 < \alpha < 1 \), \( 0 < \mathbb{P}(T^x = \infty) < 1 \) *if* \( \theta > 0 \), *while* \( T^x < \infty \) *a. s. if* \( \theta = 0 \).
- *If* \( \alpha = 1 \), \( T^x < \infty \) *a. s. if* \( \gamma \geq \theta \), *while* \( 0 < \mathbb{P}(T^x = \infty) < 1 \) *if* \( \gamma < \theta \).
- *If* \( \alpha > 1 \), \( T^x < \infty \) *a. s.*
Height of the continuous tree

**Theorem**

- *If* \( \alpha \leq 1 \), *then* \( T^x \to \infty \) *a. s., as* \( x \to \infty \).
- *If* \( \alpha > 1 \), *then* \( \mathbb{E} [ \sup_{x>0} T^x ] < \infty \).
Proof of Theorem 5 for $\alpha > 1$

We first need to establish some preliminary results on SDEs with infinite initial condition. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz and such that

$$\lim_{x \to \infty} \frac{|f(x)|}{x^\alpha} = 0. \tag{2}$$

Theorem

Let $\alpha > 1$, $\gamma > 0$ and $f$ satisfy the assumption (2). Then there exists a minimal $X \in C((0, +\infty); \mathbb{R})$ which solves

$$\begin{cases}
    dX_t = [f(X_t) - \gamma(X_t)^\alpha]1_{\{X_t \geq 0\}} dt + dW_t; \\
    X_t \to \infty, \text{ as } t \to 0.
\end{cases} \tag{3}$$

Moreover, if $T_0 := \inf\{t > 0, X_t = 0\}$, then $\mathbb{E}[T_0] < \infty$. 

Proof of the Theorem for $\alpha > 1$

The process $Y_t^x := \sqrt{Z_t^x}$ solves the SDE

$$dY_t^x = \left[\frac{\theta}{2} Y_t^x - \frac{\gamma}{2} (Y_t^x)^{2\alpha-1} - \frac{1}{8 Y_t^x} \right] dt + dW_t, \quad Y_0^x = \sqrt{x}.$$

By a well–known comparison theorem, $Y_t^x \leq U_t^x$, where $U_t^x$ solves

$$dU_t^x = \left[\frac{\theta}{2} U_t^x - \frac{\gamma}{2} (U_t^x)^{2\alpha-1} \right] dt + dW_t, \quad U_0^x = \sqrt{x}.$$

The result follows from the previous Theorem.
The result is equivalent to the fact that the time to reach 1, starting from $x$, goes to $\infty$ as $x \to \infty$. But when $Z_t^x \geq 1$, a comparison of SDEs for various values of $\alpha$ shows that it suffices to consider the case $\alpha = 1$. But in that case, $T^n$ is the maximum of the extinction times of $n$ mutually independent copies of $Z^1_t$, hence the result.
Length of the continuous tree

**Theorem**

If $\alpha \leq 2$, then $S^x \to \infty$ a. s. as $x \to \infty$.

If $\alpha > 2$, then $\mathbb{E} \left[ \sup_{x>0} S^x \right] < \infty$. 

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Proof of the Theorem for $\alpha > 2$

- Time change of $Z^x$:

$$A_t = \int_0^t Z_s^x \, ds, \quad \eta(t) = \inf\{s > 0, A_s > t\}. \quad t \geq 0, \text{ and } U_t^x = Z^x \circ \eta(t)$$

- The process $U^x$ solves the SDE

$$dU_t^x = \left[\theta - \gamma(U_t^x)^{\alpha-1}\right] dt + 2dW_t, \quad U_0^x = x. \quad (4)$$

- Let $\tau^x := \inf\{t > 0, U_t^x = 0\}$. It follows from the above that $\eta(\tau^x) = T^x$, hence $S^x = \tau^x$.

The result follows for $\alpha > 2$. 
Proof of the Theorem for $\alpha \leq 2$

It suffice to consider the case $\alpha = 2$. In that case, we have

$$U_t^x = e^{-\gamma t} x + \int_0^t e^{-\gamma(t-s)}[\theta ds + 2dW_s],$$

hence

$$S^x = \inf \left\{ t > 0, \int_0^t e^{\gamma s}(\theta ds + 2dW_s) \leq -x \right\},$$

which clearly goes to infinity, as $x \to \infty$. 
Merci pour votre attention