A time reversal duality for branching processes and applications

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Outline

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Birth-death (BD) process

Individuals
- have i.i.d. life durations \( \sim \text{Exp}(d) \)
- reproduce at constant rate \( b \) during their life
- behave independently from one another
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We consider for a fixed time $T$:

- $\mathcal{T}$: the BD tree starting from one ancestor
- $\mathcal{T}^{(T)}$: the BD tree truncated up to time $T$
- $(\xi_t (\mathcal{T}), t \geq 0)$: the population size process
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Random forests

Forest $\mathcal{F}$:
A finite sequence of i.i.d BD trees $(\mathcal{T}_1, \ldots, \mathcal{T}_n)$
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For any forest $\mathcal{F}$, the population size process is denoted by,

$$(\xi_t(\mathcal{F}), t \geq 0)$$
Time-reversal duality

Fix $b \geq d$

$\mathcal{F}^*: = \text{Supercritical}\ (b, d)$
Time-reversal duality

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$F^* := \text{Supercritical } (b, d)$  \hspace{1cm} \tilde{F}^* := \text{Subcritical } (d, b)$
Time-reversal duality

Fix \( b \geq d \)

\[ F^* := \text{Supercritical} \ (b, d) \quad \quad \quad \quad \tilde{F}^* := \text{Subcritical} \ (d, b) \]

[Athreya and Ney 1972]

A supercritical BD process conditioned to die out is a subcritical BD process, obtained by swapping birth and death rates.
Time reversal for birth-death processes

Time-reversal duality

Fix $b \geq d$

$F^* := \text{Supercritical } (b, d) \quad \tilde{F}^* := \text{Subcritical } (d, b)$

[Athreya and Ney 1972]

A supercritical BD process conditioned to die out is a subcritical BD process, obtained by swapping birth and death rates.

Theorem

We have the following identity in distribution,

$\left(\xi_{T-t} (F^*), 0 \leq t \leq T\right) \overset{d}{=} \left(\xi_t (\tilde{F}^*), 0 \leq t \leq T\right)$
Time reversal duality
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Conditional on the reduced tree: applications in epidemiology

We want to characterize the population size process conditional on the coalescence times between individuals at present time $T$. 
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Conditional on the reduced tree: applications in epidemiology

When we return the time, thanks to the duality property, coalescence times become life durations.
Conditional on the reduced tree: applications in epidemiology

Idea:

The population size process conditional on the coalescence times to be $t_1, \ldots, t_{\tilde{N}+1}$, backward in time, is that of a sum of $\tilde{N}$ BD trees, each conditioned on dying out before $t_i$ for $1 \leq i \leq \tilde{N}$, plus an additional tree conditioned on surviving up until time $T$. 
Generalization for splitting trees

Splitting trees

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A splitting tree is characterized by a $\sigma$-finite measure $\Pi$ on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge r) \Pi(dr) < \infty$$

(the lifespan measure).
Splitting trees

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\[
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We consider $\Pi$ finite with mass $b$: individuals give birth at rate $b$ and have life durations distributed as $\Pi(\cdot)/b$. 
Define for $\Pi$:

- The Laplace exponent: $\psi(\lambda) := \lambda - \int_0^\infty (1 - e^{-\lambda r}) \Pi(dr), \; \lambda \geq 0$
- $\eta$ the largest root of $\psi$
- A new measure $\tilde{\Pi}(dr) := e^{-\eta r} \Pi(dr)$
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The scale function $W$:

The unique continuous function $W : [0, +\infty) \to [0, +\infty)$, characterized by its Laplace transform,

$$\int_0^+ \lambda x W(x) = \frac{1}{\psi(\lambda)}, \quad \lambda > \eta$$
Generalization for splitting trees

Time reversal duality for splitting trees

Define for Π:

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- \( \eta \) the largest root of \( \psi \)
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The unique continuous function \( W : [0, +\infty) \to [0, +\infty) \), characterized by its Laplace transform,

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\int_0^{+\infty} e^{-\lambda x} W(x) = \frac{1}{\psi(\lambda)}, \quad \lambda > \eta
\]

Define:

\[
\gamma = \frac{1}{W(T)} \quad \tilde{\gamma} = \frac{1}{\tilde{W}(T)}
\]
Forest $\mathcal{F}^p$:

A sequence of i.i.d. splitting trees $(T_1, \ldots, T_{N_p}, T_{N_p+1}) \perp N_p$, where,

- $N_p$: a geometric random variable with $\mathbb{P}(N_p = k) = (1 - p)^k p$, $k \geq 0$
- $T_1, \ldots, T_{N_p}$: are conditioned on extinction before $T$
- $T_{N_p+1}$: is conditioned on survival up until time $T$
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$\mathcal{F}_T^p, \mathcal{F}_\perp^p$:

$\sim \mathcal{F}^p$, but lifetimes of the ancestors have a specific distribution $(\top, \perp)$, $\neq$ from $\Pi(\cdot)/b$
Generalization for splitting trees

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$\mathcal{F}^p_{\top}, \mathcal{F}^p_{\perp}$:

$\sim \mathcal{F}^p$, but lifetimes of the ancestors have a specific distribution $(\top, \perp) \neq \Pi(\cdot)/b$

Claim

$\mathcal{F}^\gamma_{\perp} = $ a sequence of i.i.d. splitting trees $(\perp, \Pi)$ stopped at the first tree having survived up to time $T$.

$\tilde{\mathcal{F}}^\gamma_{\top} = $ a sequence of i.i.d. splitting trees $(\top, \tilde{\Pi})$ stopped at the first tree having survived up to time $T$. 
Claim

\( \mathcal{F}_\perp \) = a sequence of i.i.d. splitting trees \((\perp, \Pi)\) stopped at the first tree having survived up to time \(T\).

\( \tilde{\mathcal{F}}_\gamma \) = a sequence of i.i.d. splitting trees \((\top, \tilde{\Pi})\) stopped at the first tree having survived up to time \(T\).

Theorem

If the measure \(\Pi\) is supercritical (i.e. \(m := \int_0^\infty r \Pi(dr) > 1\)) then,

\[
(\xi_{T-t} (\mathcal{F}_\perp), 0 \leq t \leq T) \overset{d}{=} (\xi_t (\tilde{\mathcal{F}}_\gamma), 0 \leq t \leq T)
\]

In particular, if \(\Pi\) is subcritical, then,

\[
(\xi_{T-t} (\mathcal{F}_\perp), 0 \leq t \leq T) \overset{d}{=} (\xi_t (\mathcal{F}_\gamma), 0 \leq t \leq T)
\]

and actually in this case \(\perp = \top\) since they have both density \(\frac{\tilde{\Pi}(r)}{m} dr\).
Ingredients of the proof

The jumping chronological contour process

Example of a finite splitting tree and its contour process

\(^1\)Figure from C. Delaporte - Aussois 2013
The jumping chronological contour process

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Example of a finite splitting tree and its contour process\(^1\)

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Example of a finite splitting tree and its contour process\textsuperscript{1}

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Example of a finite splitting tree and its contour process

Figure from C. Delaporte - Aussois 2013
Let $Y$ be a a finite variation Lévy process with Lévy measure $\Pi$ and drift -1.

**Theorem [Lambert 2010]**

Conditional on the lifespan of the ancestor to be $x$, the contour of $\mathcal{T}(T)$, is distributed as $Y$, started at $x \land T$, reflected below $T$ and killed upon hitting 0.

The contour of $\mathcal{T}$, conditional on extinction, has the law of $Y$ started at $x$, conditioned on, and killed upon hitting 0.
Time reversal duality for spectrally positive Lévy processes

Theorem [Bertoin 1992]

The excursion measure has the following property of invariance under time reversal: under $\mathbb{P}_0 \left( \cdot \bigg| -Y_{(\tau_0^+)-} = u \right)$ the reverted excursion, $(-Y_{(\tau_0-t)-}, 0 \leq t < \tau_0)$ has the same distribution that $(Y_t, 0 \leq t < \tau_0)$ under $\mathbb{P}_u \left( \cdot \bigg| \tau_0 < +\infty \right)$. 
Ingredients of the proof

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![Diagram of time reversal duality](image)

**Undershoot**
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Time reversal duality for splitting trees

Define for a Lévy measure $\Pi$:

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- $\eta$ the largest root of $\psi$
- A new measure $\tilde{\Pi}(dr) := e^{-\eta r} \Pi(dr)$
- $\tau_A = \{ t \geq 0 : Y_t \in A \}$ the first hitting time of the real Borel set $A$
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The scale function $W$:

The unique continuous function $W : [0, +\infty) \to [0, +\infty)$, characterized by its Laplace transform, satisfies

$$\int_0^{+\infty} e^{-\lambda x} W(x) = \frac{1}{\psi(\lambda)} , \; \lambda > \eta$$
Ingredients of the proof

**Time reversal duality for splitting trees**

**Define for a Lévy measure $\Pi$:**

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**Define:**

$$\gamma = \frac{1}{W(T)} = P_T (\tau_0 < \tau_T^+)$$

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Ingredients of the proof

Contour of a forest
Ingredients of the proof

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Contour of a forest
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Contour of a forest
Contour of a forest

Undershoot
Ingredients of the proof

Contour of a forest
References I

K.B. Athreya and P.E. Ney
Branching Processes
*Springer-Verlag, New York*, Band 196. MR0373040.

J. Bertoin
An Extension of Pitman’s Theorem for Spectrally Positive Levy Processes

A. Lambert
The contour of splitting trees is a Lévy process.
Thank You!
Undershoot and overshoot

\( \mathcal{F}_P^\top, \mathcal{F}_P^\perp: \)

Lifetimes of the ancestors have a specific distribution, different from \( \Pi(\cdot)/b: \)

The undershoot and overshoot at 0 of an excursion starting at 0 and conditional on \( \tau_0^+ < +\infty, \) are distributed as follows,

- **Overshoot (\( \perp \)):**
  \[
  \sim e^{\eta r} \frac{\bar{\Pi}(r) \, dr}{m \land 1}
  \]

- **Undershoot (\( \top \)):**
  \[
  \sim e^{-\eta a} \frac{\bar{\Pi}(a) \, da}{m \land 1}
  \]