The total external length in the evolving Kingman coalescent

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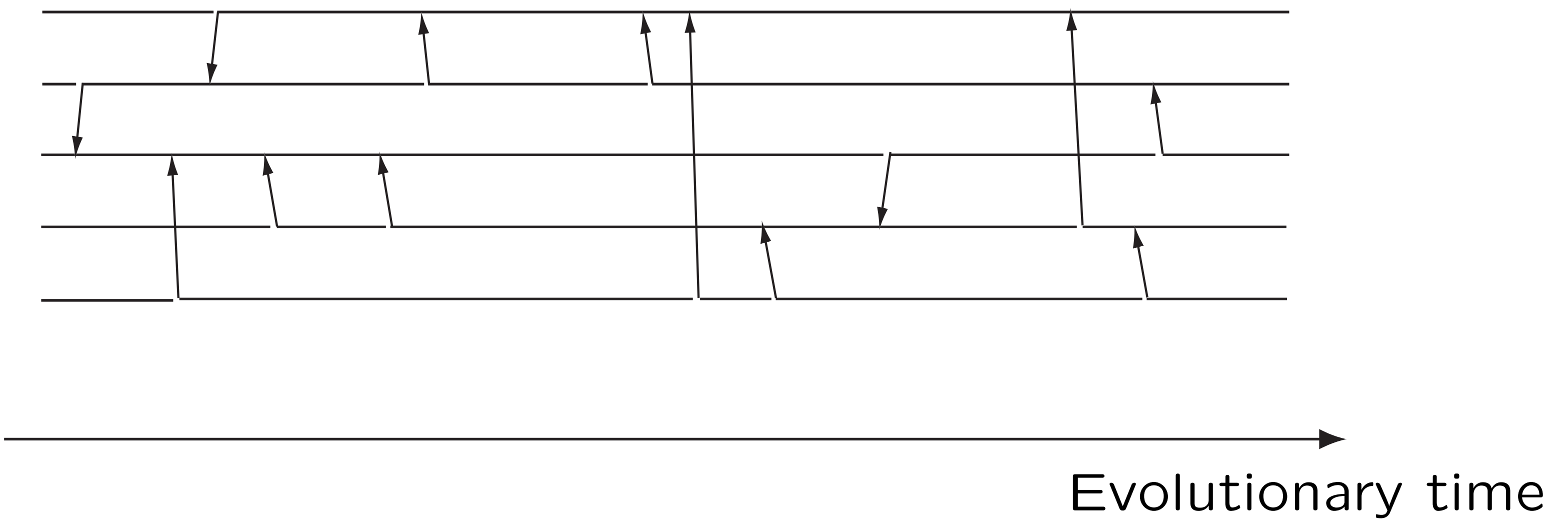
CIRM, Luminy
June 11–15 2012

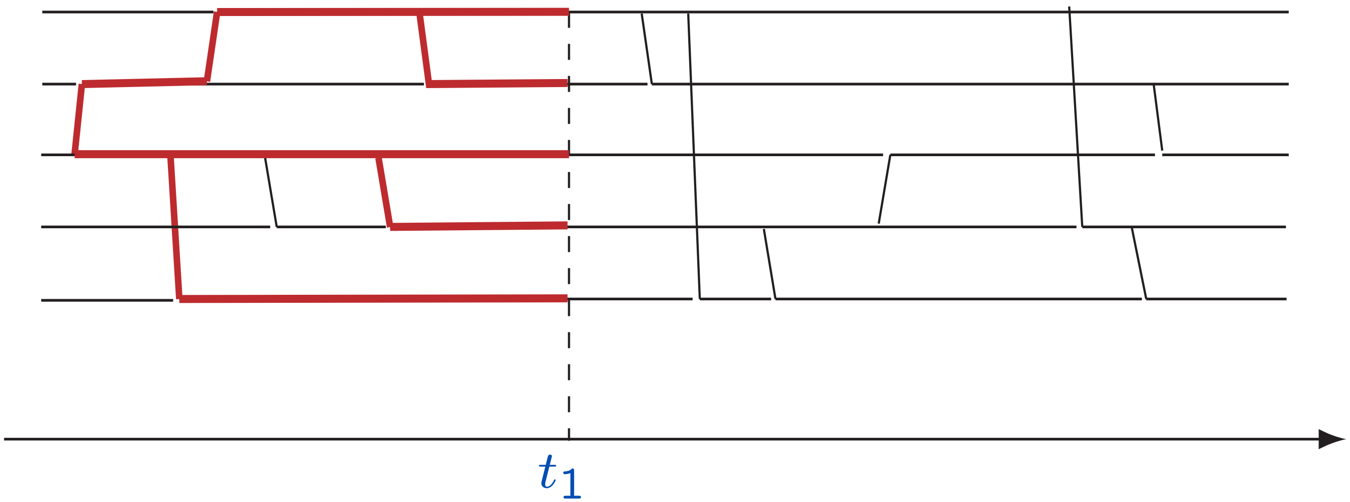
GOETHE UNIVERSITÄT FRANKFURT AM MAIN

The evolving Kingman N-coalescent ($N = 5$):



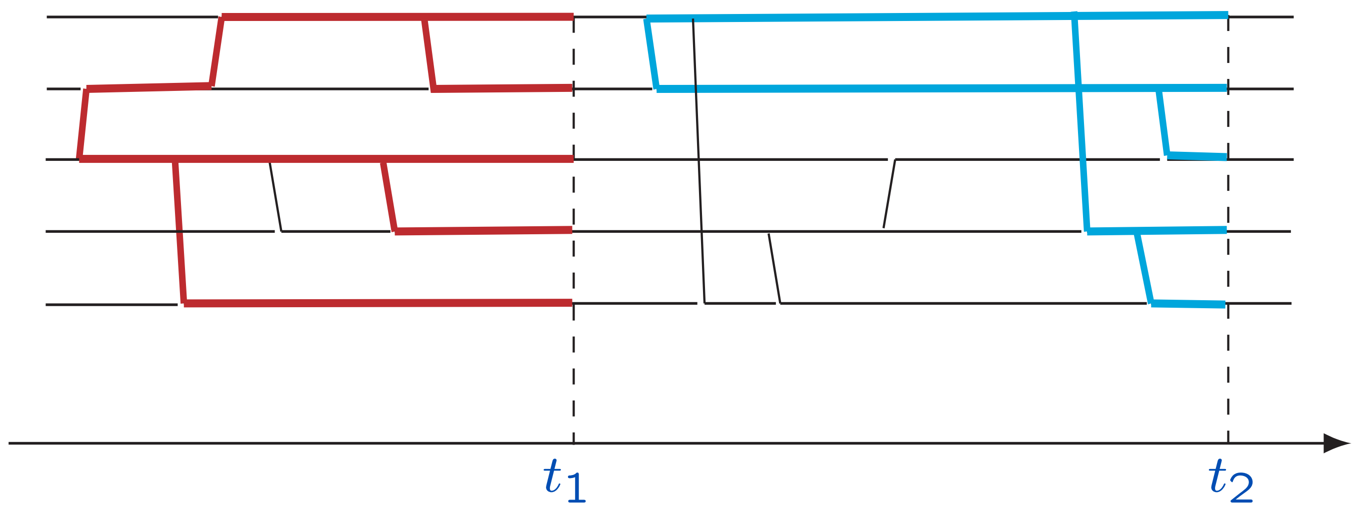
Moran's model with time $-\infty < t < \infty$:
Links between pairs of lines appear at rate 1,
independent between the different pairs.

The evolving Kingman N-coalescent:



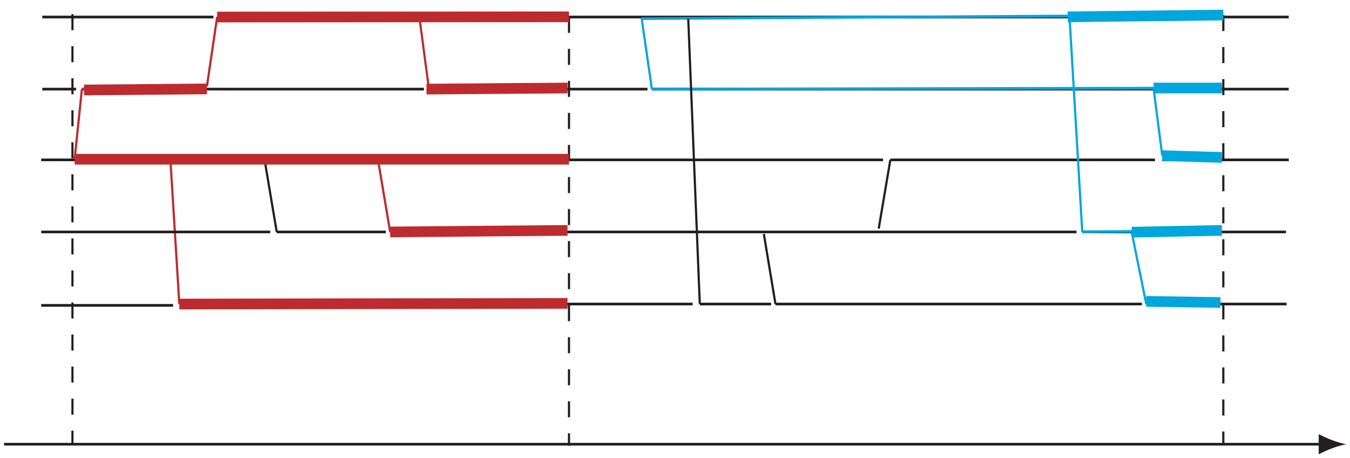
Kingman's coalescent at time t_1

The evolving Kingman N-coalescent:



The coalescent tree evolves in time.

The evolving Kingman N-coalescent:



Evolving time to MRCA

tree topology

total length

total external length

These results are rather different in nature, none covering any other.

Interesting aspects:

– limiting processes with
a.s. continuous paths versus
a.s. (compensated) pure jump paths

– different time scalings

A BASKET OF RESULTS

Theorem: (Pfaffelhuber/Wakolbinger , Donnelly/Kurtz, 2006)

Let $A_N(t)$ be the time to the MRCA of the evolving Kingman N-coalescent at time $t \in \mathbb{R}$. Then, as $N \to \infty$,

$$\Big(A_N(t)\Big)_{t\in\mathbb{R}} \quad \xrightarrow{d} \quad A \ ,$$

where the limiting process $A = (A_t)_{t\in\mathbb{R}}$ is stationary, a.s. pure jump, non-Markovian.

For a related result on the two oldest families in the genealogy see Delmas, Dhersin, and Siri-Jegousse (2010).

Theorem: (Greven, Pfaffelhuber, Winter, 2009, 2010)

Let $T_N(t)$ be the tree, induced by the evolving Kingman N-coalescent at time $t \in \mathbb{R}$ in the space of real trees furnished with the Gromov-weak topology. Then, as $N \to \infty$,

$$\big(T_N(t)\big)_{t\in\mathbb{R}} \quad \xrightarrow{d} \quad T \ ,$$

where the limiting tree-valued process $T = (T_t)_{t\in\mathbb{R}}$ is stationary, a.s. continuous, and unique solution of a martingale problem.

Theorem: (Pfaffelhuber, Wakolbinger, Weisshaupt, 2011)

Let $L'_N(t)$ be the total length of the evolving Kingman N-coalescent at time $t \in \mathbb{R}$. Then, as $N \to \infty$,

$$\Big(L'_N(t) - 2\log N\Big)_{t\in\mathbb{R}} \quad \stackrel{d}{\to} \quad L' \ ,$$

where the limiting process $L' = (L'_t)_{t\in\mathbb{R}}$ is stationary, a.s. pure jump, non-Markovian.

L' has *infinite quadratic variation* (Knobloch, Stanciu, Wakolbinger, 2011), thus fails to be a semimartingale.

Theorem: (Schweinsberg, 2011)

Let $L''_N(t)$ be the total length of the evolving *Bolthausen-Sznitman* N-coalescent at time $t \in \mathbb{R}$. Then, as $N \to \infty$,

$$\Big(\frac{(\log N)^2}{N} L''_N(\tfrac{t}{\log N}) - \log N - \log\log N\Big)_{t \in \mathbb{R}} \quad \xrightarrow{d} \quad L'' \ ,$$

where the stationary limiting process $L'' = (L''_t)_{t \in \mathbb{R}}$ solves the SDE

$$dL'' = -L''\, dt + dY$$

for a certain Lévy-process Y of index 1.

Theorem: (K., Stanciu, 2012, ongoing work)

Let $L_N(t)$ be the total *external length* of the evolving Kingman N-coalescent. Then

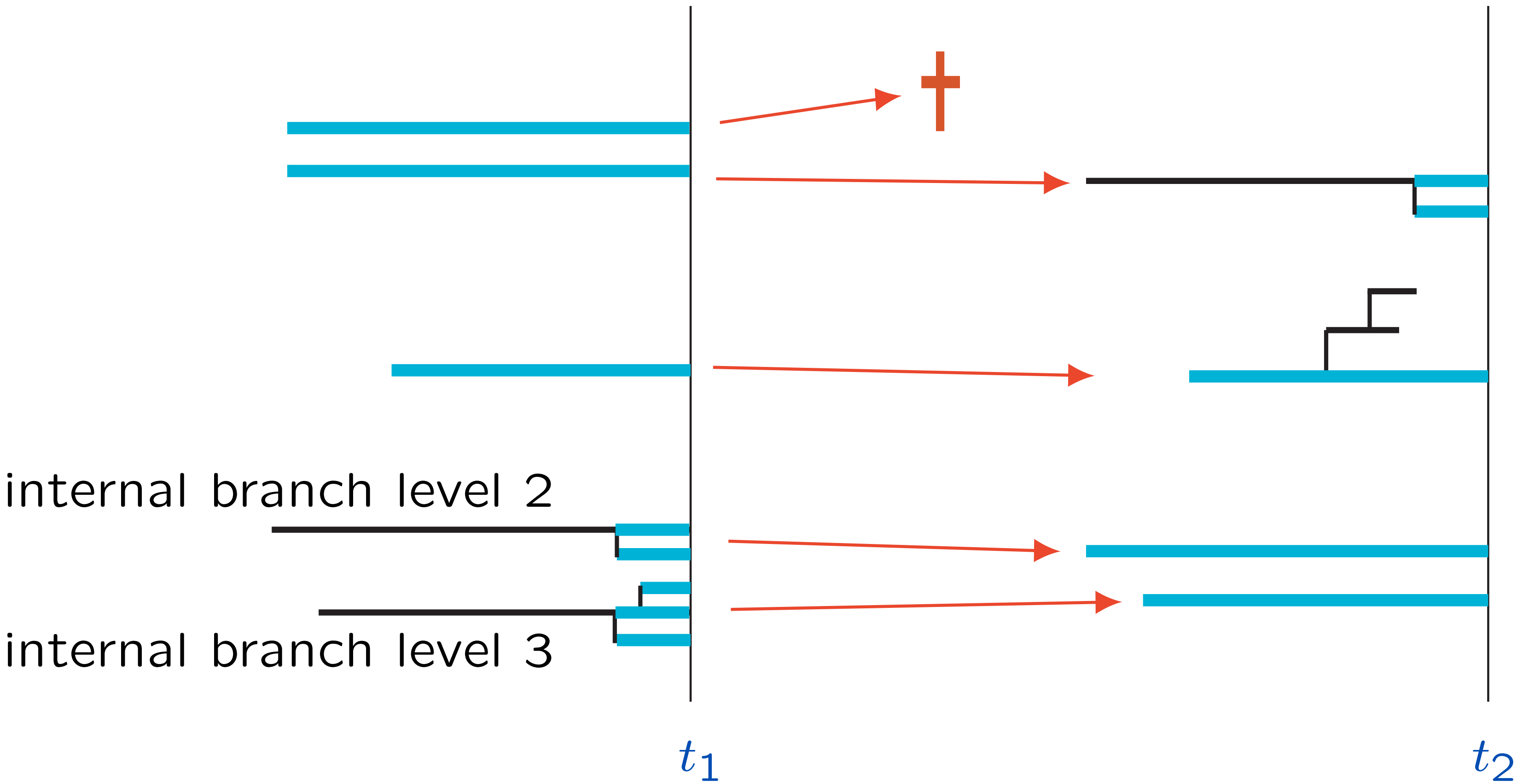
$$\left(\sqrt{\frac{N}{4\log N}}\Big(L_N(\tfrac{t}{N}) - 2\Big)\right)_{t\in\mathbb{R}} \quad \xrightarrow{d} \quad L \ ,$$

where L is a stationary, Gaussian, a.s. continuous, with covariance function

$$\mathbf{Cov}(L_s, L_t) = \Big(\frac{1}{1+|t-s|}\Big)^2 \ .$$

Note the different scaling of time (*real* instead of *evolutionary* periodes).

The dynamics of the external lengths:



So let for the *static* Kingman N-coalescent (at time $t = 0$)

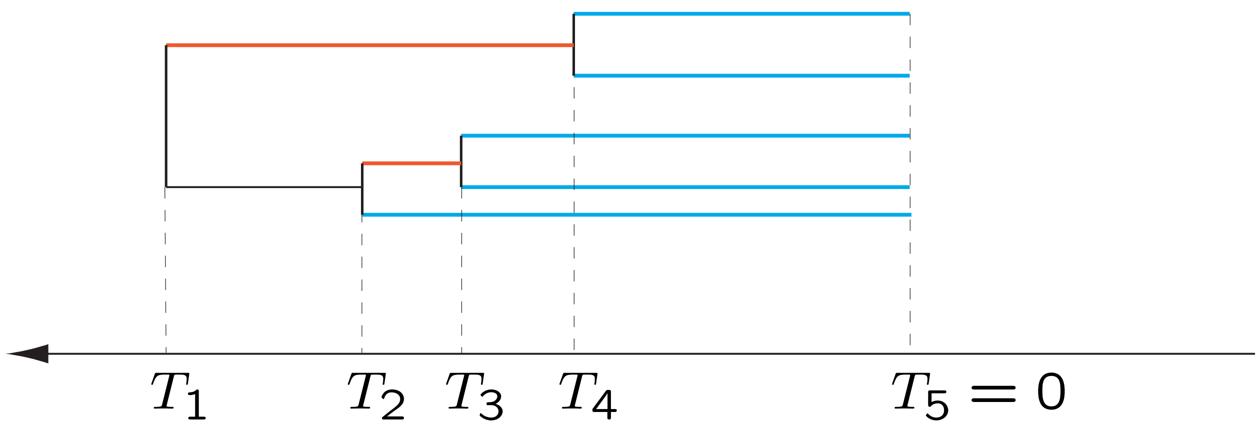
$$L_N^i = \text{ total internal branch length of level } i \ ,$$

in particular for $i = 1$

$$L_N^1 = \text{ total external branch length } .$$

How do we gain access to these quantities?

Branch numbers $V_N, \ldots, V_2$ *and* $W_N, \ldots, W_2$.



$V_2 = 0$ $\qquad$ $V_5 = 5$

$W_2 = 1$ $\qquad$ $W_5 = 0$

$$L_N^1 = \sum_{i=2}^{N} V_i (T_{i-1} - T_i) \ , \qquad L_N^2 = \sum_{i=2}^{N} W_i (T_{i-1} - T_i)$$

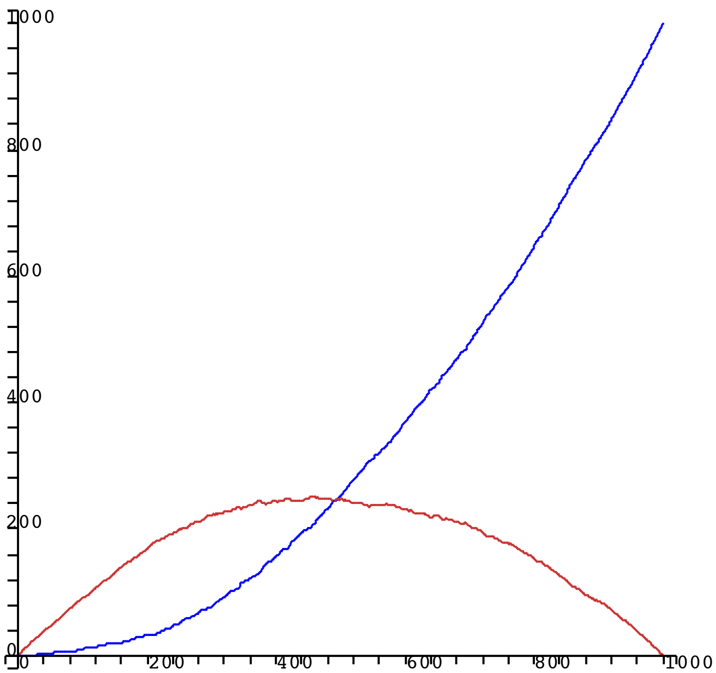
$$L_N^1 = \sum_{i=2}^{N} V_i(T_{i-1} - T_i)$$

$$= \sum_{k=1}^{N-1} T_k(V_{k+1} - V_k) + T_1 V_2$$

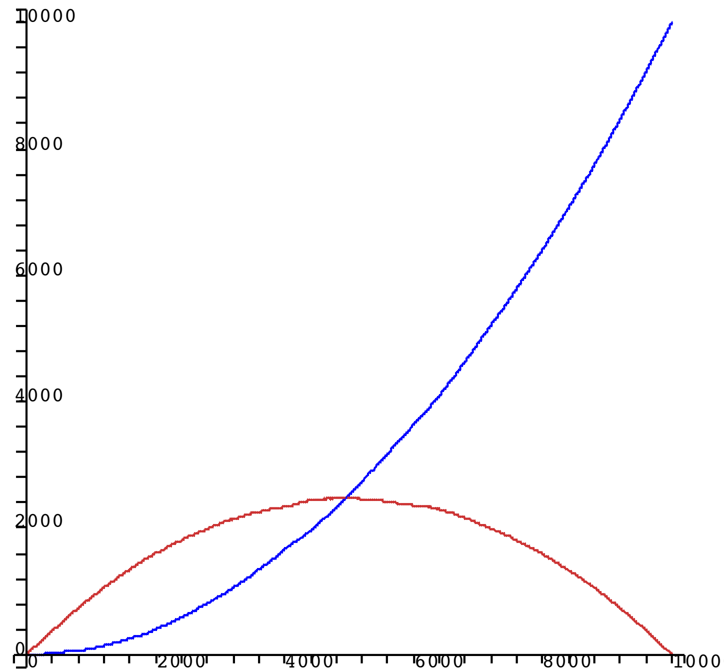
$$\approx \sum_{k=1}^{N-1} \frac{2}{k} \Delta V_k$$

ΔV_k is easy to handle for k close to N.

External branch numbers $0 = V_1, V_2, \ldots, V_N$, and (total) internal branch numbers $U_1 = 1 - V_1, U_2 = 2 - V_2, \ldots, U_N = 1 - V_N$



$N = 1000$



$N = 10000$

Let us look at the randomness within

$$V_N - N \quad (= 0)$$

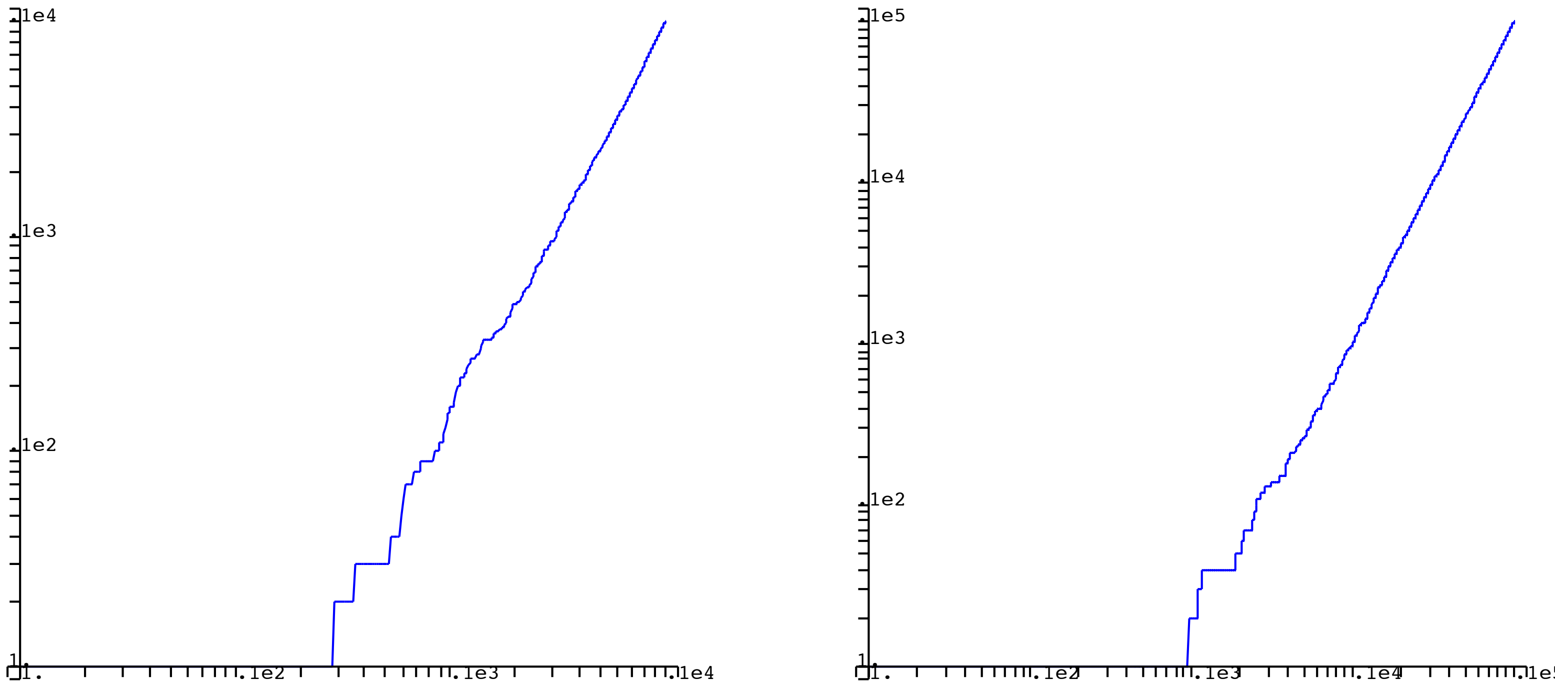
$$V_{N-1} - N + 2$$

$$\vdots$$

$$V_{N-i} - N + 2i$$

$$\vdots$$

Randomness within (V_i) close to N:
$V_N - N + 1, V_{N-1} - N + 3, \ldots$



N^0 $N^{1/2}$ N

Randomness enters (almost) independently.

However, since we consider

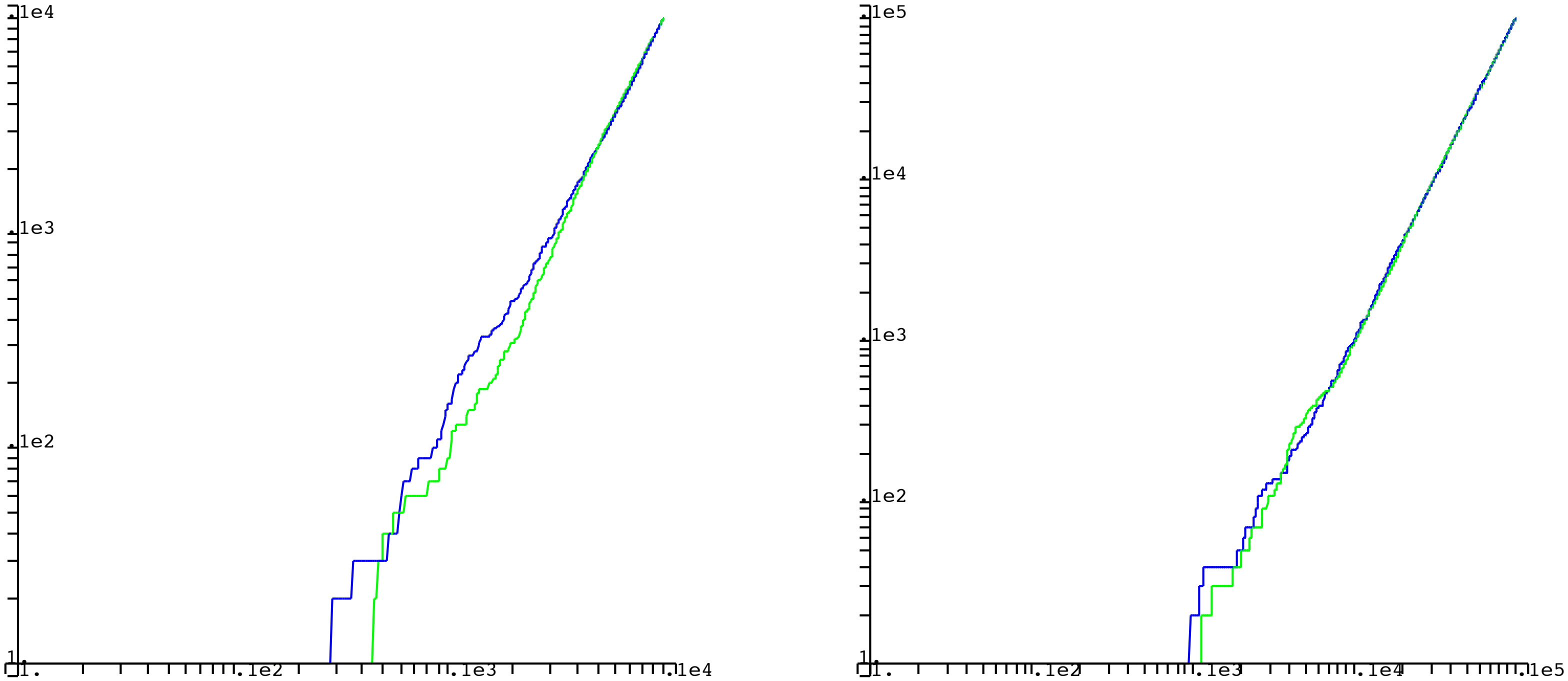
$$L_N^1 \approx \sum_{k=1}^{N-1} \frac{2}{k} \Delta V_k$$

we have to understand randomness at the beginning of $V_2, \ldots, V_N$. So let us compare

$$V_N - N, V_{N-1} - N + 2, \ldots \qquad \text{to} \qquad V_1, V_2, \ldots$$

Randomness within (V_i) close to N and 1:

$V_N - N + 1, V_{N-1} - N + 3, \ldots$ versus $V_1 + 1, V_2 + 1, \ldots$



For the (total) internal numbers $U_1 = 1 - V_1, \ldots, U_N = N - V_N$ we have reversibility:

Theorem: (Janson, K. 2011)

$$(U_1, \ldots, U_{N-1}) \stackrel{d}{=} (U_{N-1}, \ldots, U_1) \ .$$

For another proof see Knobloch, Stanciu, Wakolbinger (2011).

Theorem: (Janson, K. 2011)

$$\sqrt{\frac{N}{4 \log N}} \left(L_N^1 - 2\right) \xrightarrow{d} N(0, 1) \ .$$

The representation of the (total) internal numbers $U_1, \ldots, U_N$ *as diminishing urn:*

– Take urn with *blue* balls, altogether N balls.

– Remove them stepwise:
Successively remove a random pair of balls
and replace it by one *orange* ball.

– If i balls are left,
let U_i the number of orange balls among them
and V_i the number of blue balls.

Note:

$$V_{N-i} - N + 2i = i - U_{N-i} \quad , \quad V_i = i - U_i \ .$$

Now recall

$V_N, \ldots, V_2$ external branch numbers

$W_N, \ldots, W_2$ internal branch numbers level 2

$\vdots$

Note:

$V_N, V_{N-1}, \ldots, V_2$ is a *Markov chain* (inhomogeneous in time).

$(V_N, W_N), (V_{N-1}, W_{N-1}), \cdots, (V_2, W_2)$ is a Markov chain, or

$W_N, \ldots, W_2$ is a Markov chain,
given the *random environment* $V_N, \ldots, V_2$.

The transition probabilities:

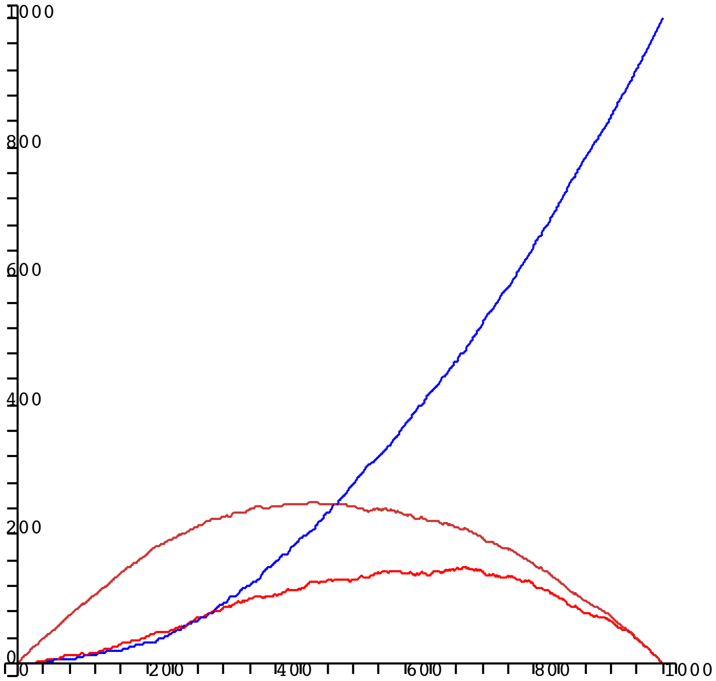
$$P^k_{v,w}(v', w') = \mathbf{P}(V_{k-1} = v', W_{k-1} = w' \mid V_k = v, W_k = w)$$

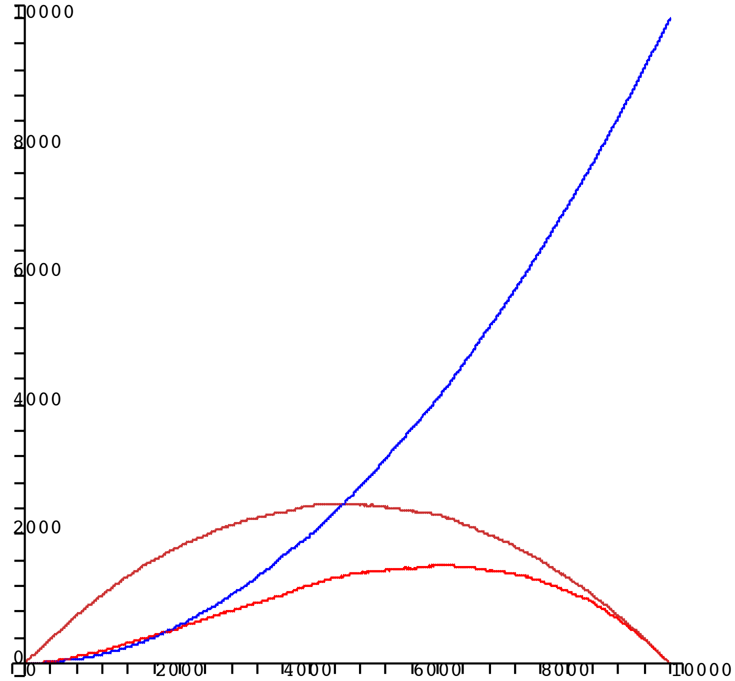
$$P^k_{v,w}(v-2, w+1) = \frac{\binom{v}{2}}{\binom{k}{2}} \qquad P^k_{v,w}(v, w-2) = \frac{\binom{w}{2}}{\binom{k}{2}}$$

$$P^k_{v,w}(v, w) = \frac{\binom{k-v-w}{2}}{\binom{k}{2}} \qquad P^k_{v,w}(v-1, w-1) = \frac{vw}{\binom{k}{2}}$$

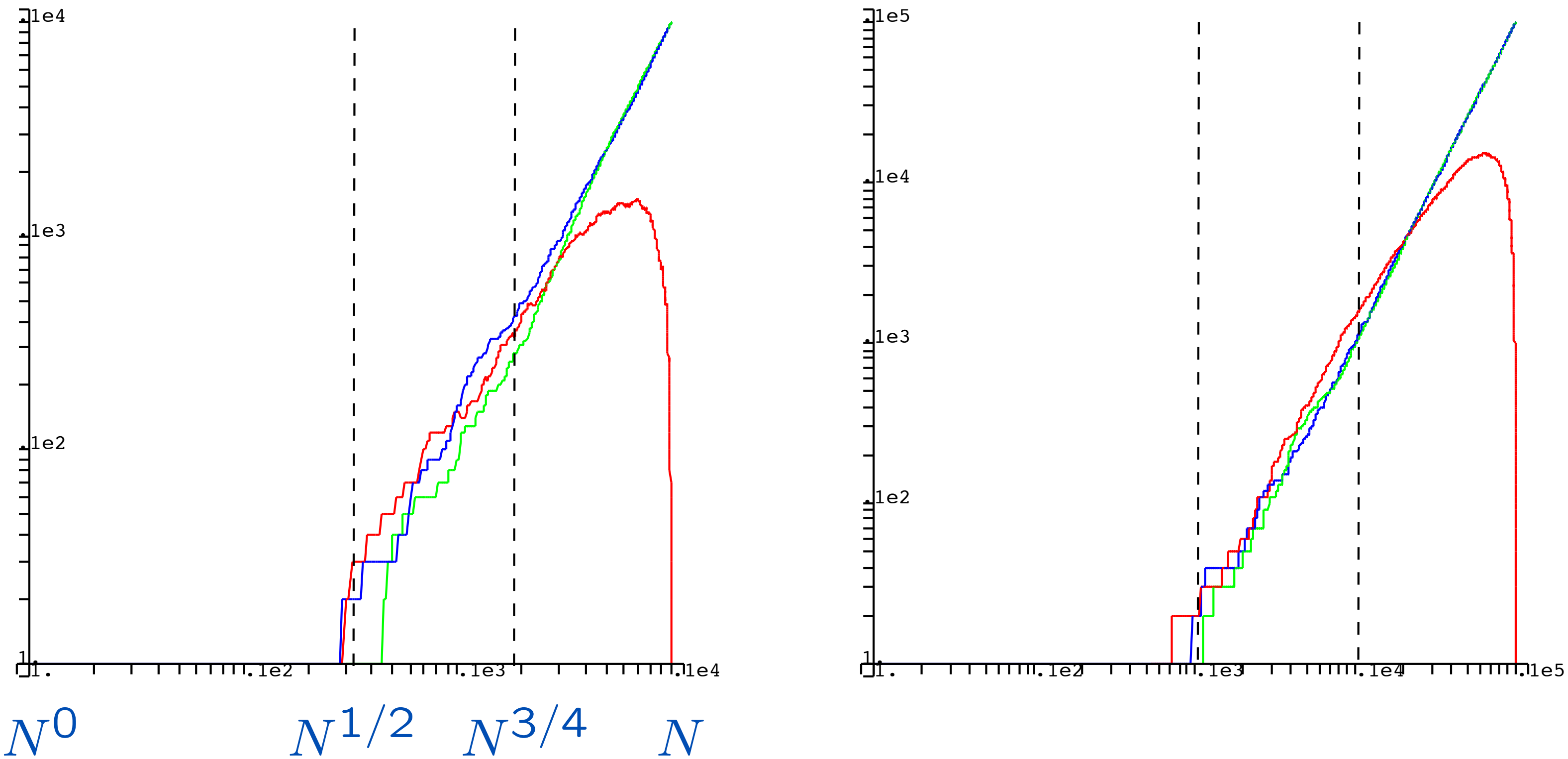
$$P^k_{v,w}(v-1, w) = \frac{v(k-v-w)}{\binom{k}{2}} \qquad P^k_{v,w}(v, w-1) = \frac{w(k-v-w)}{\binom{k}{2}}$$

External branch numbers $V_1, \ldots, V_N$, and
(total) internal branch numbers $U_1, \ldots, U_N$ and
internal branch numbers level 2 $W_1, \ldots, W_N$.





Randomness within (V_i) and (W_i):
$V_N - N + 1, V_{N-1} - N + 3, \ldots$ and $V_1 + 1, V_2 + 1, \ldots$
and $W_1 + 1, W_2 + 1, \ldots$



Theorem: (K., Stanciu 2012)

For every $k \in \mathbb{N}$

$$\sqrt{\frac{N}{4\log N}}\,\left(L_N^1 - \mu_1, \ldots, L_N^k - \mu_k\right) \xrightarrow{d} N(0, I_k) \; .$$

with the $k \times k$ identity matrix I_k and

$$\mu_i = \frac{2}{i} \; .$$

Idea of proof:

Reversing time seems no longer feasible.

We couple the Markov chain

$$(V_N, W_N), \dots, (V_2, W_2)$$

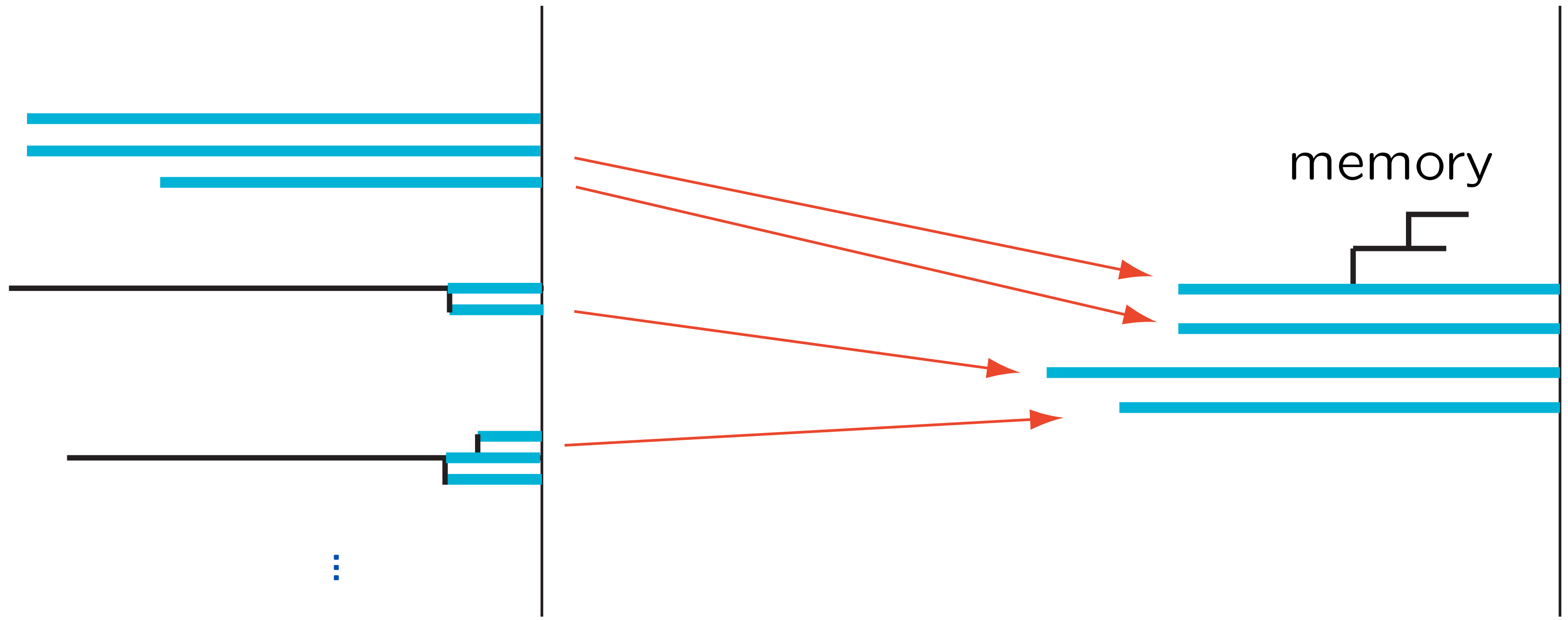
with two independent urns, i.e. with

$$(V_N, \tilde{V}_N), \dots, (V_2, \tilde{V}_2)\ ,$$

where $(\tilde{V}_N, \dots, \tilde{V}_2)$ is an independent copy of $(V_N, \dots, V_2)$.

Now the urns can be reversed.

Back to the evolving coalescent:



$\mathbf{Cov}(L_s, L_t) =$
probability that a critical binary branching process consists of exactly 1 individual at time $|t - s|$.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