# Branching random walk with a random environment in time

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**Abstract**

We consider a branching random walk on $\mathbb{R}$ with a stationary and ergodic environment $\xi = (\xi_n)$ indexed by time $n \in \mathbb{N}$. Let $Z_n$ be the counting measure of particles of generation $n$. We consider the case where the corresponding branching process $\{Z_n(\mathbb{R})\}$ $(n \in \mathbb{N})$ is supercritical. We establish large deviation principles, central limit theorems and a local limit theorem for the sequence of counting measures $\{Z_n\}$, and prove that the position $R_n$ (resp. $L_n$) of rightmost (resp. leftmost) particles of generation $n$ satisfies a law of large numbers.

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## 1 Introduction

A *random environment in time* is modeled as a stationary and ergodic sequence of random variables, $\xi_n$, indexed by the time $n \in \mathbb{N}$, taking values in some measurable space $\Theta$. Each realization of $\xi_n$ corresponds to a distribution $\eta_n = \eta(\xi_n)$ on $\mathbb{N} \times \mathbb{R} \times \mathbb{R} \times \cdots$.

When the environment $\xi = (\xi_n)$ is given, the process can be described as follows. At time 0, there is an initial particle $\emptyset$ of generation 0 located at $S_\emptyset = 0 \in \mathbb{R}$; at time 1, it is replaced by $N = N(\emptyset)$ particles of generation 1, located at $L_i = L_i(\emptyset)$, $1 \leq i \leq N$, where the random vector $X_\emptyset = (N, L_1, L_2, \cdots) \in \mathbb{N} \times \mathbb{R} \times \mathbb{R} \times \cdots$ is of distribution $\eta_0 = \eta(\xi_0)$ (given the environment $\xi$). In general, each particle $u = u_1 \cdots u_n$ of generation $n$ located at $S_u$ is replaced at time n+1 by $N(u)$ new particles $ui$ of generation $n+1$, located at

$$S_{ui} = S_u + L_i(u) \qquad (1 \leq i \leq N(u)),$$

where the random vector $X_u = (N(u), L_1(u), L_2(u), \cdots)$ is of distribution $\eta_n = \eta(\xi_n)$. Note that the values $L_i(u)$ for $i > N_u$ do not play any role for our model; we introduce them only for convenience. We can for example take $L_i(u) = 0$ for $i > N_u$. All particles behave independently conditioned on the environment $\xi$.

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment $\xi$ is fixed. As usual, $\mathbb{P}_\xi$ is called *quenched law*. The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\xi \otimes \tau$ in the sense that for all measurable and positive $g$, we have

$$\int g d\mathbb{P} = \int_{\Theta^{\mathbb{N}}} \left( \int_\Gamma g(\xi, y) d\mathbb{P}_\xi(y) \right) d\tau(\xi),$$

where $\tau$ is the law of the environment $\xi$. The total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given $\xi$.

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Let $\mathbb{U} = \{\emptyset\} \bigcup_{n\geq 1} \mathbb{N}^n$ be the set of all finite sequence $u = u_1 \cdots u_n$. By definition, under $\mathbb{P}_\xi$, the random vectors $\{X_u\}$, indexed by $u \in \mathbb{U}$, are independent of each other, and each $X_u$ has distribution $\eta_n = \eta(\xi_n)$ if $|u| = n$, where $|u|$ denotes the length of $u$.

Let $\mathbb{T}$ be the Galton-Watson tree with defining element $\{N_u\}$. We have: (a) $\emptyset \in \mathbb{T}$; (b) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$; (c) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ be the set of particles of generation $n$ and

$$Z_n = \sum_{u \in \mathbb{T}_n} \delta_{S_u}$$

be the counting measure of particles of generation $n$, so that for a subset $A$ of $\mathbb{R}$, $Z_n(A)$ is the number of particles of generation $n$ located in $A$.

For any finite sequence $u$, let

$$X(u) = \sum_{i=1}^{N_u} \delta_{L_i(u)}$$

be the counting measure corresponding to the random vector $X_u$, whose increasing points are $L_i(u)$, $1 \leq i \leq N_u$. Denote

$$X_n = X(u_0|n),$$

where $u_0 = (1, 1, \cdots)$ and $u_0|n$ is the restriction to its first $n$ terms, with the convention that $u_0|0 = \emptyset$. For simplicity, we introduce the following notations:

$$N_n = X_n(\mathbb{R}), \qquad m_n = \mathbb{E}_\xi N_n, \qquad P_0 = 1 \quad \text{and} \quad P_n = \mathbb{E}_\xi Z_n(\mathbb{R}) = \prod_{i=0}^{n-1} m_i. \tag{1.1}$$

Let

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma(\xi, (N(u), L_1(u), L_2(u), \cdots) : |u| < n) \text{ for } n \geq 1$$

be the $\sigma$-field containing all the information concerning the first $n$ generations. It is well known that the sequence $\{Z_n(\mathbb{R})/P_n\}$ is a non-negative martingale under $\mathbb{P}_\xi$ for every $\xi$ with respect to the filtration $\mathcal{F}_n$, hence it converges almost surely (a.s.) to a random variable denoted by $W$. Throughout this paper we always assume that

$$\mathbb{E} \log m_0 \in (0, \infty) \quad \text{and} \quad \mathbb{E} \frac{N}{m_0} \log^+ N < \infty. \tag{1.2}$$

The first condition means that the corresponding branching process in random environment, $\{Z_n(\mathbb{R})\}$, is *supercritical*; the second implies that $W$ is non-degenerate. Hence (see e.g. Athreya and Karlin (1971, [1]))

$$\mathbb{P}_\xi(W > 0) = \mathbb{P}_\xi(Z_n(\mathbb{R}) \to \infty) = \lim_{n\to\infty} \mathbb{P}_\xi(Z_n(\mathbb{R}) > 0) > 0 \quad a.s..$$

In this paper, we are interested in asymptotic properties of the sequence of measures $\{Z_n\}$.

Our first objective is to show a large deviation principle for $\{Z_n(n\cdot)\}$ (Theorem 3.2). Our approach uses the Gärtner-Ellis theorem. In the proof, we first demonstrate that the sequence of quenched means $\{\mathbb{E}_\xi Z_n(n\cdot)\}$ satisfies a large deviation principle, and then show that the free energy $\frac{\log \tilde{Z}_n(t)}{n}$, where $\tilde{Z}_n(t) = \sum_{u\in\mathbb{T}_n} e^{tS_n}$ denotes the partition function, converges a.s. to a limit that we calculate explicitly (Theorem 3.1). Moreover, we also show that the position $R_n$ (resp. $L_n$) of rightmost (resp. leftmost) particles of generation $n$ satisfies a law of large numbers (Theorem 3.4): $\frac{R_n}{n}$ (resp. $\frac{L_n}{n}$) converges a.s. to a limit that we determine explicitly. These results generalize those of Biggins (1977, [4]), Franchi (1995, [14]) and Chauvin & Rouault (1997, [9]) for the deterministic environment case.

Our second objective is to show central limit theorems and related results for $\{Z_n\}$. For a deterministic branching random walk, Kaplan and Asmussen (1976, [21]) proved the following central limit theorem. Assume that $m = \mathbb{E}N \in (1, \infty)$ and that $\frac{\mathbb{E}X_0(\cdot)}{m}$ has mean 0 and variance 1. If $\mathbb{E}N(\log N)^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$, then

$$m^{-n} Z_n(-\infty, \sqrt{n}x] \to \Phi(x)W \qquad a.s. \quad \forall x \in \mathbb{R}, \tag{1.3}$$

where $\Phi(x)$ is the distribution function of the standard normal distribution $\mathcal{N}(0,1)$. They also gave a local version of (1.3) under the stronger moment condition that $\mathbb{E}N(\log N)^{\gamma} < \infty$ for some $\gamma > 3/2$. The formule (1.3), which was first conjectured by Harris [16], has been studied by many authors, see e.g. Stam (1966, [32]), Kaplan & Asmussen (1976, [21]), Klebaner (1982, [23]) and Biggins (1990, [7]). We shall show the following version of (1.3) (Theorem 10.2) for a branching random walk in a random environment: under certain moment conditions, the sequence of probability measures $\frac{Z_n(b_n\cdot+a_n)}{Z_n(\mathbb{R})}$, with $(a_n, b_n)$ that we calculate explicitly, converges to the standard normal distribution $\mathcal{N}(0,1)$ in law a.s. on the survival event $\{Z_n \to \infty\}$. The technic in the proof is a mixture of Klebaner (1982) and Biggins (1990) by considering the characteristic function and choosing an appropriate truncation function. We shall also show a corresponding local limit theorem (Theorem 10.4) under stronger moment conditions, which generalizes the result of Biggins (1990, Theorem 7) on deterministic branching random walks. From Theorem 10.4 we obtain another form of local limit theorem (Corollary 10.5), which coincides with the result of Kaplan & Asmussen (1976, Theorem 2) for the deterministic environment case.

Moreover, we shall also show large deviation principles and central limit theorems for probability mesures with different normings: $\frac{\mathbb{E}_\xi Z_n(n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$, $\frac{\mathbb{E}Z_n(\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$, $\mathbb{E}\frac{Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ and $\mathbb{E}_\xi \frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$.

The rest of paper is organized as follows. In Sections 2 - 5, we consider large deviations. In Section 2, we show large deviation principles for $\mathbb{E}_\xi Z_n(n\cdot)$, $\mathbb{E}Z_n(n\cdot)$ and $\mathbb{E}\frac{Z_n(n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$. In Section 3, we state a convergence result for the free energy, a large deviation principle for $Z_n(n\cdot)$ and laws of large numbers for $R_n$ and $L_n$. In Section 4, we prove the results of Section 3. In Section 5, we show a large deviation principle for $\mathbb{E}_\xi \frac{Z_n(n\cdot)}{Z_n(\mathbb{R})}$. In Sections 6 - 12, we study central limit theorems. In Section 6, we consider a branching random walk in a varying environment and present the corresponding limit theorems. In Sections 7 and 8, we prove the results of Section 6. From Sections 9 to 12, we return to a branching random walk in a random environment: in Section 9, we show central limit theorems for $\frac{\mathbb{E}_\xi Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$, $\frac{\mathbb{E}Z_n(\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$ and $\mathbb{E}\frac{Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$; in Section 10, we present a central limit theorem and a local limit theorem for $\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$, which are proved in Section 11; in Section 12, we show central limit theorems for $\mathbb{E}_\xi \frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$ and $\mathbb{E}\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$.

# 2 Large deviations for $\mathbb{E}_\xi Z_n(n\cdot)$, $\mathbb{E}Z_n(n\cdot)$ and $\mathbb{E}\frac{Z_n(n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$

To study large deviations of $Z_n$, we begin with the study of its quenched and annealed means. For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, let

$$m_n(t) := \mathbb{E}_\xi \int e^{itx} X_n(dx) = \mathbb{E}_\xi \sum_{i=1}^{N(u)} e^{tL_i(u)} \quad (u \in \mathbb{T}_n), \tag{2.1}$$

be the Laplace transform of the counting measure describing the evolution of the system at time $n$. In particular,

$$m_0(t) = \mathbb{E}_\xi \sum_{i=1}^{N} e^{tL_i}, \quad m_0(0) = \mathbb{E}_\xi N = m_0.$$

We assume that

$$E|L_1| < \infty, \ \mathbb{E}|\log m_0(t)| < \infty \text{ and } \mathbb{E}|\frac{m_0'(t)}{m_0(t)}| < \infty \tag{2.2}$$

for all $t \in \mathbb{R}$. The last two moment conditions imply that

$$\Lambda(t) := \mathbb{E}\log m_0(t) \text{ and } \Lambda'(t) := \mathbb{E}\frac{m_0'(t)}{m_0(t)}$$

are well defined as real numbers, that $\Lambda(t)$ is differentiable everywhere on $\mathbb{R}$ with $\Lambda'(t)$ as its derivative (this can be easily verified by the dominated convergence theorem, using the fact that the function $t \mapsto \frac{m_0'(t)}{m_0(t)}$ is increasing). Let

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{xt - \Lambda(t)\}$$

be the Legendre transform of $\Lambda$. Then

$$\Lambda^*(x) = \begin{cases} t\Lambda'(t) - \Lambda(t) & if \quad x = \Lambda'(t) \text{ for some } t \in \mathbb{R}, \\ +\infty & if \quad x \geq \Lambda'(+\infty) \text{ or } x \leq \Lambda'(-\infty), \end{cases}$$

and

$$\min_x \Lambda^*(x) = \Lambda^*(\Lambda'(0)) = -\Lambda(0) = -\mathbb{E}\log m_0 < 0.$$

With these notations, now we can state our large deviation principle for the quenched means $\mathbb{E}_\xi Z_n(n\cdot)$, which will leads to a large deviation principle about $Z_n(n\cdot)$.

**Theorem 2.1** (Large deviation principle for quenched means $\mathbb{E}_\xi Z_n(n\cdot)$)**.** *Assume (2.2). For almost every $\xi$, the sequence of finite measures $A \mapsto \mathbb{E}_\xi Z_n(nA)$ satisfies a principle of large deviation with rate function $\Lambda^*$: for each measurable subset $A$ of $\mathbb{R}$,*

$$\begin{aligned} -\inf_{x\in A^o}\Lambda^*(x) &\leq \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}_\xi Z_n(nA) \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}_\xi Z_n(nA) \leq -\inf_{x\in\bar{A}}\Lambda^*(x), \end{aligned}$$

*where $A^o$ denotes the interior of $A$, and $\bar{A}$ its closure.*

*Proof.* Notice that the measures $q_n(\cdot) = \mathbb{E}_\xi Z_n(\cdot)$ satisfy

$$\tilde{q}_n(t) := \int e^{tx} q_n(dx) = \mathbb{E}_\xi \sum_{u\in\mathbb{T}_n} e^{tS_u} = m_0(t)...m_{n-1}(t).$$

By the ergodic theorem,

$$\lim_{n\to\infty}\frac{1}{n}\log\tilde{q}_n(t) = \Lambda(t) := \mathbb{E}\log m_0(t) \quad \text{a.s..}$$

Therefore, applying the Gärtner-Ellis theorem ([11], p.53, Exercise 2.3.20) to the sequence of normalized probability measures $q_n(n\cdot)/q_n(\mathbb{R})$, we obtain the desired result. □

If the environment is *i.i.d.*, similar results can be established for annealed means. Let

$$\Lambda_a(t) = \log\mathbb{E}m_0(t),$$

and $\Lambda_a^*$ be its Legendre transform. Then we have:

**Theorem 2.2** (Large deviation principle for annealed means $\mathbb{E}Z_n(n\cdot)$)**.** *Assume that $\xi_n$ are i.i.d.. If $\mathbb{E}m_0(t) \in (0,\infty)$ for all $t \in \mathbb{R}$, then the sequence of finite measures $A \mapsto \mathbb{E}Z_n(nA)$ satisfies a principle of large deviation with rate function $\Lambda_a^*$: for each measurable subset $A$ of $\mathbb{R}$,*

$$\begin{aligned} -\inf_{x\in A^o}\Lambda_a^*(x) &\leq \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}Z_n(nA) \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}Z_n(nA) \leq -\inf_{x\in\bar{A}}\Lambda_a^*(x), \end{aligned}$$

*where $A^o$ denotes the interior of $A$, and $\bar{A}$ its closure.*

**Remark.** It is easy to see that

$$\Lambda_a(t) \geq \Lambda(t) \quad \text{and} \quad \Lambda_a^*(x) \leq \Lambda^*(x).$$

*Proof of Theorem 2.2.* The proof is similar to that of Theorem 2.1, with $q_n(\cdot) = \mathbb{E}Z_n(\cdot)$. Notice that when $\xi_n$ are i.i.d.,

$$\tilde{q}_n(t) := \int e^{tx} q_n(dx) = \mathbb{E} \sum_{u \in \mathbb{T}_n} e^{tS_u} = (\mathbb{E}m_0(t))^n .$$

□

If we consider the measures $\mathbb{E}\frac{Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ instead of $\frac{\mathbb{E}Z_n(\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$, we can obtain another large deviation principle.

**Theorem 2.3** (Large deviation principle for $\mathbb{E}\frac{Z_n(n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$). *Assume that $\xi_n$ are i.i.d.. Let $\bar{\Lambda}_a(t) = \log \mathbb{E}\frac{m_0(t)}{m_0}$ and $\bar{\Lambda}_a^*$ be its Legendre transform. If $\mathbb{E}\frac{m_0(t)}{m_0} \in (0, \infty)$ for all $t \in \mathbb{R}$, then the sequence of finite measures $A \mapsto \mathbb{E}\frac{Z_n(nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ satisfies a principle of large deviation with rate function $\bar{\Lambda}_a^*$: for each measurable subset $A$ of $\mathbb{R}$,*

$$\begin{aligned} -\inf_{x \in A^o} \bar{\Lambda}_a^*(x) &\leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{E}\frac{Z_n(nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}\frac{Z_n(nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \leq -\inf_{x \in \bar{A}} \bar{\Lambda}_a^*(x), \end{aligned}$$

*where $A^o$ denotes the interior of $A$, and $\bar{A}$ its closure.*

*Proof.* The proof is still similar to that of Theorem 2.1, with $q_n(\cdot) = \mathbb{E}\frac{Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ whose Laplace transform is

$$\tilde{q}_n(t) := \int e^{tx} q_n(dx) = \left(\mathbb{E}\frac{m_0(t)}{m_0}\right)^n .$$

□

## 3 Convergence of the free energy; large deviations for $Z_n(n\cdot)$; positions of rightmost and leftmost particles

Now we consider large deviations for the sequence of measures $\{Z_n(n\cdot)\}$. Let

$$\tilde{Z}_n(t) := \int e^{tx} Z_n(dx) = \sum_{u \in \mathbb{T}_n} e^{tS_u} \tag{3.1}$$

be the Laplace transform of $Z_n$, also called *partition function* by physicians. We are interested in the convergence of the *free energy* $\frac{\log \tilde{Z}_n(t)}{n}$. To this end we define two critical values $t_-$ and $t_+$. Let

$$\rho(t) = t\Lambda'(t) - \Lambda(t), \qquad t \in \mathbb{R}.$$

Notice that $\rho'(t) = t\Lambda''(t)$. Therefore $\rho(t)$ decreases on $(-\infty, 0]$, increases on $[0, \infty)$, and attains its minimum at 0:

$$\min_t \rho(t) = \rho(0) = -\Lambda(0) < 0.$$

Let

$$t_- = \inf\{t \in \mathbb{R} : t\Lambda'(t) - \Lambda(t) \leq 0\},$$

$$t_+ = \sup\{t \in \mathbb{R} : t\Lambda'(t) - \Lambda(t) \le 0\}.$$

Then $-\infty \le t_- < 0 < t_+ \le \infty$, $t_-$ and $t_+$ are two solutions of $t\Lambda'(t) - \Lambda(t) = 0$ if they are finite. For simplicity, we also assume that

$$N \ge 1 \qquad a.s., \tag{3.2}$$

so that $Z_n(\mathbb{R}) \to \infty$ a.s..

**Theorem 3.1** (Convergence of the free energy)**.** *It is a.s. that for all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} = \tilde{\Lambda}(t) := \begin{cases} \Lambda(t) & if \quad t \in (t_-, t_+), \\ t\Lambda'(t_+) & if \quad t \ge t_+, \\ t\Lambda'(t_-) & if \quad t \le t_-. \end{cases} \tag{3.3}$$

For the deterministic environment case, see Chauvin & Rouault (1997, [9]) and Franchi (1995, [14]).

Let $\tilde{\Lambda}^*(x)$ be the Legendre transform of $\tilde{\Lambda}(t)$. By Theorem 3.1 and the Gärtner- Ellis' theorem, we immediately obtain the following large deviation principe for $Z_n(n\cdot)$.

**Theorem 3.2** (Large deviation principle for $Z_n(n\cdot)$)**.** *It is a.s. that the sequence of finite measures $A \mapsto Z_n(nA)$ satisfies a principle of large deviation with rate function $\tilde{\Lambda}^*$: for each measurable subset $A$ of $\mathbb{R}$,*

$$\begin{aligned} -\inf_{x\in A^o} \tilde{\Lambda}^*(x) &\le \liminf_{n\to\infty} \frac{1}{n} \log Z_n(nA) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log Z_n(nA) \le -\inf_{x\in \bar{A}} \tilde{\Lambda}^*(x), \end{aligned}$$

*where $A^o$ denotes the interior of $A$, and $\bar{A}$ its closure.*

**Remark.** It can be seen that $\tilde{\Lambda}(t) \le \Lambda(t)$, so that $\tilde{\Lambda}^*(x) \ge \Lambda^*(x)$. Moreover,

$$\tilde{\Lambda}^*(x) = \begin{cases} \Lambda^*(x) & if \quad x \in [\Lambda'(t_-), \Lambda'(t_+)], \\ +\infty & if \quad x < \Lambda'(t_-) \text{ or } x > \Lambda'(t_+), \end{cases}$$

**Corollary 3.3.** *It is a.s. that*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n[nx, \infty) = -\Lambda^*(x) > 0 \ \textit{if}\ x \in (\Lambda'(0), \Lambda'(t_+)),$$

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(-\infty, nx] = -\Lambda^*(x) > 0 \ \textit{if}\ x \in (\Lambda'(t_-), \Lambda'(0)).$$

For deterministic branching random walks, see Biggins (1977, [4]) and Chauvin & Rouault (1997, [9]).

**Remark.**

$$x \in (\Lambda'(0), \Lambda'(t_+)) \text{ if and only if } x > \Lambda'(0) \text{ and } \Lambda^*(x) < 0.$$

$$x \in (\Lambda'(t_-), \Lambda'(0)) \text{ if and only if } x < \Lambda'(0) \text{ and } \Lambda^*(x) < 0.$$

If the set $\mathbb{T}_n \ne \emptyset$, let

$$L_n = \min_{u\in\mathbb{T}_n} S_u \quad (resp. \quad R_n = \max_{u\in\mathbb{T}_n} S_u)$$

be the position of leftmost (resp. rightmost) particles of generation $n$. We can see that $L_n$ (resp. $R_n$) satisfies a law of large numbers.

**Theorem 3.4** (Asymptotic properties of $L_n$ and $R_n$)**.** *It is a.s. that*

$$\lim_{n\to\infty} \frac{L_n}{n} = \Lambda'(t_-),$$

$$\lim_{n\to\infty} \frac{R_n}{n} = \Lambda'(t_+).$$

For deterministic branching random walks, see Biggins (1977) and Chauvin & Rouault (1997).

# 4 Proofs of Theorems 3.1 and 3.4

Let us give the proofs of Theorems 3.1 and 3.4 which are composed by some lemmas. Similar arguments have been used in Franchi (1995, [14]) and Chauvin & Rouault (1997).

Observe that

$$W_n(t) := \frac{\tilde{Z}_n(t)}{\mathbb{E}_\xi \tilde{Z}_n(t)} = \frac{\sum_{u\in\mathbb{T}_n} e^{tS_u}}{m_0(t)...m_{n-1}(t)}$$

is a martingale, therefore it converges a.s. to a random variable $W(t) \in [0,\infty)$. In the deterministic environment case, this martingale has been studied by Kahane & Peyrière (1976), Biggins (1977), Durrett & Liggett (1983), Guivarc'h (1990), Lyons (1997) and Liu (1997, 1998, 2000, 2001), etc. in different contexts.

The following lemma concerns the non degeneration of $W(t)$.

**Lemma 4.1.** *If $t \in (t_-, t_+)$ and $\mathbb{E}W_1(t)\log^+ W_1(t) < \infty$, then*

$$W(t) > 0 \quad a.s.$$

*If $t \le t_-$ or $t \ge t_+$, then*

$$W(t) = 0 \quad a.s.$$

Notice that $t \in (t_-, t_+)$ is equivalent to $t\Lambda'(t) - \Lambda(t) < 0$. Therefore the lemma is an immediate consequence of Theorem 7.2 of Biggins and Kyprianous (2004) on a branching process in a random environment, or of a result of Kuhlbusch (2004, [22]) on weighted branching processes in random environment.

**Lemma 4.2.** *If $t \in (t_-, t_+)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{Z}_n(t) = \Lambda(t) \ \ a.s.. \tag{4.1}$$

*Proof.* If $\mathbb{E}W_1(t)\log^+ W_1(t) < \infty$, by Lemma 4.1, $W(t) > 0$ a.s.. Consequently,

$$\frac{1}{n}\log \tilde{Z}_n(t) = \frac{1}{n}\log W_n(t) + \frac{1}{n}\sum_{i=0}^{n-1} \log m_i(t) \to \mathbb{E}\log m_0(t) = \Lambda(t) \ \ a.s..$$

We now consider the general case where $\mathbb{E}W_1(t)\log^+ W_1(t)$ may be infinite. We only consider the case where $t \in [0, t_+)$ (the case where $t \in (t_-, 0]$) can be considered in a similar way, or by considering $(-L_u)$ instead of $(L_u)$).

For the lower bound, we use an truncating argument. For $c \in \mathbb{N}$, we construct a new branching random walk in a random environment (BRWRE) using $X^c(u) = (N(u)\wedge c, L_1(u), L_2(u), \cdots)$ instead of $X(u) = (N(u), L_1(u), L_2(u), \cdots)$, where and throughout we write $a \wedge b = \min(a,b)$. We shall apply Lemma 4.1 to the new BRWRE. We define $m_n^c(t)$, $W_n^c(t)$, $\Lambda_c(t)$ and $t_+^c$ for the new BRWRE just as just as $m_n(t)$ $W_n(t)$, $\Lambda(t)$ and $t^c$ were defined for the original BRWRE.

We first show that $\Lambda_c(t) := \mathbb{E}\log m_0(t) \uparrow \Lambda(t)$ as $c \uparrow \infty$. Clearly, $m_0^c(t) = \mathbb{E}_\xi \sum_{i=1}^{N\wedge c} e^{tL_i} \uparrow m_0(t)$ as $c \uparrow \infty$. This leads to $\mathbb{E}\log^+ m_0^c(t) \uparrow \mathbb{E}\log^+ m_0(t)$ by the monotone convergence theorem. On the other hand, for $c \ge 1$, we have

$$\log^- m_0^c(t) \le \log^- m_0^1(t) = \log^- \mathbb{E}_\xi e^{tL_1} \le t\mathbb{E}_\xi |L_1|$$

(as $\mathbb{E}_\xi e^{tL_1} \ge e^{-t\mathbb{E}_\xi |L_1|}$ by Jensen's inequality). Therefore by the condition $E|L_1| < \infty$ and the dominated convergence theorem, $\mathbb{E}\log^- m_0^c(t) \downarrow \mathbb{E}\log^- m_0(t)$.

We next prove that for $c > 0$ large enough, $t \in [0, t_+^c)$, which is equivalent to $t\Lambda_c'(t) - \Lambda_c(t) < 0$. Recall that $t \in [0,, t_+)$ is equivalent to $t\Lambda'(t) - \Lambda(t) < 0$. By the definition of $\Lambda'(t)$, there exists a $h > 0$ such that

$$t\frac{\Lambda(t+h) - \Lambda(t)}{h} - \Lambda(t) < 0.$$

Since $\Lambda_c \uparrow \Lambda$ as $c \uparrow \infty$, we have for $c$ large enough,

$$t\frac{\Lambda_c(t+h) - \Lambda_c(t)}{h} - \Lambda_c(t) < 0. \tag{4.2}$$

The convexity of $\Lambda_c(t)$ shows that

$$\Lambda'_c(t) \leq \frac{\Lambda_c(t+h) - \Lambda_c(t)}{h}. \tag{4.3}$$

Combing (4.4) with (4.2) we obtain for $c$ large enough,

$$t\Lambda'_c(t) - \Lambda_c(t) < 0. \tag{4.4}$$

We finally prove that $\mathbb{E}W_1^c(t) \log^+ W_1^c(t) < \infty$. Let $Y = W_1^c(t)$. we define a random variable $X$ whose distribution is determined by

$$\mathbb{E}_\xi g(X) = \mathbb{E}_\xi Y g(Y)$$

for all bounded and measurable function $g$ (notice that $\mathbb{E}_\xi Y = 1$ by definition). For $x \in \mathbb{R}$, let

$$l(x) = \begin{cases} x/e & \text{if } x < e, \\ \log x & \text{if } x \geq e. \end{cases}$$

It is clear that $l$ is concave and $\log^+ x \leq l(x) \leq 1 + \log^+ x$ for all $x \in \mathbb{R}$. Thus

$$\begin{aligned} \mathbb{E}_\xi Y \log^+ Y = \mathbb{E}_\xi \log^+ X &\leq \mathbb{E}_\xi l(x) \\ &\leq l(\mathbb{E}_\xi X) = l(\mathbb{E}_\xi Y^2) \\ &\leq 1 + \log^+ \mathbb{E}_\xi Y^2 \\ &\leq 1 + \log^+ \left( \frac{c m_0^c(2t)}{m_0^c(t)^2} \right), \end{aligned}$$

where the last inequality holds as $(\sum_{i=1}^{N \wedge c} e^{tL_i})^2 \leq (N \wedge c) \sum_{i=1}^{N \wedge c} e^{2tL_i}$. Taking expectation in the above inequality , we get

$$\begin{aligned} \mathbb{E}W_1^c(t) \log^+ W_1^c(t) = \mathbb{E}Y \log^+ Y &\leq 1 + \mathbb{E} \log^+ \left( \frac{c m_0^c(2t)}{m_0^c(t)^2} \right) \\ &\leq 1 + \log c + \mathbb{E} \log^+ m_0(2t) + 2\mathbb{E} \log^- m_0^c(t) < \infty. \end{aligned}$$

We have therefore proved that for $c > 0$ large enough, the new BRWRE satisfies the conditions of Lemma 4.1, so that

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{Z}_n^c(t) = \mathbb{E} \log m_0^c(t) = \Lambda_c(t) \ \ a.s..$$

Notice that $\tilde{Z}_n(t) \geq \tilde{Z}_n^c(t)$. It follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log \tilde{Z}_n(t) \geq \Lambda_c(t) \ \ a.s..$$

Letting $c \uparrow \infty$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \tilde{Z}_n(t) \geq \Lambda(t) \ \ a.s..$$

For th upper bound, from the decomposition $\frac{1}{n} \log \tilde{Z}_n(t) = \frac{1}{n} \log W_n(t) + \frac{1}{n} \sum_{i=0}^{n-1} \log m_i(t)$ and the fact that $W_n(t) \to W(t) < \infty$ a.s., we obtain that

$$\limsup_{n\to\infty} \frac{1}{n} \log \tilde{Z}_n(t) \leq \Lambda(t) \ \ a.s..$$

This completes the proof. □

**Lemma 4.3.** *It is a.s. that*

$$\limsup_{n\to\infty} \frac{R_n}{n} \le \Lambda'(t_+).$$

*Proof.* For $a > \Lambda'(t_+)$, we have $\Lambda^*(a) > 0$. By Theorem 2.1,

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_\xi Z_n[an, \infty) = -\Lambda^*(a) < 0 \ \ a.s..$$

This leads to $\sum_n \mathbb{P}_\xi(Z_n[an, \infty) \ge 1) < \infty$ a.s.. It follows that by Borel-Cantelli's lemma, $\mathbb{P}_\xi$ a.s. ,

$$Z_n[an, \infty) = 0 \quad \text{for } n \text{ large enough.}$$

Therefore $R_n < an$, so that a.s.,

$$\limsup_{n\to\infty} \frac{R_n}{n} \le a.$$

Letting $a \downarrow \Lambda'(t_+)$, we obtain the desired result. □

**Lemma 4.4.** *If $t \ge t_+$, then a.s.,*

$$\lim_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} = t\Lambda'(t_+). \tag{4.5}$$

*Proof.* For the upper bound, we only consider the case where $t_+ < \infty$. Choose $0 < t_0 < t_+ \le t$. Since $S_u \le R_n$ for $u \in \mathbb{T}_n$, we have

$$tS_u \le t_0 S_u + (t - t_0) R_n,$$

so that

$$\tilde{Z}_n(t) \le \tilde{Z}_n(t_0) e^{(t-t_0)R_n}.$$

Thus

$$\frac{\log \tilde{Z}_n(t)}{n} \le \frac{\log \tilde{Z}_n(t_0)}{n} + (t - t_0)\frac{R_n}{n}.$$

Letting $n \to \infty$ and using Lemma 4.3, we get a.s.,

$$\limsup_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \le \Lambda(t_0) + (t - t_0)\Lambda'(t_+).$$

Letting $t_0 \uparrow t_+$ and using $\Lambda(t_+) - t_+\Lambda'(t_+) = 0$, we obtain a.s.,

$$\limsup_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \le t\Lambda'(t_+).$$

For the lower bound, as $\log \tilde{Z}_n(t)$ is a convex function of $t$, for $t_- < t_0 < t_1 < t_+ \le t$, we have

$$\frac{\log \tilde{Z}_n(t) - \log \tilde{Z}_n(t_0)}{t - t_0} \ge \frac{\log \tilde{Z}_n(t_1) - \log \tilde{Z}_n(t_0)}{t_1 - t_0}.$$

Dividing the inequality by $n$ and applying Lemma 4.2 to $t_0$ and $t_1$, we obtain a.s.,

$$\liminf_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \ge \Lambda(t_0) + \frac{t - t_0}{t_1 - t_0}(\Lambda(t_1) - \Lambda(t_0)).$$

Letting $t_1 \downarrow t_0$, we get a.s.,

$$\liminf_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \ge \Lambda(t_0) + (t - t_0)\Lambda'(t_0).$$

Letting $t_0 \uparrow t_+$ and using $\Lambda(t_+) - t_+\Lambda'(t_+) = 0$, we obtain a.s.,

$$\liminf_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \ge t\Lambda'(t_+).$$

This completes the proof. □

**Lemma 4.5.** *It is a.s. that*

$$\liminf_{n\to\infty} \frac{R_n}{n} \geq \Lambda'(t_+).$$

*Proof.* Notice that $S_u \leq R_n$ for $u \in \mathbb{T}_n$, we have

$$\tilde{Z}_n(t) \leq Z_n(\mathbb{R}) e^{tR_n},$$

so that for each $0 < t < \infty$,

$$\frac{\log \tilde{Z}_n(t)}{n} \leq \frac{\log Z_n(\mathbb{R})}{n} + t\frac{R_n}{n}. \tag{4.6}$$

If $t_+ < \infty$, then by Lemma 4.4, the above inequality gives for $t > t_+$, a.s.,

$$\Lambda'(t_+) \leq \frac{1}{t}\mathbb{E}\log m_0 + \liminf_{n\to\infty} \frac{R_n}{n}.$$

Letting $t \uparrow \infty$, we obtain the desired result. If $t_+ = \infty$, then by Lemma 4.2, the inequality (4.6) gives for $t > 0$, a.s.,

$$\frac{\Lambda(t)}{t} \leq \frac{1}{t}\mathbb{E}\log m_0 + \liminf_{n\to\infty} \frac{R_n}{n}.$$

Letting $t \uparrow \infty$, we get a.s.,

$$\liminf_{n\to\infty} \frac{R_n}{n} \geq \Lambda'(\infty) = \Lambda'(t_+).$$

□

The conclusions for $t \leq t_-$ and $L_n$ can be obtained in a similar way, or by applying the obtained results for $t \geq t_+$ and $R_n$ to the opposite branching random walk $-S_u$. Hence Theorem 3.4 holds, and (3.3) holds a.s. for each fixed $t \in \mathbb{R}$. So a.s. (3.3) holds for all rational $t$, and therefore for all real t by the convexity of $\log \tilde{Z}_n(t)$. This ends the proof of Theorem 3.1.

# 5 Large deviations for $\mathbb{E}_\xi \frac{Z_n(n\cdot)}{Z_n(\mathbb{R})}$

Using the lower bound in Therorem 3.2 and the upper bound Theorem 2.1, we have the following theorem.

**Theorem 5.1.** *If a.s. $\mathbb{P}_\xi(N \leq 1) = 0$ and $\mathbb{E}_\xi N^{1+\delta} \leq K$ for some constants $\delta > 0$ and $K > 0$, then a.s., for each measurable subset $A$ of $\mathbb{R}$,*

$$\begin{aligned} -\inf_{x\in A^o} \tilde{\Lambda}^*(x) - \mathbb{E}\log m_0 &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi \frac{Z_n(nA)}{Z_n(\mathbb{R})} \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi \frac{Z_n(nA)}{Z_n(\mathbb{R})} \leq -\inf_{x\in \bar{A}} \Lambda^*(x) - \mathbb{E}\log m_0, \end{aligned}$$

*where $A^o$ denotes the interior of $A$, and $\bar{A}$ its closure.*

Notice that $\tilde{\Lambda}^*(x) = \Lambda^*(x)$ for $x \in (\Lambda'(t_-), \Lambda'(t_+))$. From Theorem 5.1 we obtain

**Corollary 5.2.** *If a.s. $\mathbb{P}_\xi(N \leq 1) = 0$ and $\mathbb{E}_\xi N^{1+\delta} \leq K$ for some constants $\delta > 0$ and $K > 0$, then a.s.,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi \frac{Z_n[nx,\infty)}{Z_n(\mathbb{R})} = -\Lambda^*(x) - \mathbb{E}\log m_0 \ \textit{if}\ x \in (\Lambda'(0), \Lambda'(t_+)),$$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi \frac{Z_n(-\infty, nx]}{Z_n(\mathbb{R})} = -\Lambda^*(x) - \mathbb{E}\log m_0 \ \textit{if}\ x \in (\Lambda'(t_-), \Lambda'(0)).$$

Theorem 5.1 is a combination of Lemmas 5.1 and 5.4 below.

**Lemma 5.1** (Lower bound)**.** *It is a.s. that for each measurable subset $A$ of $\mathbb{R}$,*

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}_\xi\left(\frac{Z_n(nA)}{Z_n(\mathbb{R})}\right)\geq-\inf_{x\in A^o}\tilde{\Lambda}^*(x)-E\log m_0. \tag{5.1}$$

*Proof.* By Therorem 3.2, a.s.

$$\liminf_{n\to\infty}\frac{1}{n}\log Z_n(nA)\geq-\inf_{x\in A^o}\tilde{\Lambda}^*(x),$$

which implies that for each $\varepsilon>0$, a.s.

$$\lim_{n\to\infty}\mathbb{P}_\xi\left(\frac{1}{n}\log\frac{Z_n(nA)}{Z_n(\mathbb{R})}\geq-\tilde{\Lambda}^*(x)-\mathbb{E}\log m_0-\varepsilon\right)=1.$$

Write $f(A)=-\inf_{x\in A^o}\tilde{\Lambda}^*(x)-\mathbb{E}\log m_0$. Notice that

$$\begin{aligned}\mathbb{E}_\xi\left(\frac{Z_n(nA)}{Z_n(\mathbb{R})}\right)&\geq\mathbb{E}_\xi\left(\frac{Z_n(nA)}{Z_n(\mathbb{R})}1_{\{\frac{Z_n(nA)}{Z_n(\mathbb{R})}\geq\exp(n(f(A)-\varepsilon))\}}\right)\\&\geq\exp(n(f(A)-\varepsilon))\mathbb{P}_\xi\left(\frac{1}{n}\log\frac{Z_n(nA)}{Z_n(\mathbb{R})}\geq f(A)-\varepsilon\right).\end{aligned}$$

We have a.s.

$$\frac{1}{n}\log\mathbb{E}_\xi\left(\frac{Z_n(nA)}{Z_n(\mathbb{R})}\right)\geq f(A)-\varepsilon+\frac{1}{n}\log\mathbb{P}_\xi\left(\frac{1}{n}\log\frac{Z_n(nA)}{Z_n(\mathbb{R})}\geq f(A)-\varepsilon\right).$$

Taking inferior limit and letting $\varepsilon\to0$, we obtain (5.1). □

To obtain the upper bound, we need certain moment conditions.

**Lemma 5.2** ([18], Theorem 3.1)**.** *If a.s. $\mathbb{P}_\xi(N\leq1)=0$ and $\mathbb{E}_\xi N^{1+\delta}\leq K$ for some constants $\delta>0$ and $K>0$, then for each $s>0$, there exists a constants $C_s>0$ such that $\mathbb{E}_\xi W^{-s}\leq C_s$ a.s..*

**Lemma 5.3.** *If a.s. $\mathbb{P}_\xi(N\leq1)=0$ and $\mathbb{E}_\xi N^{1+\delta}\leq K$ for some constants $\delta>0$ and $K>0$, then a.s.*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}_\xi(Z_n(\mathbb{R})\leq e^{(\mathbb{E}\log m_0-\varepsilon)n})=-\infty. \tag{5.2}$$

*Proof.* Denote $W_n=Z_n(\mathbb{R})/P_n$. Notice that $\forall s>0$, $\sup_n\mathbb{E}_\xi W_n^{-s}=\mathbb{E}_\xi W^{-s}$. Lemma 5.2 shows that $\mathbb{E}_\xi W^{-s}<\infty$ a.s.. By Markov's inequality, a.s.

$$\begin{aligned}&\mathbb{P}_\xi\left(Z_n(\mathbb{R})\leq e^{(\mathbb{E}\log m_0-\varepsilon)n}\right)\\\leq\ &\mathbb{E}_\xi W_n^{-s}\exp\left(s\left((\mathbb{E}\log m_0-\varepsilon)n-\sum_{i=0}^{n-1}\log m_i\right)\right)\\\leq\ &\mathbb{E}_\xi W^{-s}\exp\left(s\left((\mathbb{E}\log m_0-\varepsilon)n-\sum_{i=0}^{n-1}\log m_i\right)\right).\end{aligned}$$

Hence a.s.

$$\frac{1}{n}\log\mathbb{P}_\xi\left(Z_n(\mathbb{R})\leq e^{(\mathbb{E}\log m_0-\varepsilon)n}\right)\leq\frac{1}{n}\log\mathbb{E}_\xi W^{-s}+s\left(\mathbb{E}\log m_0-\varepsilon-\frac{1}{n}\sum_{i=0}^{n-1}\log m_i\right).$$

Taking superior limit, we get a.s.

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_\xi\left(Z_n(\mathbb{R})\leq e^{(E\log m_0-\varepsilon)n}\right)\leq-\varepsilon s.$$

Letting $s\to\infty$, we obtain (5.2). □

**Lemma 5.4** (Upper bound)**.** *If a.s. $\mathbb{P}_\xi(N \le 1) = 0$ and $\mathbb{E}_\xi N^{1+\delta} \le K$ for some constant $\delta > 0$ and $K > 0$, then it is a.s. that for each measurable subset $A$ of $\mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi \left( \frac{Z_n(nA)}{Z_n(\mathbb{R})} \right) \le - \inf_{x\in\bar{A}} \Lambda^*(x) - E\log m_0. \tag{5.3}$$

*Proof.* Notice that for each $\varepsilon > 0$, a.s.

$$\begin{aligned} \mathbb{E}_\xi \left( \frac{Z_n(nA)}{Z_n(\mathbb{R})} \right) &= \mathbb{E}_\xi \left( \frac{Z_n(nA)}{Z_n(\mathbb{R})} 1_{\{Z_n(\mathbb{R}) > e^{(\mathbb{E}\log m_0 - \varepsilon)n}\}} \right) + \mathbb{E}_\xi \left( \frac{Z_n(nA)}{Z_n(\mathbb{R})} 1_{\{Z_n(\mathbb{R}) \le e^{(\mathbb{E}\log m_0 - \varepsilon)n}\}} \right) \\ &\le e^{-(\mathbb{E}\log m_0 - \varepsilon)n} \mathbb{E}_\xi Z_n(nA) + \mathbb{P}_\xi \left( Z_n(\mathbb{R}) \le e^{(\mathbb{E}\log m_0 - \varepsilon)n} \right). \end{aligned}$$

Hence a.s.

$$\frac{1}{n} \log \mathbb{E}_\xi \left( \frac{Z_n(nA)}{Z_n(\mathbb{R})} \right) \le \frac{1}{n} \log \left( e^{-(\mathbb{E}\log m_0 - \varepsilon)n} \mathbb{E}_\xi Z_n(nA) + \mathbb{P}_\xi \left( Z_n(\mathbb{R}) \le e^{(\mathbb{E}\log m_0 - \varepsilon)n} \right) \right).$$

Taking superior limit in the above inequality, and using Theorem 2.1 and Lemma 5.3, we obtain a.s.

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi \left( \frac{Z_n(nA)}{Z_n(\mathbb{R})} \right) \\ \le\ & \max \left\{ \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi Z_n(nA) - \mathbb{E}\log m_0 + \varepsilon,\ \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_\xi \left( Z_n(\mathbb{R}) \le e^{(\mathbb{E}\log m_0 - \varepsilon)n} \right) \right\} \\ =\ & \max\{ - \inf_{x\in\bar{A}} \Lambda^*(x) - \mathbb{E}\log m_0 + \varepsilon,\ -\infty \} = - \inf_{x\in\bar{A}} \Lambda^*(x) - \mathbb{E}\log m_0 + \varepsilon. \end{aligned}$$

Then let $\varepsilon \to 0$. □

# 6 Branching random walk in varying environment

Kaplan and Asmussen (1976, [21]) showed that under certain moment conditions, the probability measures $\frac{Z_n(b_n\cdot + a_n)}{Z_n(\mathbb{R})}$ satisfy a central limit and a local limit theorem for a branching random walk in deterministic environment for some sequence $(a_n, b_n)$. Biggins (1990, [7]) proved the same results under weaker moments conditions. We want to generalize these results to branching random walk with random environment in time. But instead of studying the case of random environment directly, we first introduce branching random walk with varying environment in time and give some related results.

A branching random walk with a varying environment in time is modeled in a similar way as the branching random walk with a random environment in time. Let $\{X_n\}$ be a sequence of point processes on $\mathbb{R}$. The distribution of $X_n$ is denoted by $\eta_n$. At time 0, there is an initial particle $\emptyset$ of generation 0 located at $S_\emptyset = 0$; at time 1, it is replaced by $N = N(\emptyset)$ particles of generation 1, located at $L_i = L_i(\emptyset)$, $1 \le i \le N$, where the point process $X_\emptyset = (N, L_1, L_2, \cdots)$ is an independent copy of $X_0$ . In general, each particle $u = u_1 \cdots u_n$ of generation $n$ located at $S_u$ is replaced at time n+1 by $N(u)$ new particles $ui$ of generation $n+1$, located at

$$S_{ui} = S_u + L_i(u) \qquad (1 \le i \le N(u)),$$

where the point process formulated by the number of offspring and there displacements, $\{X(u) = (N(u), L_1(u), L_2(u), \cdots)\}$, is an independent copy of $X_n$. All particles behave independently, namely, the point processes $\{X(u)\}$ are independent of each other. In particular, $\{X(u) : u \in \mathbb{T}_n\}$ are independent of each other and have a common distribution $\eta_n$. Let $Z_n = \sum_{u\in\mathbb{T}_n} \delta_{S_u}$ be the counting measure of particles of generation $n$. As the case of random environment, we introduce the following notations:

$$N_n = X_n(\mathbb{R}), \qquad m_n = \mathbb{E}N_n, \qquad P_0 = 1 \quad \text{and} \quad P_n = \mathbb{E}Z_n(\mathbb{R}) = \prod_{i=0}^{n-1} m_i. \tag{6.1}$$

Assume that

$$0 < m_n < \infty, \quad \liminf_{n\to\infty} \frac{1}{n}\log P_n > 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{n}\log m_n = 0. \tag{6.2}$$

Thus for some $c > 1$, there exists an integer $n_0$ depending on $c$ such that

$$P_n > c^n \qquad \text{for all } n > n_0 \tag{6.3}$$

Denote $\Gamma$ the probability space under which the process is defined. Let $\mathcal{F}_0 = \{\emptyset, \Gamma\}$ and $\mathcal{F}_n = \sigma((N(u), L_1(u), L_2(u), \cdots) : |u| < n)$ for $n \geq 1$ be the $\sigma$-field containing all the information concerning the first $n$ generations, then the sequence $\{Z_n(\mathbb{R})/P_n\}$ forms a non-negative martingale with respect to the filtration $\mathcal{F}_n$ and converges a.s. to a random variable $W$.

Let $\nu_n$ be the intensity measure of the point process $\frac{X_n}{m_n}$ in the sense that for a subset $A$ of $\mathbb{R}$,

$$\nu_n(A) = \frac{\mathbb{E}X_n(A)}{m_n},$$

and let $\phi_n$ be the corresponding characteristic function, i.e.

$$\phi_n(t) = \int e^{itx}\nu_n(dx) = \frac{1}{m_n}\mathbb{E}\int e^{itx}X_n(dx). \tag{6.4}$$

The characteristic function of $\frac{Z_n}{c_n}$ is defined as

$$\Psi_n(t) = \frac{1}{P_n}\int e^{itx}Z_n(dx) = \frac{1}{P_n}\sum_{u\in\mathbb{T}_n} e^{itS_u}. \tag{6.5}$$

It is not difficult to see that $\phi_i$ and $\Psi_n$ have the following relation:

$$\mathbb{E}\Psi_n(t) = \prod_{i=0}^{n-1}\phi_i(t). \tag{6.6}$$

Furthermore, denote

$$\upsilon_n(\varepsilon) = \sum_{i=0}^{n-1}\int |x|^{\varepsilon}\nu_i(dx). \tag{6.7}$$

**Condition (A).** *There is a non-degenerate probability distribution $L(x)$ and constants $\{a_n, b_n\}$ with $b_n \to \infty$ such that*

$$e^{-ita_n}\prod_{i=0}^{n-1}\phi_i(t/b_n) \to g(t) = \int e^{itx}L(dx).$$

Similar conditions were posed by Klebaner (1982, [23]) and Biggins (1990, [7]). If additionally $b_{n+1}/b_n \to 1$, then the limit distribution would be in What Feller (1971, [13]) calls the class $L$, also known as the self-decomposable distributions.

Denote $G_n(x) = \nu_0 * \cdots * \nu_{n-1}(x)$, we introduce another condition:

**Condition (B).** *There exist constants $\{a_n, b_n\}$ with $b_n \to \infty$ such that $G_n(b_nx + a_n)$ converges to a non-degenerate probability distribution $L(x)$ .*

It is clear that if (B) holds with $\{a_n, b_n\}$, then (A) holds with $a_n' = \frac{a_n + o(b_n)}{b_n}$ and $b_n' = b_n$. Let $\mu_n = \int x\nu_n(dx)$ and $\sigma_n^2 = \int |x-\mu_n|^2\nu_n(dx)$. Take $a_n = \sum_{i=0}^{n-1}\mu_i$ and $b_n = (\sum_{i=0}^{n-1}\sigma_i^2)^{1/2}$, if moreover $b_n$ satisfying $b_{n+1}/b_n \to 1$, then $G_n(b_nx + a_n) \to L(x)$. In particular, if $\{\nu_n\}$ satisfies Lindeberg or Liapounoff conditions, then the limiting distribution $L$ is standard normal, i.e. $L(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$.

We have the following result:

**Theorem 6.1.** *For a branching random walk in a varying environment satisfying (6.2), assume that for some $\delta > 0$,*

$$\sum_n \frac{1}{m_n n(\log n)^{1+\delta}} \mathbb{E} N_n \log^+ N_n (\log^+ \log^+ N_n)^{1+\delta} < \infty, \tag{6.8}$$

*for some $\varepsilon > 0$ and $\gamma_1 < \infty$,*

$$\upsilon_n(\varepsilon) = o(n^{\gamma_1}), \tag{6.9}$$

*and for some $\gamma_2 > 0$,*

$$b_n^{-1} = o(n^{-\gamma_2}), \tag{6.10}$$

*then*

$$\Psi_n(t/b_n) - W \prod_{i=0}^{n-1} \phi_i(t/b_n) \to 0 \qquad a.s.. \tag{6.11}$$

*If in addition (A) holds, then*

$$e^{-ita_n} \Psi_n(t/b_n) \to g(t) W \qquad a.s., \tag{6.12}$$

*and for $x$ a continuity point of $L$,*

$$P_n^{-1} Z_n(-\infty, b_n(x + a_n)] \to L(x) W \qquad a.s.. \tag{6.13}$$

*The null set can be taken to be independent of $t$ in (6.12) and $x$ in (6.13) respectively, and (6.12) holds uniformly for $t$ in compact sets.*

*Remark.* The above conclusions were obtained by Biggins (1990, [7], Theorem 1 and 2) under similar hypothesis with (6.8) replaced by a condition $\int x \log x F(dx) < \infty$, where $F(x) := \sum_{k=0}^{[x]} \sup_n P(N_n = k)$. In homogeneous case, $F$ is simply the distribution function which determines the offspring's number, but in general, $F$ has not such a concrete expression as (6.8).

The following theorem is a local limit theorem. We use the notation $a_n \sim b_n$ to signify that $a_n/b_n \to 1$ as $n \to \infty$.

**Theorem 6.2.** *For a branching random walk in a varying environment satisfying (6.2), assume that (A) holds with $b_n \sim \theta n^\gamma$ for some constants $0 < \gamma \le \frac{1}{2}$ and $\theta > 0$, $g$ is integrable and for some $\iota > 0$,*

$$\sup_i \sup_{|u| \ge \iota} |\phi_i(t)| =: c_\iota < 1. \tag{6.14}$$

*If (6.9) holds and*

$$\sum_n \frac{1}{m_n n(\log n)^{1+\delta}} \mathbb{E} N_n (\log^+ N_n)^{1+\beta} < \infty \tag{6.15}$$

*for some $\delta > 0$ and $\beta > \gamma$, then*

$$\sup_{x \in \mathbb{R}} \left| b_n P_n^{-1} Z_n(x, x+h) - W h p_L(x/b_n - a_n) \right| \to 0 \qquad a.s., \tag{6.16}$$

*where $p_L(x)$ denotes the density function of $L$.*

# 7 Proof of Theorem 6.1

To prove Theorem 6.1, we only need to show (6.11), for it is obvious that (6.12) is directly from (6.11), and (6.13) is from (6.12) by applying the continuity theorem. The rest assertions are according to Biggins (1990, [7], Theorem 2) . We remark here that our proof is inspired by Biggins (1990, [7]) and Klebaner (1982, [23]).

We will use a truncation method. Let $\kappa > 0$ be a constant. Let $\tilde{X}_{n,\kappa}$ be equal to $X_n$ on $\{N_n(\log N_n)^\kappa \le P_{n+1}\}$ and be empty otherwise; the rest of the notations is extended similarly. Let $I_n(x) = 1_{\{x(\log x)^\kappa \le P_{n+1}\}}$ and $I_n^c = 1 - I_n$, so

$$\tilde{m}_{n,\kappa} = \mathbb{E} N_n I_n(N_n),$$

and

$$\tilde{\phi}_{n,\kappa}(t) = \int e^{itx} \tilde{\nu}_{n,\kappa}(dx) = \frac{1}{\tilde{m}_{n,\kappa}} \mathbb{E} \int e^{itx} X_n(dx) I_n(N_n).$$

The proof of Theorem 6.1 is composed of several lemmas.

**Lemma 7.1.** *Let $\beta \ge 0$. If $\sum_n \frac{1}{m_n n(\log n)^{1+\delta}} \mathbb{E} N_n (\log^+ N_n)^{1+\beta} (\log^+ \log^+ N_n)^{1+\delta} < \infty$ holds for some $\delta > 0$, then for all $\kappa$, $\sum_n n^\beta (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$.*

*Proof.* We can calculate

$$\begin{aligned} \sum_n n^\beta \left(1 - \frac{\tilde{m}_{n,\kappa}}{m_n}\right) &= \sum_n \frac{n^\beta}{m_n} (m_n - \tilde{m}_{n,\kappa}) \\ &= \sum_n \frac{n^\beta}{m_n} \mathbb{E} N_n I_n^c(N_n) \\ &= \sum_n \frac{n^\beta}{m_n} \mathbb{E} N_n I_n^c(N_n) 1_{\{N_n > a\}} + \sum_n \frac{n^\beta}{m_n} \mathbb{E} N_n I_n^c(N_n) 1_{\{N_n \le a\}}, \end{aligned}$$

where $a$ is a constant. Since $P_n \to \infty$, the convergence of the second series above is obvious. It suffices to show that of the first series for suitable $a$. Take $f(x) = (\log x)^{1+\beta} (\log\log x)^{(1+\delta)}$. $f(x)$ is increasing and positive on $(a, +\infty)$. Noticing (6.3), we have for $n$ large enough,

$$\begin{aligned} & \frac{n^\beta}{m_n} \mathbb{E} N_n I_n^c(N_n) 1_{\{N_n > a\}} \\ \le\ & \frac{n^\beta}{m_n} \mathbb{E} \frac{f(N_n (\log N_n)^\kappa)}{f(P_{n+1})} 1_{\{N_n > a\}} \\ \le\ & \frac{C}{m_n n (\log n)^{1+\delta}} \mathbb{E} N_n (\log N_n)^{1+\beta} (\log\log N_n)^{1+\delta} 1_{\{N_n > a\}} \\ \le\ & \frac{C}{m_n n (\log n)^{1+\delta}} \mathbb{E} N_n (\log^+ N_n)^{1+\beta} (\log^+ \log^+ N_n)^{1+\delta}, \end{aligned}$$

where $C$ is a constant, and like $a$, in general, it does not necessarily stand for the same constant throughout. The convergence of the series $\sum_n \frac{1}{m_n n(\log n)^{1+\delta}} \mathbb{E} N_n (\log^+ N_n)^{1+\beta} (\log^+ \log^+ N_n)^{1+\delta}$ implies that of the series $\sum_n \frac{n^\beta}{m_n} \mathbb{E} N_n I_n^c(N_n) 1_{\{N_n > a\}}$. □

**Lemma 7.2** ([7], Lemma 3 (ii))**.** *If $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$, then*

$$\prod_{i=0}^{n-1} \tilde{\phi}_{i,\kappa}(t/b_n) - \prod_{i=0}^{n-1} \phi_i(t/b_n) \to 0, \qquad \text{as } n \to \infty. \tag{7.1}$$

The formula (7.1) shows that we can prove (6.11) with $\tilde{\phi}_{i,\kappa}$ in place of $\phi_i$. For simplicity, let

$$\zeta_n(t) = \tilde{\phi}_{n,\kappa}(t) \qquad \text{and} \qquad \omega_n = \frac{\tilde{m}_{n,\kappa}}{m_n},$$

where the value of $\kappa$ will be fixed to be suitably large later.

Let $\Psi_u^{(1)}(t) := m_n^{-1} \int e^{itx} X(u)(dx)$ if $u \in \mathbb{T}_n$. Then

$$\begin{aligned} & \Psi_{n+1}(t) - \omega_n \zeta_n(t) \Psi_n(t) \\ = & \frac{1}{P_n} \sum_{u \in \mathbb{T}_n} e^{itS_u} \Psi_u^{(1)}(t) I_n^c(N(u)) + \frac{1}{P_n} \sum_{u \in \mathbb{T}_n} e^{itS_u} \left( \Psi_u^{(1)}(t) I_n(N(u)) - \omega_n \zeta_n(t) \right) \\ = & : A_n(t) + B_n(t). \end{aligned}$$

By iteration, we obtain

$$\Psi_n\left(\frac{t}{b_n}\right) - \Psi_k\left(\frac{t}{b_n}\right) \prod_{i=k}^{n-1} \omega_i \zeta_i \left(\frac{t}{b_n}\right) = \sum_{i=k}^{n-1} \left( A_i\left(\frac{t}{b_n}\right) + B_i\left(\frac{t}{b_n}\right) \right) \prod_{j=i+1}^{n-1} \omega_j \zeta_j \left(\frac{t}{b_n}\right). \tag{7.2}$$

Thus

$$\begin{aligned} & \Psi_n(t/b_n) - W \prod_{i=0}^{n-1} \zeta_i(t/b_n) \\ = & \sum_{i=k}^{n-1} A_i(t/b_n) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(t/b_n) + \sum_{i=k}^{n-1} B_i(t/b_n) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(t/b_n) \\ & + \left( \Psi_k(t/b_n) \prod_{i=k}^{n-1} \omega_i \zeta_i(t/b_n) - W \prod_{i=0}^{n-1} \zeta_i(t/b_n) \right). \end{aligned} \tag{7.3}$$

Let $\alpha > 1$. Take $k = J(n) = j$ if $j^\alpha \le n < (j+1)^\alpha$, so that $k^\alpha \sim n$, which means $k$ goes to infinity more slowly than $n$. For this $k$, we will show that each term in the right side of (7.3) is negligible.

**Lemma 7.3.** *If $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$, then*

$$\sum_{i=k}^{n-1} A_i(t/b_n) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(t/b_n) \to 0 \qquad a.s., \qquad as\ n \to \infty. \tag{7.4}$$

*Proof.* Notice that

$$\left| \sum_{i=k}^{n-1} A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j \right| \le \sum_{i=k}^{n-1} |A_i| \le \sum_{i=k}^{n-1} \frac{1}{P_i m_i} \sum_{|u|=i} N(u) I_i^c(N(u)). \tag{7.5}$$

Since

$$\mathbb{E}\left( \sum_{i=0}^{\infty} \frac{1}{P_i m_i} \sum_{|u|=i} N(u) I_i^c(N(u)) \right) = \sum_{i=0}^{\infty} \frac{1}{m_i} \mathbb{E} N_i I_i^c(N_i) = \sum_i (1 - \frac{\tilde{m}_{i,\kappa}}{m_i}) < \infty.$$

we get

$$\sum_{i=0}^{\infty} \frac{1}{P_i m_i} \sum_{|u|=i} N(u) I_i^c(N(u)) < \infty.$$

which implies (7.4), combined with (7.5). □

**Lemma 7.4.** *If for some $\delta_1 > 0$ ,*

$$\sum_n \frac{1}{m_n n^{1+\delta_1}} \mathbb{E} N_n \log^+ N_n < \infty, \tag{7.6}$$

*then*

$$\sum_{i=k}^{n-1} B_i(t/b_n) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(t/b_n) \to 0 \qquad a.s., \qquad as\ n \to \infty. \tag{7.7}$$

*Remark.* Obviously (6.8) implies (7.6).

*Proof.* Let

$$C_n = \sum_{i=k}^{n-1} B_i(t/b_n) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(t/b_n).$$

We want to show that $\sum_{n=1}^{\infty} \mathbb{E}|C_n|^2 < \infty$, which implies (7.6). Since $E(B_i|\mathcal{F}_i) = 0$, we have

$$\mathbb{E}|C_n|^2 = var(C_n) = var\left(\sum_{i=k}^{n-1} B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j\right) \leq \sum_{i=k}^{n-1} var(B_i),$$

where the notation $var$ denotes variance. Moreover,

$$var(B_i) \quad = \quad \mathbb{E}(var(B_i|\mathcal{F}_i)) \leq \frac{1}{P_i} var(\Psi_i^{(1)} I_i(N_i)) \leq \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i),$$

where $\Psi_n^{(1)}(t) := m_n^{-1} \int e^{itx} X_n(dx)$. We denote $J^{-1}$ be the inverse mapping of $J$, $J^{-1}(j) = \{n : J(n) = j\}$ and $|J^{-1}(j)|$ be the number of the elements in $J^{-1}(j)$. It is not difficult to see that $|J^{-1}(j)| = O(j^{\alpha-1})$ and $\sum_{j=1}^{i} |J^{-1}(j)| = O(i^{\alpha})$. Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{E}|C_n|^2 &\leq \sum_{n=1}^{\infty} \sum_{i=k}^{n-1} \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&= \sum_{j=1}^{\infty} \sum_{n \in J^{-1}(j)} \sum_{i=j}^{n-1} \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&\leq \sum_{j=1}^{\infty} |J^{-1}(j)| \sum_{i=j}^{\infty} \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} |J^{-1}(j)| \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&\leq C \sum_{i=1}^{\infty} \frac{i^{\alpha}}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&= C \sum_{i=1}^{\infty} \frac{i^{\alpha}}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) 1_{\{N_i > a\}} + C \sum_{i=1}^{\infty} \frac{i^{\alpha}}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) 1_{\{N_i \leq a\}}.
\end{aligned}$$

The second series above converges, since $\sum_i \frac{i^{\alpha}}{P_i m_i^2} < \infty$. For the first series above, take $f(x) = x(\log x)^{-(\alpha+1+\delta_1)}$. $f(x)$ is increasing and positive on $(a, +\infty)$. We have for $i$ large enough,

$$\begin{aligned}
& \frac{i^{\alpha}}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) 1_{\{N_i > a\}} \\
\leq \quad & \frac{i^{\alpha}}{P_i m_i^2} \mathbb{E} N_i^2 \frac{f(P_{i+1})}{f(\{N_i (\log N_i)^{\kappa})} 1_{\{N_i > a\}} \\
\leq \quad & \frac{C}{m_i i^{1+\delta_1}} \mathbb{E} N_i (\log N_i)^{\alpha+1+\delta_1-\kappa} 1_{\{N_i > a\}} \\
\leq \quad & \frac{C}{m_i i^{1+\delta_1}} \mathbb{E} N_i \log^+ N_i,
\end{aligned}$$

if we take $\kappa \geq \alpha + \delta_1$. Then by (7.6), it follows the convergence of the series $\sum_i \frac{i^{\alpha}}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) 1_{\{N_i > a\}}$. □

**Lemma 7.5.** *If $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ and (6.9), (6.10) hold, then*

$$\Psi_k(t/b_n) \prod_{i=k}^{n-1} \omega_i \zeta_i(t/b_n) - W \prod_{i=0}^{n-1} \zeta_i(t/b_n) \to 0 \qquad a.s., \qquad as\ n \to \infty. \tag{7.8}$$

*Proof.* $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ implies that $\sum_{i=k}^{n-1} \omega_i \to 1$, so the factor $\prod_{i=k}^{n-1} \omega_i$ in (2.8) can be ignored. Notice that

$$\Psi_k \prod_{i=k}^{n-1} \zeta_i - W \prod_{i=0}^{n-1} \zeta_i = \left(\Psi_k - \frac{Z_k(\mathbb{R})}{P_k}\right) \prod_{i=k}^{n} \zeta_i + \left(\frac{Z_k(\mathbb{R})}{P_k} - W\right) \prod_{i=k}^{n-1} \zeta_i + W \left(\prod_{i=k}^{n-1} \zeta_i - \prod_{i=0}^{n-1} \zeta_i\right).$$

It suffices to prove that

$$\Psi_k(t/b_n) - \frac{Z_k(\mathbb{R})}{P_k} \to 0 \qquad a.s., \qquad as\ k \to \infty. \tag{7.9}$$

and

$$\prod_{i=k}^{n-1} \zeta_i(t/b_n) - \prod_{i=0}^{n-1} \zeta_i(t/b_n) \to 0, \qquad as\ k \to \infty. \tag{7.10}$$

Since $|e^{itx} - 1| \leq C|tx|^\varepsilon$, we have

$$\begin{aligned}
\left|\Psi_k\left(t/b_n\right) - \frac{Z_n(\mathbb{R})}{P_k}\right| &\leq \frac{1}{P_k} \int \left|e^{tb_n^{-1}x} - 1\right| Z_k(dx) \\
&\leq C|u|^\varepsilon b_n^{-\varepsilon} \frac{1}{P_k} \int |x|^\varepsilon Z_k(dx).
\end{aligned}$$

Assume that $0 < \varepsilon \leq 1$ (the proof for the case of $\varepsilon > 1$ is similar). Taking expectation in the above inequality, we obtain

$$\begin{aligned}
\mathbb{E}\left|\Psi_k(t/b_n) - \frac{Z_k(\mathbb{R})}{P_k}\right| &\leq C|t|^\varepsilon \sup\{b_n^{-\varepsilon} : k^\alpha \leq n < (k+1)^\alpha\} \int |x|^\varepsilon \nu_0 * \cdots * \nu_{k-1}(dx) \\
&\leq C|t|^\varepsilon \sup\{b_n^{-\varepsilon} : k^\alpha \leq n < (k+1)^\alpha\} \sum_{i=0}^{k-1} \int |x|^\varepsilon \nu_i(dx) \\
&= C|t|^\varepsilon \sup\{b_n^{-\varepsilon} : k^\alpha \leq n < (k+1)^\alpha\} \upsilon_k(\varepsilon) = |u|^\varepsilon o(k^{\gamma_1 - \alpha\varepsilon\gamma_2}).
\end{aligned}$$

Hence (7.9) holds if we take $\alpha$ large. By Lemma 7.2, we can prove (7.10) with $\phi_i$ in place of $\zeta_i$, which holds directly by noticing that

$$\begin{aligned}
\left|\prod_{i=k}^{n-1} \phi_i(t/b_n) - \prod_{i=0}^{n-1} \phi_i(t/b_n)\right| &= \left|\prod_{i=k}^{n-1} \phi_i(t/b_n) \left(1 - \prod_{i=0}^{k-1} \phi_i(t/b_n)\right)\right| \leq \left|1 - \prod_{i=0}^{k-1} \phi_i(t/b_n)\right| \\
&= \left|\mathbb{E}\left(\Psi_k(t/b_n) - \frac{Z_k(\mathbb{R})}{P_k}\right)\right| \leq \mathbb{E}\left|\Psi_k(t/b_n) - \frac{Z_k(\mathbb{R})}{P_k}\right|.
\end{aligned}$$

Thus (7.8) holds. □

# 8 Proof of Theorem 6.2

We will go along the proof by following the lines in [7]. Let

$$K(x) = \frac{1}{2\pi} \left(\frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)^2 \qquad K_a(x) = \frac{1}{a} K(\frac{x}{a}) \ (a > 0).$$

Then

$$\int_{\mathbb{R}} K(x)dx = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} K_a(x)dx = 1.$$

The characteristic function of $K_a$ is denoted by $k_a$, which vanishes outside $(-\frac{1}{a}, \frac{1}{a})$, so that the characteristic function of $\frac{Z_n}{P_n} * K_a$ is integrable and so $\frac{Z_n}{P_n} * K_a$ has a density function $D_a^{(n)}$. We will get our result through the asymptotic property of $D_a^{(n)}$.

**Lemma 8.1** (see [10])**.** *If $f(t)$ is a characteristic function such that $|f(t)| \leq \kappa$ as soon as $b \leq |u| \leq 2b$, then we have for $|u| < b$,*

$$|f(t)| \leq 1 - (1-\kappa^2)\frac{t^2}{8b^2}.$$

**Lemma 8.2.** *Under the conditions of Theorem 6.2,*

$$\sup_{x\in\mathbb{R}} |b_n D_a^{(n)}(b_n(x+a_n)) - Wp_L(x)| \to 0 \; a.s., \qquad as \;\; n \to \infty. \tag{8.1}$$

*Proof.* Let $A$ be a positive constant. By the Fourier inversion theorem,

$$2\pi \left| b_n D_a^{(n)}(b_n(x+a_n)) - Wp_L(x) \right| = \left| \int \left( \Psi_n(\frac{t}{b_n}) k_a(\frac{t}{b_n}) e^{-ita_n} - Wg(t) \right) dt \right|.$$

Split the integral of the right side into $|t| < A$ and $|t| \geq A$. Using Theorem 6.1 and noticing that $\lim_n k_a(t/b_n) = 1$, we have

$$\begin{aligned} &\left| \int_{|t|<A} \left( \Psi_n(\frac{t}{b_n}) k_a(\frac{t}{b_n}) e^{-iua_n} - Wg(t) \right) dt \right| \\ = \;& \int_{|t|<A} \left| \left( \Psi_n(\frac{t}{b_n}) e^{-ita_n} - Wg(t) \right) k_a(\frac{t}{b_n}) \right| dt + \int_{|t|<A} \left| (1 - k_a(\frac{t}{b_n})) W g(t) \right| dt \\ \leq \;& 2A \sup_{|t|\leq A} \left| \Psi_n(\frac{t}{b_n}) e^{-ita_n} - Wg(t) \right| + W \int_{|t|<A} \left| 1 - k_a(\frac{t}{b_n}) \right| dt \to 0 \; a.s., \quad as \; n \to \infty. \end{aligned}$$

For $A$ large, the integral of $g(t)$ over $|t| \geq A$ is small. So to show (8.1), it remains to consider

$$\left| \int_{|t|\geq A} \Psi_n(\frac{t}{b_n}) k_a(\frac{t}{b_n}) dt \right| = \left| \int_U b_n \Psi_n(t) k_a(t) dt \right|,$$

where $U = \{t : \frac{A}{b_n} \leq |t| \leq \frac{1}{a}\}$. By the decomposition (7.2),

$$b_n \Psi_n k_a = b_n \Psi_k \prod_{i=k}^{n-1} \omega_i \zeta_i k_a + b_n \sum_{i=k}^{n-1} A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a + b_n \sum_{i=k}^{n-1} B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a.$$

Take $k = J(n)$ the same as the proof of Theorem 6.1,we need to show that

$$\left| b_n \sum_{i=k}^{n-1} \int_U A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a dt \right| \to 0 \quad a.s., \qquad as \; n \to \infty. \tag{8.2}$$

and the similar result with $B_i$ in place of $A_i$.

Firstly, for $n$ large enough,

$$\begin{aligned} \left| b_n \sum_{i=k}^{n-1} \int_U A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a dt \right| &\leq b_n \sum_{i=k}^{n-1} \int_U |A_i| dt \\ \leq \;& Ck^{\alpha\gamma} \sum_{i=k}^{n-1} \frac{1}{P_i m_i} \sum_{|u|=i} N(u) I_i^c(N(u)) \\ \leq \;& C \sum_{i=k}^{n-1} \frac{i^{\alpha\gamma}}{P_i m_i} \sum_{|u|=i} N(u) I_i^c(N(u)). \end{aligned}$$

Like the proof of Lemma 7.3, we obtain

$$\mathbb{E}\left(\sum_{i=0}^{\infty}\frac{i^{\alpha\gamma}}{P_i m_i}\sum_{|u|=i}N(u)I_i^c(N(u))\right)=\sum_{i=0}^{\infty}i^{\alpha\gamma}\left(1-\frac{\tilde{m}_{i,k}}{m_i}\right)<\infty$$

from Lemma 7.1, if we take $\alpha$ sufficiently near 1 such that $\alpha\gamma<\beta$. Hence (8.2) is proved.

Secondly, to prove (8.2) with $B_i$ in place of $A_i$, like the proof of Lemma 7.4, we set

$$C_n=b_n\sum_{i=k}^{n-1}\int_U B_i\prod_{j=i+1}^{n-1}\omega_j\zeta_j k_a dt.$$

Since $\mathbb{E}(B_i|\mathcal{F}_i)=0$, for $n$ large enough,

$$\begin{aligned}
\mathbb{E}|C_n|^2 &= var\left(b_n\sum_{i=k}^{n-1}\int_U B_i\prod_{j=i+1}^{n-1}\omega_j\zeta_j k_a dt\right)\\
&= b_n^2\sum_{i=k}^{n-1}var\left(\int_U B_i\prod_{j=i+1}^{n-1}\omega_j\zeta_j k_a dt\right)\\
&= b_n^2\sum_{i=k}^{n-1}\mathbb{E}\left|\int_U B_i\prod_{j=i+1}^{n-1}\omega_j\zeta_j k_a dt\right|^2\\
&\leq b_n^2\sum_{i=k}^{n-1}\mathbb{E}\left(\int_U dt\right)\left(\int_U |B_i\prod_{j=i+1}^{n-1}\omega_j\zeta_j k_a|^2 dt\right)\\
&\leq \frac{2}{a}b_n^2\sum_{i=k}^{n-1}\int_U\mathbb{E}|B_i|^2 dt\\
&= \frac{2}{a}b_n^2\sum_{i=k}^{n-1}\int_U var|B_i|^2 dt\\
&\leq C\sum_{i=k}^{n-1}\frac{i^{2\alpha\gamma}}{P_i m_i^2}\mathbb{E}N_i^2 I_i(N_i).
\end{aligned}$$

Following the last part of the proof of Lemma 7.4, we obtain that $\sum_{n=1}^{\infty}\mathbb{E}|C_n|^2<\infty$ provided $\kappa$ large enough, which implies that $C_n\to 0$ a.s..

Finally, we consider $b_n\int_U\Psi_k\prod_{i=k}^{n-1}\omega_i\zeta_i k_a dt$. Clearly,

$$\left|b_n\int_U\Psi_k\prod_{i=k}^{n-1}\omega_i\zeta_i k_a dt\right|\leq\frac{Z_k(\mathbb{R})}{P_k}b_n\int_U\left|\prod_{i=k}^{n-1}\zeta_i\right|dt \tag{8.3}$$

Since $\frac{Z_k(\mathbb{R})}{P_k}\to W$ a.s. as $k\to\infty$, it remains to consider $b_n\int_U|\prod_{i=k}^{n-1}\zeta_i|dt$. It suffices to show that

$$\limsup_{n\to\infty}b_n\int_U\left|\prod_{i=k}^{n-1}\zeta_i(t)\right|dt\leq\limsup_{n\to\infty}b_n\int_U\prod_{i=k}^{n-1}|\phi_i(t)|dt, \tag{8.4}$$

and there exists a constant $\theta_1>0$ (not depending on $A$) such that

$$\limsup_{n\to\infty}b_n\int_U\prod_{i=k}^{n-1}|\phi_i(t)|dt\leq\int_{|t|\geq A}e^{-\theta_1 t^2}du\qquad\text{for any } A. \tag{8.5}$$

Notice that

$$b_n \int_U \left|\prod_{i=k}^{n-1} \zeta_i\right| dt \le b_n \int_U \left|\prod_{i=k}^{n-1} \zeta_i - \prod_{i=k}^{n-1} \phi_i\right| dt + b_n \int_U \prod_{i=k}^{n-1} |\phi_i| dt.$$

The proof of [7] Lemma 3 gives $|\zeta_i - \phi_i| \le 2(1 - \tilde{m}_{i,k}/m_i)$, so we have

$$\begin{aligned} b_n \int_U \left|\prod_{i=k}^{n-1} \zeta_i - \prod_{i=k}^{n-1} \phi_i\right| dt &\le b_n \int_U \sum_{i=k}^{n-1} |\zeta_i - \phi_i| dt \\ &\le \frac{4}{a} b_n \sum_{i=k}^{n-1} \left(1 - \frac{\tilde{m}_{i,k}}{m_i}\right) \\ &\le C \sum_{i=k}^{n-1} (i+1)^{\alpha\gamma} \left(1 - \frac{\tilde{m}_{i,k}}{m_i}\right) \to 0 \quad as\ n \to \infty, \end{aligned}$$

provided $\alpha\gamma < \beta$. Hence (8.4) holds. Now we turn to prove (8.5). Split the set $U$ into two parts: $U_1 = \{t : A/b_n \le t \le \epsilon\}$ and $U_2 = \{t : \epsilon \le t \le \frac{1}{a}\}$. Since for some $\iota > 0$, $|\phi_i(t)| \le c_\iota < 1$ for all $|t| \ge \iota$, by Lemma 8.1, we have for all $|t| < \iota$,

$$|\phi_i(t)| \le 1 - \frac{1 - c_\iota^2}{8\iota^2} t^2 \le e^{-\gamma_1 t^2},$$

where $\gamma_1 = \frac{1-c_\iota^2}{8\iota^2}$. Thus

$$\sup_i \sup_{|t| \ge \epsilon} |\phi_i(t)| = \max\{e^{-\gamma_1 \epsilon^2}, c_\iota\} =: c_\iota' < 1.$$

It follows that

$$b_n \int_{U_2} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \frac{2}{a} b_n (c_\iota')^{n-k-1} \to 0\ as\ n \to \infty, \tag{8.6}$$

and

$$b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t| \ge A} \exp(-b_n^{-2}(n-k-1)\gamma_1 t^2) dt. \tag{8.7}$$

It is easy to see that

$$\lim_{n\to\infty} \frac{n-k-1}{b_n^2} = \begin{cases} \frac{1}{\theta^2} & \text{if } \gamma = \frac{1}{2}, \\ \infty & \text{if } 0 < \gamma < \frac{1}{2}. \end{cases}$$

So there exists a constant $\theta_1 > 0$ such that $b_n^{-2}(n-k-1)\gamma_1 > \theta_1$ for $n$ large enough. Thus

$$\limsup_{n\to\infty} b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t| \ge A} e^{-\theta_1 t^2} dt \qquad \text{for any } A. \tag{8.8}$$

Consequently, (8.5) holds via (8.6) and (8.8). This completes the proof. □

By a similar argument of Stone (1965, [33]), we have the following Lemma.

**Lemma 8.3.** *If (8.1) holds, then $\forall \varepsilon > 0$, there exist $n_0 > 0$ and $\delta > 0$ such that $\forall n \ge n_0$ and $\forall 0 < h < \delta$,*

$$h(Wp_L(x) - \varepsilon) \le P_n^{-1} Z_n(b_n(x + a_n), b_n(x + a_n + h)) \le h(Wp_L(x) + \varepsilon) \quad a.s., \quad \forall x \in \mathbb{R}. \tag{8.9}$$

*The null set can be taken to be independent of $x$.*

Now we turn to the proof of Theorem 6.2:

*Proof of Theorem 6.2.* Fix $h > 0$. $\forall \varepsilon > 0$, take $0 < \varepsilon' < \varepsilon/h$. By Lemmas 8.2 and 8.3 , for this $\varepsilon' > 0$, there exist $n_0' > 0$ and $\delta' > 0$ such that $\forall n \geq n_0'$ and $\forall 0 < h' < \delta'$,

$$h'(Wp_L(x) - \varepsilon') \leq P_n^{-1} Z_n(b_n(x + a_n), b_n(x + a_n + h')) \leq h'(Wp_L(x) + \varepsilon') \quad a.s., \quad \forall x \in \mathbb{R},$$

Let $h' = h/b_n$. Then there exist $\tilde{n}_0 > 0$ such that $0 < h' < \delta'$ for $n \geq \tilde{n}_0$. Take $n_0 := \max\{n_0', \tilde{n}_0\} > 0$, we have $\forall n \geq n_0$,

$$h(Wp_L(x) - \varepsilon') \leq b_n P_n^{-1} Z^n(b_n(x + a_n), b_n(x + a_n) + h) \leq h(Wp_L(x) + \varepsilon') \quad a.s., \ \forall x \in \mathbb{R},$$

which implies that

$$\sup_{x \in \mathbb{R}} |b_n P_n^{-1} Z_n(b_n(x + a_n), b_n(x + a_n) + h) - Whp_L(x)| \leq \varepsilon' h < \varepsilon \qquad a.s.,$$

so that

$$\sup_{x \in \mathbb{R}} |b_n P_n^{-1} Z_n(x, x + h) - Whp_L(x/b_n - a_n)| < \varepsilon \qquad a.s..$$

The proof is finished. □

# 9 Central limit theorems for $\frac{\mathbb{E}_\xi Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$, $\frac{\mathbb{E} Z_n(\cdot)}{\mathbb{E} Z_n(\mathbb{R})}$ and $\mathbb{E}\frac{Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$

Now we return to consider the branching random walk with a random environment in time introduced in Section 1. When the environment $\xi$ is fixed, a branching random walk in random environment is in fact a branching random walk in varying environment introduced in Section 6. We still assume (1.2), which implies that

$$\lim_{n \to \infty} \frac{1}{n} \log P_n = \mathbb{E} \log m_0 > 0 \qquad \text{and} \qquad \lim_{n \to \infty} \frac{1}{n} \log m_n = 0 \qquad a.s.$$

by the ergodic theorem. Hence the assumption (6.2) is satisfied, so that (6.3) holds for some constant $c > 1$ and integer $n_0 = n_0(\xi)$ depending on $c$ and $\xi$. Note that all the notations and results in Section 6 are still available under the quenched law $\mathbb{P}_\xi$ and the corresponding expectation $\mathbb{E}_\xi$.

Recall that $\nu_n(\cdot) = \frac{\mathbb{E}_\xi X_n(\cdot)}{m_n}$ is the intensity measure of $\frac{X_n}{m_n}$. Let

$$\mu_n = \int x \nu_n(dx) \qquad \textit{and} \qquad \sigma_n^2 = \int |x - \mu_n|^2 \nu_n(dx). \tag{9.1}$$

We first have a central limit theorem for quenched means as follows.

**Theorem 9.1** (Central limit theorem for quenched means $\frac{\mathbb{E}_\xi Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$). *If $|\mu_0| < \infty$ a.s. and $\mathbb{E}\sigma_0^2 \in (0, \infty)$, then*

$$\frac{\mathbb{E}_\xi Z_n(-\infty, b_n x + a_n]}{\mathbb{E}_\xi Z_n(\mathbb{R})} \to \Phi(x) \quad a.s.,$$

*where $a_n = \sum_{i=0}^{n-1} \mu_i$ and $b_n = (\sum_{i=0}^{n-1} \sigma_i^2)^{1/2}$.*

*Proof.* Notice that $\frac{\mathbb{E}_\xi Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})} = \nu_0 * \cdots * \nu_{n-1}(\cdot)$. It suffices to show that $\{\nu_n\}$ satisfies Lindeberg condition, i.e., for all $t > 0$ ,

$$\lim_{n \to \infty} \frac{1}{b_n^2} \sum_{i=0}^{n-1} \int_{|x - \mu_i| > tb_n} |x - \mu_i|^2 \nu_i(dx) = 0 \qquad a.s.. \tag{9.2}$$

By the ergodic theorem,

$$\lim_{n \to \infty} \frac{b_n^2}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sigma_i^2 = \mathbb{E}\sigma_0^2 > 0 \qquad a.s.. \tag{9.3}$$

So for a positive constant $a$ satisfying $0 < a^2 < \mathbb{E}\sigma_0^2$, there exists an integer $n_0$ depending on $a$ and $\xi$ such that $b_n^2 \geq a^2 n$ for all $n \geq n_0$. Fix a constant $M > 0$ . For $n \geq \max\{n_0, M\}$, we have $b_n^2 \geq a^2 n \geq a^2 M$, so that

$$\frac{1}{b_n^2}\sum_{i=0}^{n-1}\int_{|x-\mu_i|>tb_n}|x-\mu_i|^2\nu_i(dx) \leq \frac{1}{a^2 n}\sum_{i=0}^{n-1}\int_{|x-\mu_i|>ta\sqrt{M}}|x-\mu_i|^2\nu_i(dx).$$

Taking superior limit in the above inequality , we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{b_n^2}\sum_{i=0}^{n-1}\int_{|x-\mu_i|>tb_n}|x-\mu_i|^2\nu_i(dx)\\
\leq\ & \frac{1}{a^2}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\int_{|x-\mu_i|>ta\sqrt{M}}|x-\mu_i|^2\nu_i(dx)\\
=\ & \frac{1}{a^2}\mathbb{E}\int_{|x-\mu_0|>ta\sqrt{M}}|x-\mu_0|^2\nu_0(dx).
\end{aligned}$$

Let $M \to \infty$, it obvious that $\mathbb{E}\int_{|x-\mu_0|>ta\sqrt{M}}|x-\mu_0|^2\nu_0(dx) \to 0$ by the dominated convergence theorem, since $\mathbb{E}\sigma_0^2 < \infty$. This completes the proof. □

If the environment is *i.i.d.*, we can obtain a central limit theorem for annealed means.

**Theorem 9.2** (Central limit theorem for annealed means $\frac{\mathbb{E}Z_n(\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$)**.** *Assume that* $\{\xi_n\}$ *are i.i.d.. Let* $\bar{\mu} = \frac{1}{\mathbb{E}m_0}\mathbb{E}\int xX_0(dx)$ *and* $\bar{\sigma}^2 = \frac{1}{\mathbb{E}m_0}\mathbb{E}\int(x-\bar{\mu})^2X_0(dx)$*. If* $|\bar{\mu}| < \infty$ *and* $\bar{\sigma}^2 \in (0,\infty)$*, then*

$$\frac{\mathbb{E}Z_n(-\infty, \bar{b}_n x + \bar{a}_n]}{\mathbb{E}Z_n(\mathbb{R})} \to \Phi(x),$$

*where* $\bar{a}_n = n\bar{\mu}$ *and* $\bar{b}_n = \sqrt{n}\bar{\sigma}$.

*Proof.* Denote $\bar{\nu}_n(\cdot) = \frac{\mathbb{E}Z_n(\bar{b}_n\cdot+\bar{a}_n)}{\mathbb{E}Z_n(\mathbb{R})}$. The characteristic function of $\bar{\nu}_n$ is denoted by $\bar{\varphi}_n$. We can calculate

$$\begin{aligned}
\bar{\varphi}_n(t) = \int e^{itx}\bar{\nu}_n(dx) &= (\mathbb{E}m_0)^{-n}\mathbb{E}\int e^{itx}Z_n(\bar{b}_n dx + \bar{a}_n)\\
&= (\mathbb{E}m_0)^{-n}e^{-it\bar{a}_n/\bar{b}_n}\mathbb{E}\prod_{i=0}^{n-1}\mathbb{E}_\xi\int e^{itx/\bar{b}_n}X_n(dx)\\
&= e^{-it\bar{a}_n/\bar{b}_n}\left(\frac{\mathbb{E}m_0(t/\bar{b}_n)}{\mathbb{E}m_0}\right)^n,
\end{aligned}$$

where $m_n(t) := \mathbb{E}_\xi\int e^{itx}X_n(dx)$. The last step above is from the independency of $(\xi_n)$. Denote $F(x) = \frac{\mathbb{E}X_0(x)}{\mathbb{E}m_0}$, then by the classic central limit theorem, we have

$$F^{*n}(\bar{b}_n x + \bar{a_n}) \to \Phi(x).$$

Therefore,

$$\int e^{itx}F^{*n}(\bar{b}_n dx + \bar{a_n}) \to g(t) := \int e^{itx}p(x)dx,$$

where $p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the density function of standard normal distribution. Notice that

$$\begin{aligned}
\int e^{itx}F^{*n}(\bar{b}_n dx + \bar{a_n}) &= e^{-it\bar{a}_n/\bar{b}_n}\int e^{ity/\bar{b}_n}F^{*n}(dy)\\
&= e^{-it\bar{a}_n/\bar{b}_n}\left(\int e^{ity/\bar{b}_n}F(dy)\right)^n\\
&= e^{-it\bar{a}_n/\bar{b}_n}\left(\frac{\mathbb{E}m_0(t/\bar{b}_n)}{\mathbb{E}m_0}\right)^n = \bar{\varphi}_n(t).
\end{aligned}$$

We in fact have obtained $\bar{\varphi}_n(t) \to g(t)$, it follows that $\bar{\nu}_n(x) \to \Phi(x)$ by the continuity theorem. □

By an argument similar to the proof of Theorem 9.2, we obtain a central limit theorem as follows:

**Theorem 9.3** (Central limit theorem for $\mathbb{E}\frac{Z_n(\cdot)}{\mathbb{E}_\xi(\mathbb{R})}$)**.** *Assume that $\{\xi_n\}$ are i.i.d.. Let $\bar{\mu}' = \mathbb{E}\int x\nu_0(dx)$ and $\bar{\sigma}'^2 = \mathbb{E}\int (x-\bar{\mu}')^2\nu_0(dx)$. If $|\bar{\mu}'| < \infty$ and $\bar{\sigma}'^2 \in (0,\infty)$, then*

$$\mathbb{E}\frac{Z_n(-\infty, \bar{b}_n' x + \bar{a}_n']}{\mathbb{E}_\xi Z_n(\mathbb{R})} \to \Phi(x),$$

*where $\bar{a}_n' = n\bar{\mu}'$ and $\bar{b}_n' = \sqrt{n}\bar{\sigma}'$.*

# 10 Central limit theorem and local limit theorem for $\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$

As we mentioned in last section (Section 9), we can directly use the results of Theorems 6.1 and 6.2 considering the quenched law $\mathbb{P}_\xi$ and the corresponding expectation $\mathbb{E}_\xi$. However, by the good properties of stationary and ergodic random process, we have some similar but simper and more precise results than Theorems 6.1 and 6.2.

**Theorem 10.1.** *Assume that for some $\varepsilon > 0$,*

$$\upsilon(\varepsilon) := \mathbb{E}\int |x|^\varepsilon \nu_0(dx) < \infty,$$

*and $b_n = b_n(\xi)$ satisfying*

$$b_n^{-1} = o(n^{-\gamma}) \text{ a.s.} \quad \text{for some } \gamma > 0,$$

*then*

$$\Psi_n(t/b_n) - W\prod_{i=0}^{n-1}\phi_i(t/b_n) \to 0 \qquad \text{a.s..}$$

*If in addition (A) holds wit $\{a_n(\xi), b_n(\xi)\}$ and $g_\xi$, then*

$$e^{-ita_n}\Psi_n(t/b_n) \to g_\xi(t)W \qquad \text{a.s.,} \tag{10.1}$$

*and for $x$ a continuity point of $L_\xi$,*

$$P_n^{-1}Z_n(-\infty, b_n(x+a_n)] \to L_\xi(x)W \qquad \text{a.s..}$$

*Moreover, (10.1) holds uniformly for $u$ in compact sets.*

The following result is the most important central limit theorem of this paper.

**Theorem 10.2** (Central limit theorem for $\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$)**.** *If $\mathbb{E}|\mu_0|^\varepsilon < \infty$ for some $\varepsilon > 0$ and $\mathbb{E}\sigma_0^2 \in (0,\infty)$, then*

$$\frac{Z_n(-\infty, b_n x + a_n]}{Z_n(\mathbb{R})} \to \Phi(x) \quad \text{a.s. on } \{Z_n(\mathbb{R}) \to \infty\}, \tag{10.2}$$

*where $a_n = \sum_{i=0}^{n-1}\mu_i$ and $b_n = (\sum_{i=0}^{n-1}\sigma_i^2)^{1/2}$.*

*Remark.* If $\mathbb{E}\int x^2\nu_0(dx) < \infty$, it can be easily seen that $\mathbb{E}\mu_0^2 < \infty$ and $\mathbb{E}\sigma_0^2 < \infty$.

Theorem 10.2 is an extension of the results of Kaplan and Asmussen (1976, II, Theorem 1) and Biggins (1990) on deterministic branching random walks.

Similarly to the case of varying environment, we also have the local limit theorems corresponding to Theorems 10.1 and 10.2 respectively.

**Theorem 10.3.** *Assume that $\nu_0$ is non-lattice a.s., (A) holds with $\{a_n(\xi), b_n(\xi)\}$ satisfying $b_n \sim \theta n^\gamma$ a.s. for some constants $0 < \gamma \le \frac{1}{2}$ and $\theta > 0$, and $g_\xi$ is integrable. If $\upsilon(\varepsilon) < \infty$ for some $\varepsilon > 0$, and*

$$\mathbb{E}\frac{N}{m_0}(\log^+ N)^{1+\beta} < \infty \tag{10.3}$$

*for some $\beta > \gamma$, then $\forall h > 0$,*

$$\sup_{x\in\mathbb{R}} |b_n P_n^{-1} Z_n(x, x+h) - W h p_L(x/b_n - a_n)| \to 0 \qquad a.s.,$$

*where $p_L$ is the density function of $L_\xi$.*

Theorem 10.4 below is a direct consequence of Theorem 10.3. To verify the conditions of Theorem 10.3, see the proof of Theorem 10.2.

**Theorem 10.4** (Local limit theorem for $\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$). *Assume that $\nu_0$ is non-lattice a.s.. If $\mathbb{E}|\mu_0|^\varepsilon < \infty$ for some $\varepsilon > 0$, $\mathbb{E}\sigma_0^2 \in (0, \infty)$, and*

$$\mathbb{E}\frac{N}{m_0}(\log^+ N)^\beta < \infty$$

*for some $\beta > \frac{3}{2}$, then $\forall h > 0$,*

$$\sup_x |b_n \frac{Z_n(x, x+h)}{Z_n(\mathbb{R})} - hp(\frac{x - a_n}{b_n})| \to 0 \quad a.s. \text{ on } \{Z_n(\mathbb{R}) \to \infty\},$$

*where $a_n = \sum_{i=0}^{n-1} \mu_i$, $b_n = (\sum_{i=0}^{n-1} \sigma_i^2)^{1/2}$, and $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density function of standard normal distribution.*

For the deterministic environment case, similar result was showed by Biggins (1990).

From Theorem 10.4, we immediately obtain the following corollary.

**Corollary 10.5.** *Under the conditions of Theorem 10.4, we have $\forall a < b$,*

$$b_n \frac{Z_n(a + a_n, b + a_n)}{Z_n(\mathbb{R})} \to \frac{1}{\sqrt{2\pi}}(b - a) \quad a.s. \text{ on } \{Z_n(\mathbb{R}) \to \infty\},$$

*where $a_n = \sum_{i=0}^{n-1} \mu_i$ and $b_n = (\sum_{i=0}^{n-1} \sigma_i^2)^{1/2}$.*

Corollary 10.5 coincide with a result of Kaplan and Asmussen (1976, II, Theorem 2) on deterministic branching random walks.

## 11 Proofs of Theorems 10.1-10.3

Before of the proof of Theorem 10.1, we prove a lemma at first.

**Lemma 11.1.** *Let $\beta \ge 0$. If $\mathbb{E}\frac{N_0}{m_0}(\log^+ N_0)^{1+\beta} < \infty$, then for all $\kappa$, $\sum_n n^\beta (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ a.s..*

*Proof.* As the proof of Lemma 7.1, we have

$$\sum_n \left(1 - \frac{\tilde{m}_{n,\kappa}}{m_n}\right) = \sum_n \frac{1}{m_n} \mathbb{E}_\xi N_n I_n^c(N_n).$$

By (6.3), for n large enough,

$$\mathbb{E}_\xi N_n I_n^c(N_n) \le \mathbb{E}_\xi N_n 1_{\{N_n(\log N_n)^\kappa > c^{n+1}\}}.$$

Taking expectation for the series $\sum \frac{n^\beta}{m_n}\mathbb{E}_\xi N_n 1_{\{N_n(\log N_n)^\kappa > c^{n+1}\}}$, we have

$$\begin{aligned} & \mathbb{E}\left(\sum_n \frac{n^\beta}{m_n}\mathbb{E}_\xi N_n 1_{\{N_n(\log N_n)^\kappa > c^{n+1}\}}\right) \\ = & \sum_n n^\beta \mathbb{E}\frac{N_0}{m_0} 1_{\{N_0(\log N_0)^\kappa > c^{n+1}\}} \\ = & \mathbb{E}\frac{N_0}{m_0}\sum_n n^\beta 1_{\{N_0(\log N_0)^\kappa > c^{n+1}\}} \\ \leq & C\mathbb{E}\frac{N_0}{m_0}(\log^+ N_0)^{1+\beta} < \infty, \end{aligned}$$

so that $\sum_n n^\beta(1-\tilde{m}_{n,\kappa}/m_n) < \infty$ a.s.. □

*Proof of Theorem 10.1.* From the proof of Theorem 6.1, we know that in fact, instead of (6.8), we only need (7.6) and $\sum_n(1-\tilde{m}_{n,\kappa}/m_n) < \infty$ for the suitable $\kappa$. For the branching random walk in a stationary and ergodic random environment, Lemma 11.1 tells us that the condition $\mathbb{E}\frac{N_0}{m_0}\log^+ N_0 < \infty$ ensures $\sum_n(1-\tilde{m}_{n,\kappa}/m_n) < \infty$. And it also ensures (7.6), since for any $\delta_1 > 0$,

$$\mathbb{E}\left(\sum_n \frac{1}{m_n n^{1+\delta_1}}\mathbb{E}_\xi N_n \log^+ N_n\right) = \sum_n \frac{1}{n^{1+\delta_1}}\mathbb{E}\frac{N_0}{m_0}\log^+ N_0 < \infty.$$

By the ergodic theorem,

$$\lim_n \frac{\upsilon_n(\varepsilon)}{n} = \upsilon(\varepsilon) < \infty \qquad a.s..$$

Hence the condition (6.9) holds. Thus Theorem 10.1 is just a direct consequence of Theorem 6.1. □

*Proof of Theorem 10.2.* We will use Theorem 10.1 to prove Theorem 10.2. Assume that $0 < \varepsilon \leq 2$ (otherwise, consider $\min\{\varepsilon, 2\}$ instead of $\varepsilon$), then

$$\upsilon(\varepsilon) = \mathbb{E}\int |x|^\varepsilon \nu_0(dx) \leq C_\varepsilon\left(\mathbb{E}\int |x-\mu_0|^\varepsilon \nu_0(dx) + \mathbb{E}|\mu_0|^\varepsilon\right) < \infty.$$

By (9.3), $b_n \sim \mathbb{E}\sigma_0^2\sqrt{n}$ a.s., which implies that for any $0 < \gamma < \frac{1}{2}$, $b_n^{-1} = o(n^{-\gamma})a.s.$. The proof of Theorem 9.1 show that $\{\nu_n\}$ satisfies Lindeberg condition, so that (A) holds with $a_n' = a_n/b_n$ and $b_n' = b_n$. By Theorem 10.1,

$$P_n^{-1}Z_n(-\infty, b_n x + a_n] \to \Phi(x)W \quad a.s.$$

Notice that $Z_n(\mathbb{R})/P_n \to W$ a.s. and $\mathbb{P}(W>0) = \mathbb{P}(Z_n(\mathbb{R}) \to \infty)$. Thus (10.2) holds. □

**Lemma 11.2.** *Let $A > 0$ be a constant. Assume that $b_n \sim \theta n^\gamma a.s.$ for some constants $0 < \gamma \leq \frac{1}{2}$. If $\nu_0$ is non-lattice a.s.,then there exists a constant $\theta_1 > 0$ (not depending on A) such that*

$$\limsup_{n\to\infty} b_n \int_U \prod_{i=k}^{n-1}|\phi_i(t)|dt \leq \int_{|t|\geq A} e^{-\theta_1 t^2}dt \quad a.s., \tag{11.1}$$

*where $k = J(n)$ the same as the proof of Theorem 6.1 and $U = \{t : \frac{A}{b_n} \leq |t| \leq \frac{1}{a}\}$.*

*Proof.* Take $0 < 2\epsilon < \frac{1}{a}$. Like the last part of the proof of Theorem 6.2, split $U$ into $U_1$ and $U_2$, so

$$b_n\int_U \prod_{i=k}^{n-1}|\phi_i(t)|dt = b_n\int_{U_1}\prod_{i=k}^{n-1}|\phi_i(t)|dt + b_n\int_{U_2}\prod_{i=k}^{n-1}|\phi_i(t)|dt.$$

Since $\nu_i$ is non-lattice a.s., we have

$$\sup_{\epsilon \le |t| \le a^{-1}} |\phi_i(t)| =: c_i(\epsilon, a) = c_i < 1 \qquad a.s.. \tag{11.2}$$

Hence by Lemma 8.1, for $|t| < \epsilon$,

$$|\phi_i(t)| \le 1 - \frac{1-c_i^2}{8\epsilon^2} t^2 \le \exp(-\frac{1-c_i^2}{8\epsilon^2} t^2) = e^{-\alpha_i t^2} \qquad a.s., \tag{11.3}$$

where $\alpha_i = \frac{1-c_i^2}{8\epsilon^2} > 0$ a.s.. Using (11.2), we immediately get

$$b_n \int_{U_2} \prod_{i=k}^{n-1} |\phi_i(t)| \le \frac{2}{a} b_n \prod_{i=k}^{n-1} c_i \to 0 \qquad a.s., \tag{11.4}$$

since

$$\lim_{n\to\infty} \frac{\log b_n + \sum_{i=k}^{n-1} \log c_i}{n} = \mathbb{E} \log c_0 < 0 \qquad a.s..$$

Observe that

$$\lim_{n\to\infty} \frac{\sum_{i=k}^{n-1} \alpha_i}{b_n^2} = \begin{cases} \frac{1}{\theta^2}\mathbb{E}\alpha_0 > 0 & \text{if } \gamma = \frac{1}{2} \\ \infty & \text{if } 0 < \gamma < \frac{1}{2} \end{cases} \qquad a.s..$$

Take $0 < \theta_1 < \frac{1}{\theta^2}\mathbb{E}\alpha_0$. Using (11.3), we have for $n$ large,

$$b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t|\ge A} \exp(-b_n^{-2} \sum_{i=k}^{n-1} \alpha_i t^2) du \le \int_{|t|\ge A} e^{-\theta_1 t^2} dt \qquad a.s.. \tag{11.5}$$

(11.4) and (11.5) yield (11.1). □

*Proof of Theorem 10.3.* In the proof of Lemma 8.2, the condition (6.14) is just used to ensure (8.5) (i.e.(11.1) in random environment), which always holds in random environment if $\nu_i$ is non-lattice a.s., by Lemma 11.2. Besides,

$$\mathbb{E}\left(\sum_n \frac{1}{m_n n (\log n)^{1+\delta}} \mathbb{E}_\xi N_n (\log^+ N_n)^{1+\beta}\right) = \sum_n \frac{1}{n(\log n)^{1+\delta}} \mathbb{E}\frac{N_0}{m_0} (\log^+ N_0)^{1+\beta} < \infty.$$

So (10.3) implies (6.15). Theorem 10.3 is a consequence of Theorem 6.2. □

## 12 Central limit theorems for $\mathbb{E}_\xi \frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$ and $\mathbb{E}\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$

From Theorem 10.2, it is not hard to obtain the following central limit theorems for the probability measures $\mathbb{E}_\xi(\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}|Z_n(\mathbb{R}) > 0)$ and $\mathbb{E}(\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}|Z_n(\mathbb{R}) > 0)$:

**Theorem 12.1** (Central limit theorems for $\mathbb{E}_\xi \frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$ and $\mathbb{E}\frac{Z_n(\cdot)}{Z_n(\mathbb{R})}$)**.** *If $\mathbb{E}|\mu_0|^\varepsilon < \infty$ for some $\varepsilon > 0$ and $\mathbb{E}\sigma_0^2 \in (0, \infty)$, then*

$$\mathbb{E}_\xi \left( \frac{Z_n(-\infty, b_n x + a_n]}{Z_n(\mathbb{R})} \middle| Z_n(\mathbb{R}) > 0 \right) \to \Phi(x) \qquad a.s., \tag{12.1}$$

$$\mathbb{E} \left( \frac{Z_n(-\infty, b_n x + a_n]}{Z_n(\mathbb{R})} \middle| Z_n(\mathbb{R}) > 0 \right) \to \Phi(x), \tag{12.2}$$

*where $a_n = \sum_{i=0}^{n-1} \mu_i$ and $b_n = (\sum_{i=0}^{n-1} \sigma_i^2)^{1/2}$.*

*Proof.* Theorem 12.1 is a consequence of Theorem 10.2. We only prove (12.2), the proof for (12.1) is similar. By Theorem 10.2,

$$\left(\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} - \Phi(x)\right) 1_{\{Z_n(\mathbb{R})\to\infty\}} \to 0 \quad a.s.. \tag{12.3}$$

The condition $\mathbb{E}\frac{N}{m_0}\log^+ N < \infty$ ensures that

$$\lim_{n\to\infty} \mathbb{P}(Z_n(\mathbb{R}) > 0) = \mathbb{P}(Z_n(\mathbb{R}) \to \infty) > 0.$$

Observing that

$$\begin{aligned}
&\left|\mathbb{E}\left(\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})}\middle| Z_n(\mathbb{R}) > 0\right) - \Phi(x)\right| \\
= & \frac{1}{\mathbb{P}(Z_n(\mathbb{R}) > 0)}\left|\mathbb{E}1_{\{Z_n(\mathbb{R})>0\}}\left(\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} - \Phi(x)\right)\right| \\
\leq & \frac{1}{\mathbb{P}(Z_n(\mathbb{R}) > 0)}\left|\mathbb{E}\left(1_{\{Z_n(\mathbb{R})>0\}} - 1_{\{Z_n(\mathbb{R})\to\infty\}}\right)\left(\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} - \Phi(x)\right)\right| \\
& + \frac{1}{\mathbb{P}(Z_n(\mathbb{R}) > 0)}\left|\mathbb{E}1_{\{Z_n(\mathbb{R})\to\infty\}}\left(\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} - \Phi(x)\right)\right|,
\end{aligned}$$

we only need to show that the two terms in the right side of the inequality above tend to zero as $n$ tends to infinity. Since

$$0 \leq \frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} \leq 1 \quad \text{and} \quad 0 \leq \Phi(x) \leq 1,$$

we have

$$\left|\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} - \Phi(x)\right| \leq 1.$$

Notice (12.3), by the dominated convergence theorem, we get

$$\left|\mathbb{E}1_{\{Z_n(\mathbb{R})\to\infty\}}\left(\frac{Z_n(Z_n(-\infty, b_nx + a_n])}{Z_n(\mathbb{R})} - \Phi(x)\right)\right| \to 0.$$

For the first term, we have

$$\begin{aligned}
&\left|\mathbb{E}(1_{\{Z_n(\mathbb{R})>0\}} - 1_{\{Z_n(\mathbb{R})\to\infty\}})\left(\frac{Z_n(-\infty, b_nx + a_n]}{Z_n(\mathbb{R})} - \Phi(x)\right)\right| \\
\leq & \mathbb{E}|1_{\{Z_n(\mathbb{R})>0\}} - 1_{\{Z_n(\mathcal{R})\to\infty\}}| \\
= & \mathbb{P}(Z_n(\mathbb{R}) > 0) - \mathbb{P}(Z_n(\mathbb{R}) \to \infty) \to 0.
\end{aligned}$$

This completes the proof. □

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