Small positive values for supercritical branching processes in random environment

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Behavior of $\mathbb{P}(Z_n = k)$ as $n \to \infty$, with $k \geq 1$

Let Z_n be a GW process with reproduction law specified by the p.g.f f. Then

$$\mathbb{E}(s^{Z_n}) = f^{\circ n}(s) \qquad (s \in [0,1])$$

In the supercritical case ($f'(1) > 1$),

$$\mathbb{P}(Z_n \to \infty) > 0, \quad \mathbb{P}(Z_n \to \infty \text{ or } \exists n \in \mathbb{N} : Z_n = 0) = 1$$

What about

$$\{Z_n = k\} \qquad k = 1, \ldots$$

and its probability $f_n^{(k)}(0)/k$?

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Proofs of the asymptotic behavior $\mathbb{P}_1(Z_n = 1)$

- easy if $\mathbb{P}_1(Z_1 = 0) = 0$ ($p_e = 0$)
- analytical proofs [Athreya,Ney 70s]
- The reduced tree (i.e. keeping only the survival branches) of a supercritical GW is a supercritical GW (without extinction !) [see e.g. Peres Lyons's book]
- spine decomposition [Lyons Peres Pemantle 95, Geiger 99]
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Conclusion : if $\mathbb{P}_1(Z_1 = 1) > 0$

$$\mathbb{P}_1(Z_n = 1) \sim cf'(p_e)^n \qquad (n \to \infty)$$

where $p_e = \mathbb{P}(\exists n : Z_n = 0) = \inf\{s \in [0,1] : f(s) = s\}$

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Motivations for random environments

- To evaluate the number $N_n(k)$ of infected cells with k parasites in Kimmel's branching model

$$N_n(k) \sim 2^n \mathbb{P}(Z_n = k) \qquad (n \to \infty)$$

where Z_n is a branching process in random environment.

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Branching processes in random environment (BPRE) generalize Galton Watson processes [Smith, Wilkinson 69] :

In each generation, one pick in an i.i.d. manner an environment which gives the reproduction law of each individual.

Description of a BPRE $(Z_n)_{n\geq 0}$

Now, in each generation, we pick randomly an environment in an i.i.d. manner :

$$\mathcal{E}_i = \text{environment in generation } i.$$

The reproduction law in environment e is given by the r.v. N_e :

$$f_e(s) := \mathbb{E}(s^{N(e)}), \qquad m(e) := \mathbb{E}(N(e)) = f_e'(1).$$

For every $n \in \mathbb{N}$, conditionally on

$$\mathcal{E}_n = e,$$

we have

$$Z_{n+1} = \sum_{i=1}^{Z_n} N_i,$$

where $(N_i)_{i\in\mathbb{N}}$ are i.i.d. r.v. distributed as $N(e)$.

Z becomes extincted a.s. iff $\mathbb{E}[\log(m(\mathcal{E}))] \leq 0$. [Athreya, Karlin 71].

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Supercritical BPRE with $\mathbb{P}(Z_1 = 0) > 0$

Let us note $Q_{\mathcal{E}}$ the random reproduction law, i.e. the law of $N(\mathcal{E})$,

$$\mathcal{I} := \{j \geq 1 \ : \ \mathbb{P}(Q_{\mathcal{E}}(j) > 0, Q_{\mathcal{E}}(0) > 0) > 0\}$$

and introduce the set $Cl(\mathcal{I})$ of integers which can be reached from $\mathcal{I}$ by Z :

$$Cl(\mathcal{I}) := \{k \geq 1 : \exists n \geq 0 \text{ and } j \in \mathcal{I} \text{ with } \mathbb{P}_j(Z_n = k) > 0\}.$$

Finally, the reproduction between generation i and n is given by

$$f_{i,n} = f_{\mathcal{E}_i} \circ \cdots \circ f_{\mathcal{E}_{n-1}}$$

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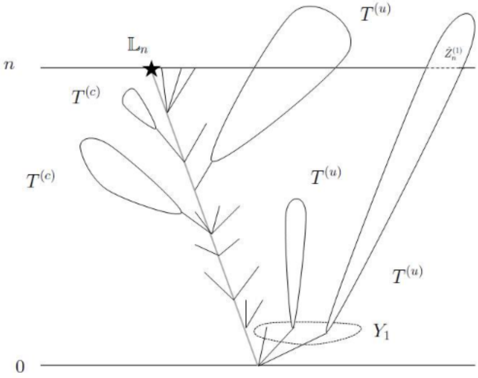
Theorem

The following limits exist and coincide for all $k, j \in Cl(\mathcal{I})$,

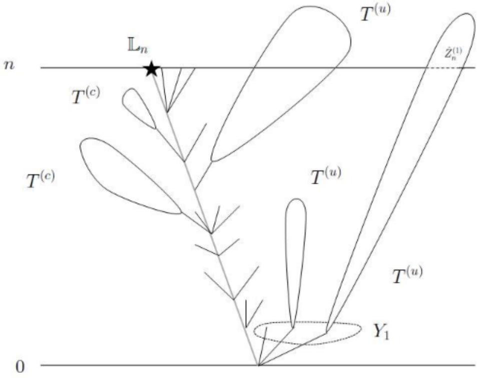
$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[f_{0,n}(0)^{z_0 - 1} \Pi_{i=1}^{n-1} f'_{\mathcal{E}_i}\left(f_{i+1,n}(0)\right)\right]$$

where z_0 is the smallest element of $\mathcal{I}$.
This common limit is denoted ϱ and $\varrho \in (0, \infty)$.



[Geiger 99] construction with $T^{(c)}$ trees conditioned on extinction and $T^{(u)}$ unconditioned trees.

To get $\rho > 0$, we use an estimation of $f_{i,n}(0)$ due to Agresti, which gives a lower bound using the random walk $S_n = \sum_{i=0}^{n-1} \log m(\mathcal{E}_i)$



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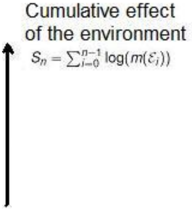

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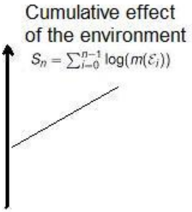

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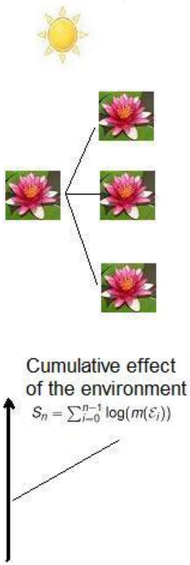

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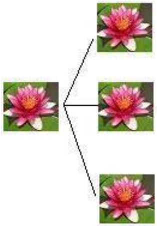

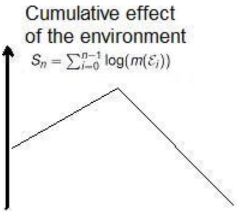

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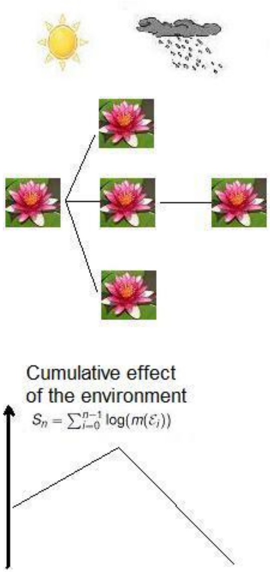

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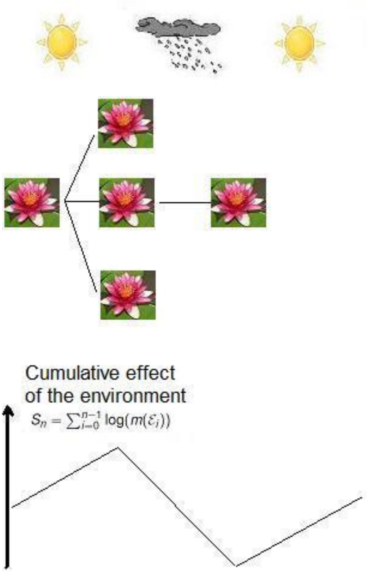

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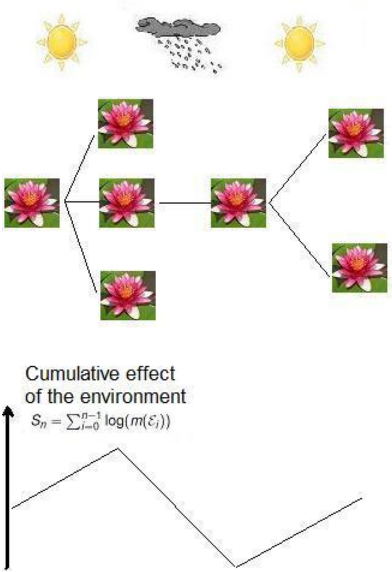

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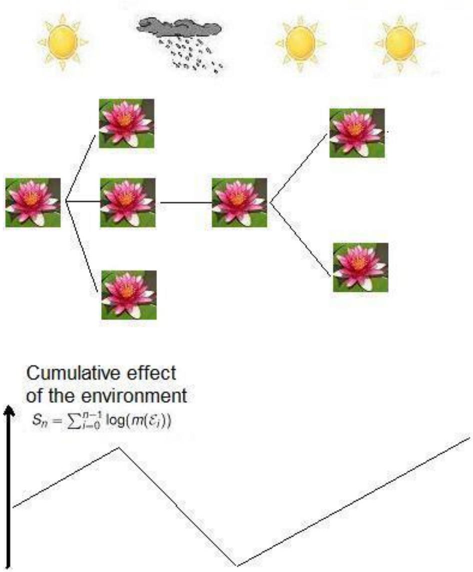

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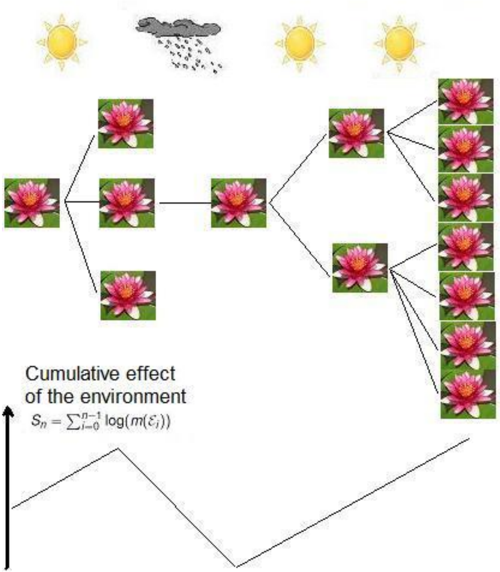

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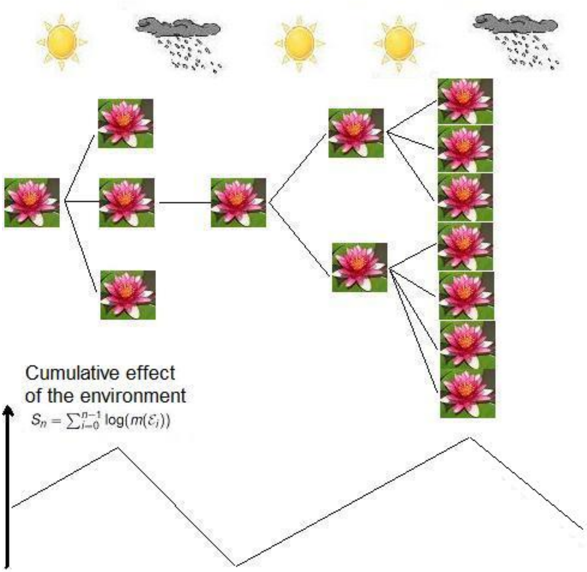

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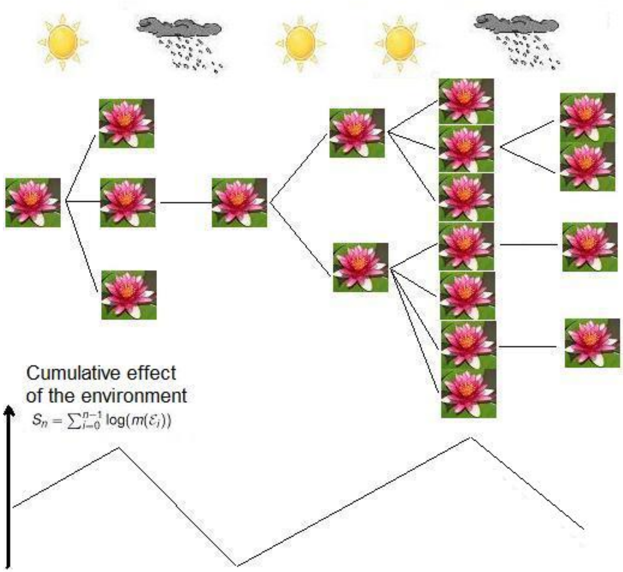

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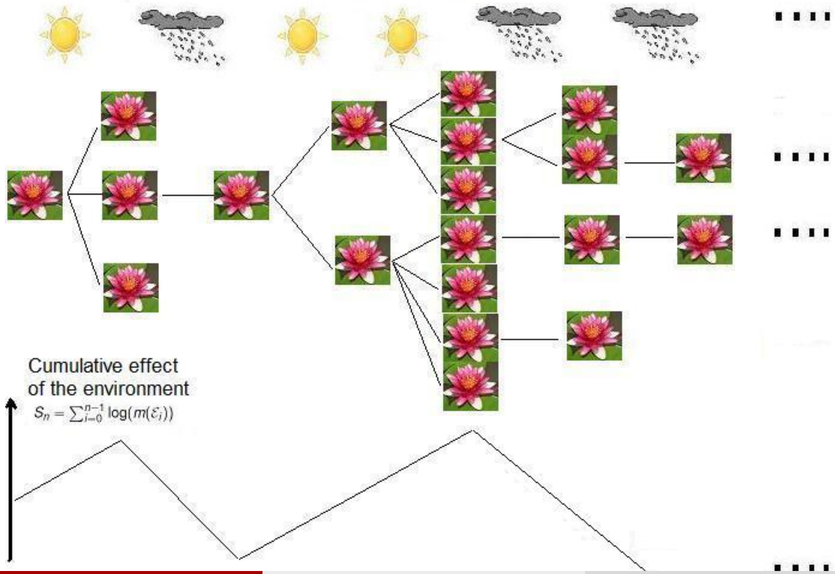

What about environmental stochasticity ?

We note $S_n = \sum_{i=0}^{n-1} \log m(\mathcal{E}_i)$ and

$$\mathbb{E}(Z_n \mid \mathcal{E}_0, \cdots \mathcal{E}_{n-1}) = \Pi_{i=0}^{n-1} m(\mathcal{E}_i) = \exp(S_n)$$

and

$$\Lambda(x) = \sup\{tx - \log \mathbb{E}(\exp(tX)) : t \in \mathbb{R}\}$$

Proposition (Environmental stochasticity scenario)

If truncated moment assumption (or $\mathbb{P}(m(\mathcal{E}) \geq 1) = 1$) is fulfilled, then

$$\rho \leq \Lambda(0)$$

Informally : we focus on $\{S_n \approx 0\}$, so the probability that the population survives without tending to ∞ decreases polynomially.
Proof : Change of probability so that the r.w. S becomes critical and

$$\{1 \leq Z_n \leq k_n\}" \supset " \{\min_{i=0 \cdots n-1} S_i \geq 0, S_n \leq C\}$$

for k_n not growing fast.

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Some special class of reproduction laws

We recall that a probability generating function is linear fractional (LF) if there exist positive real numbers m and b such that

$$f(s) = 1 - (1-s)/(m^{-1} + bm^{-2}(1-s)/2).$$

The good news

- This family of p.g.f is stable by composition
- $z_0 = 1$
- it is representative ?

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Supercritical regimes

Theorem

If $N(\mathcal{E})$ is a.s. linear fractional, then for every $k \geq 1$, $-\varrho$ is

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n = k) = -\varrho = \begin{cases} \log \mathbb{E}\left[m(\mathcal{E})^{-1}\right] & \text{, if } \mathbb{E}[\log(m(\mathcal{E}))/m(\mathcal{E})] \geq 0 \\ -\Lambda(0) & \text{, else} \end{cases}$$

Theorem (Dekking 88 ; D'Souza, Hambly 97 ; Guivarc'h, Liu 01 ; Geiger, Kersting, Vatutin 03)

In the subcritical case, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n > 0) = \begin{cases} \log \mathbb{E}\left[m(\mathcal{E})\right] & , \text{ if } \mathbb{E}[\log(m(\mathcal{E}))m(\mathcal{E})] \leq 0 \\ -\Lambda(0) & , \text{ else} \end{cases}.$$

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$MRCA_n$= most recent common ancestor of individuals living at time n.

Proposition

(i) If $\mathbb{E}[\log(m(\mathcal{E}))/m(\mathcal{E})] > 0$*, then for every* $\delta \in (0,1]$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(MRCA_n > \delta n | Z_n = 2) < 0.$$

(ii) If $\mathbb{E}[\log(m(\mathcal{E}))/m(\mathcal{E})] < 0$*, then*

$$\liminf_{n\to\infty} \mathbb{P}_1(MRCA_n = n | Z_n = 2) > 0 \ ; \ \liminf_{n\to\infty} \mathbb{P}_1(MRCA_n = 1 | Z_n = 2) > 0.$$

(iii) If $\mathbb{E}[\log(m(\mathcal{E}))/m(\mathcal{E})] = 0$*, then for every sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in [1,n]$*, we have*

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(iii) If $\mathbb{E}[\log(m(\mathcal{E}))/m(\mathcal{E})] = 0$, *then for every sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in [1,n]$, *we have*

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$MRCA_n$= most recent common ancestor of individuals living at time n.

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Now...lower larger deviations

In the supercritical regime, on the survival event, with $N(\mathcal{E}) \log N(\mathcal{E})$ moment assumption [Athreya Karlin 71]

$$Z_n \sim W \exp(S_n) \approx \exp(\mathbb{E}(\log m(\mathcal{E})n)) \quad n \to \infty, \quad W > 0$$

Let us now focus on

$$\{0 < Z_n \leq \exp(\theta n)\}$$

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case without extinction, with J. Berestycki 09

Here, $\mathbb{P}(Z_1 = 0) = 0$ and Z grows a.s.

Theorem

If the mean and variance of reproduction law are bounded a.s.

$$\frac{1}{n}\log \mathbb{P}(0 < Z_n \leq \exp(\theta n)) \overset{n\to\infty}{\longrightarrow} -\chi(\theta)$$

where

$$\chi(\theta) = \inf_{t\in[0,1]}\{-t\log(\mathbb{P}_1(Z_1 = 1)) + (1-t)\Lambda(c/(1-t))\}.$$

+ uniform dimensional convergence of the trajectory

case with possible extinction, with C. Boeinghoff

Theorem

Moment assumptions about the mean offspring.

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$$\chi(\theta) = \inf_{t\in[0,1]} \{t\rho + (1-t)\Lambda(\theta/(1-t))\}.$$

+ finite dimensional convergence of the trajectory

...and cell infection model

In Kimmel's (general) branching model

- the cell divides in discrete time and the population is a binary tree.
- the parasites population grows inside the cells following a Galton Watson process
- the parasites are shared randomly in the two daughter's cells (for example, by a binomial repartition with a random parameter P picked in a iid manner for every cell)

The number of parasites in a random cell line is a BPRE (a GW process iff $P = 1/2$ a.s).
Motivations come from experiments in TaMaRa's laboratory, which note a strong asymmetry.
A random environment (in time) can be added (for growth and sharing).

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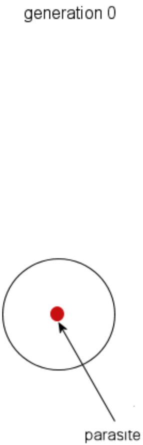
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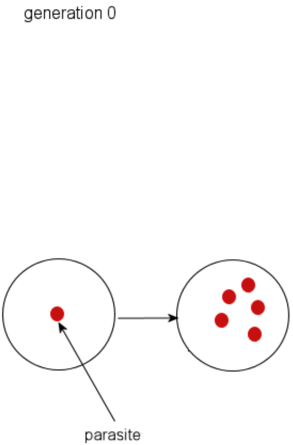
parasite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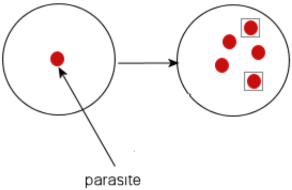

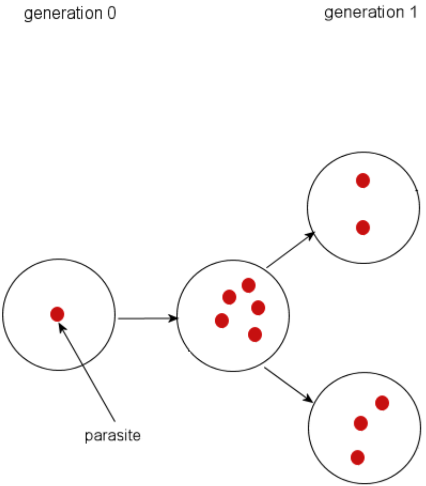
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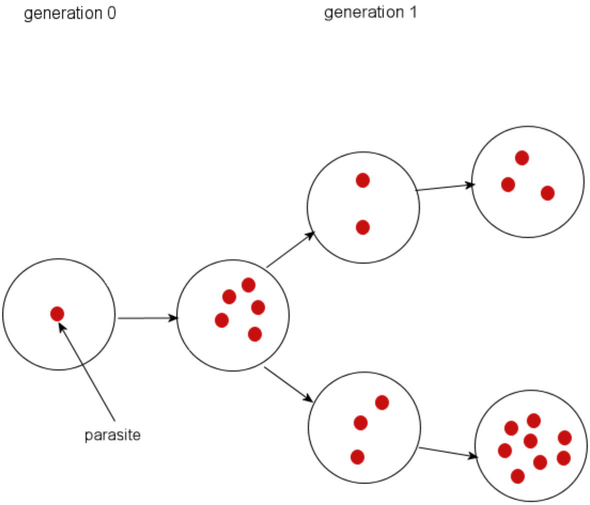
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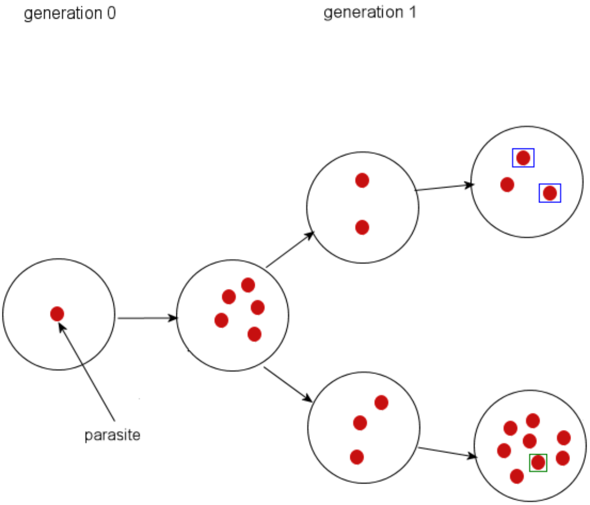
generation 0
generation 1
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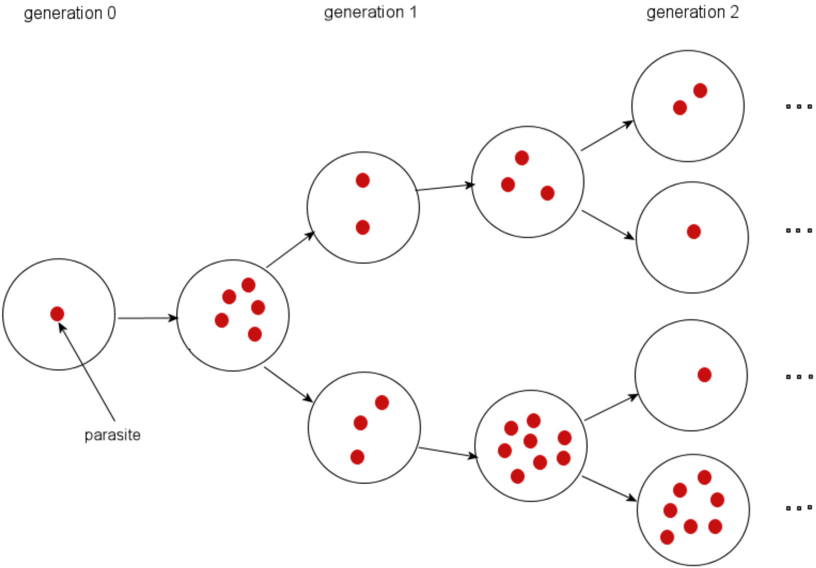
generation 0
generation 1
generation 2
parasite
...
...
...
...

Counting cells...

Let us call G_n the cells in generation n and

$$N_n(A) = \#\{i \in G_n : Z_i \in A\}$$

the number of cells with k parasites in generation n. Then

$$\mathbb{E}(N_n(A)) = 2^n \mathbb{P}(Z_n \in A)$$

and in particular

$$\frac{1}{n} \log \mathbb{E} N_n[\exp(n\theta), \infty) \stackrel{n\to\infty}{\longrightarrow} 2 - \chi(\theta)$$

$$\frac{1}{n} \log \mathbb{E} N_n\{k\} \stackrel{n\to\infty}{\longrightarrow} 2 - \varrho$$

-> The two stochasticities of the model (growth and sharing) appear along the lineage (separately or combined).

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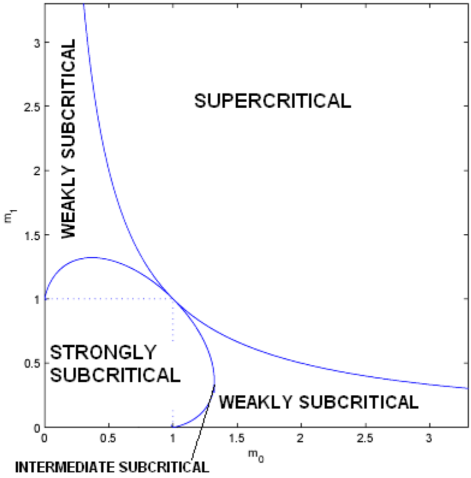
Different regimes for the cell infection

$$P = p \text{ a.s.} \qquad m_0 = mp \qquad m_1 = m(1-p)$$



Conclusion Rate function χ for LD of BPRE

- Kozlov [06, Discrt. Math. Appl.] : geometric offspring distributions, upper rate function $\chi(\theta) = \Lambda(\theta)$.
- B. & Beresticky [09, MPRF] : $\mathbb{P}(Z_1 = 0) = 0$, lower rate function : $\chi(c) = \inf_{t\in[0,1]}\{-t\log(\mathbb{P}_1(Z_1 = 1)) + (1-t)\Lambda(c/(1-t))\}$.
- Kersting & Boeinghoff [10, SPA] : Geometric tail offspring distribution upper rate function $\chi(\theta) = \inf_{t\in[0,1]}\left\{t\gamma + (1-t)\Lambda((\theta-u)/(1-t))\right\}$
- Kozlov [10, TPA] : Geometric offspring distributions. Finer estimates for upper large deviations.
- B. & Boeinghoff [11, EJP] : Possible heavy tails, upper rate function $\chi(\theta) = \inf_{t\in[0,1],u\in[0,\theta]}\left\{t\gamma + \beta u + (1-t)\Lambda((\theta-u)/(1-t))\right\}$
- B. & Boeinghoff [12] : lower large deviations and probability to stay bounded without extinction $\chi(\theta) = \inf_{t\in[0,1]}\{t\varrho + (1-t)\Lambda(\theta/(1-t))\}$.