LogFeller et Ray–Knight

Etienne Pardoux

joint work with V. Le and A. Wakolbinger

Feller's branching diffusion with logistic growth

- We consider the diffusion

$$dZ_t = Z_t(\theta - \gamma Z_t)dt + 2\sqrt{Z_t}\,dW_t, \quad t \geq 0, \tag{1}$$

with $Z_0 = x$, θ, $\gamma > 0$.

- Z_t is not a branching process : the quadratic term introduces interactions between the branches. However there is still an interpretation in terms of the evolution of a population of the solution of the "Fellog" process Z_t, see A. Lambert (2005).
- Consider

$$\begin{cases} dH_s = \left(\frac{\theta}{2} - \gamma L_s(H_s)\right) ds + dB_s + \frac{1}{2} dL_s(0), \quad s \geq 0, \\ H_0 = 0, \end{cases} \tag{2}$$

where B is a standard Brownian motion. In this SDE, the term $dL_s(0)/2$ takes care of the reflection of H at the origin.

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Results

- Using Girsanov's theorem, one can show

Proposition

The SDE (2) *has a unique weak solution.*

- H being the solution of (2), $L_s(t)$ denoting its local time, let

$$S_x := \inf\{s > 0 : L_s(0) > x\}.$$

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Discrete approximation

Our proof of the above extended Ray–Knight theorem is based on an approximation by finite population.

- For $N \in \mathbb{N}$, let $Z^{N,x}$ be the total mass of a population of individuals, each of which has mass $1/N$. The initial mass is $Z_0^{N,x} = \lfloor Nx \rfloor / N$, and $Z_t^{N,x}$ follows a Markovian jump dynamics : from its current state k/N,

$$Z^{N,x} \text{ jumps to } \begin{cases} (k+1)/N \text{ at rate } 2kN + k\theta \\ (k-1)/N \text{ at rate } 2kN + k(k-1)\gamma/N. \end{cases}$$

- For $\gamma = 0$, this is a GW process in cont. time : each individual independently spawns a child at rate $2N + \theta$, and dies (childless) at rate $2N$. For $\gamma \neq 0$, the quadratic death rate destroys independence. Viewing the individuals alive at time t as being arranged "from left to right", and by decreeing that each of the pairwise fights (which happens at rate 2γ) is won by the individual to the left, we arrive at the additional death rate $2\gamma \mathcal{L}_i(t)/N$ for individual i, where $\mathcal{L}_i(t)$ denotes the number of indiv. currently living to the left of i at time t.

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- The just described reproduction dynamics gives rise to a forest F_N of *trees of descent*. At any branch point, we imagine the "new" branch being placed to the right of the mother branch. Because of the asymmetric killing, the trees further to the right have a tendency to stay smaller : they are "under attack" by the trees to their left.
- For a given realization of F^N, we read off a continuous and piecewise linear $\mathbb{R}_+$-valued path H^N (called the *exploration path* associated with F^N) in the following way :
- Starting $(0, 0)$, H^N goes upwards at speed $2N$ until the top of the first mother branch (which is the leftmost leaf of the tree) is hit. There H^N turns and goes downwards, again at speed $2N$, until arriving at the closest branch point (which is the last birth time of a child of the first ancestor before his death). From there one goes upwards into the (yet unexplored) next branch, and proceeds in a similar fashion until being back at height 0, which means that the exploration of the leftmost tree is completed. Then explore the next tree, etc., until the exploration of the forest F^N is completed.

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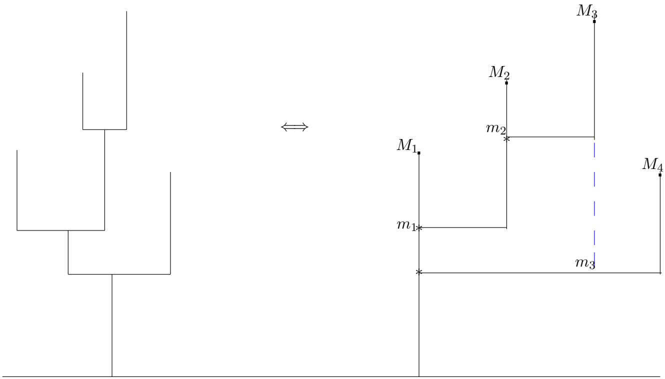
$\Longleftrightarrow$
M_3
M_2
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The exploration process H^N in case $\theta = \gamma = 0$

- We have the following SDE ((H^N, V^N) takes values in $\mathbb{R}_+ \times \{-1, 1\}$)

$$\frac{dH_s^N}{ds} = 2NV_s^N, H_0^N = 0,$$
$$dV_s^N = 2\mathbf{1}_{\{V_{s-}^N=-1\}}dP_s^N - 2\mathbf{1}_{\{V_{r-}^N=1\}}dP_r^N + 2NdL_s^N(0), V_0^N = 1,$$

  where $\{P_s^N,\ s \geq 0\}$ is a Poisson processes with intensity $4N^2$, i. e. $M_s^N = P_s^N - 4N^2 s$ is a martingale, and

- $$L_s^N(t) := \text{the local time accumulated by } H^N \text{ at level } t \text{ up to time } s$$
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$$S_x^N := \inf\{s > 0 : \ L_s^N(0) \geq [Nx]/N\},$$

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$$Z_t^{N,x} = L_{S_x^N}^N(t), \quad t \geq 0.$$

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Taking the limit

- As $N \to \infty$, V^N oscillates faster and faster. In order to study the limit of H^N, we consider for $f \in C^2(\mathbb{R})$, the "perturbed test function", see e. g. Ethier, Kurtz (1986)

$$f^N(h, v) = f(h) + \frac{v}{4N} f'(h).$$

- Implementing this with $f(h) = h$, we get

$$H_s^N + \frac{V_s^N}{4N} = M_s^{1,N} - M_s^{2,N} + \frac{1}{2} L_s^N(0),$$

where

$$M_s^{1,N} = \frac{1}{2N} \int_0^s \mathbf{1}_{\{V_{r-}^N = -1\}} dM_r^N \quad \text{and} \quad M_s^{2,N} = \frac{1}{2N} \int_0^s \mathbf{1}_{\{V_{r-}^N = 1\}} dM_r^N$$

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Taking the limit as $N \to \infty$

We obtain the following joint convergence

Theorem

For any $x > 0$, *as* $n \to \infty$,

$$\left(\{H^N_s, M^{1,N}_s, M^{2,N}_s, s \geq 0\}, \{L^N_s(t), s, t \geq 0\}, S^N_x\right)$$
$$\Rightarrow$$
$$\left(\{H_s, B^1_s, B^2_s, s \geq 0\}, \{L_s(t), s, t \geq 0\}, S_x\right),$$

where B^1 *and* B^2 *are two mutually independent B. M.s, if* $B = (\sqrt{2})^{-1}(B^1 - B^2)$, *H is B reflected above 0, L its local time, and* $S_x = \inf\{s > 0;\ L_s(0) > x\}$.

A Girsanov transformation 1

- Let

$$X_s^{N,1} := \int_0^s \frac{\theta}{2N} \mathbf{1}_{\{V_{r-}^N=-1\}} dM_r^N,$$
$$X_s^{N,2} := \int_0^s \frac{\gamma L_r^N(H_r^N)}{N} \mathbf{1}_{\{V_{r-}^N=1\}} dM_r^N,$$
$$X^N := X^{N,1} + X^{N,2},$$

- and $Y^N := \mathcal{E}(X^N)$ denote the Doléans exponential of X^N, i. e. Y^N solves the SDE

$$Y_s^N = 1 + \int_0^s Y_{r-}^N dX_r^N, \quad s \geq 0.$$

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Proposition

Y^N is a martingale ($\Leftrightarrow \mathbb{E}Y_s^N = 1, \forall s > 0$).

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- Define new probability measures $\tilde{\mathbb{P}}^N$ s. t. for all $s > 0$,

$$\frac{d\tilde{\mathbb{P}}^N \,|_{\mathcal{F}_s}}{d\mathbb{P}\,|_{\mathcal{F}_s}} = Y_s^N, \quad \forall s \geq 0$$

- Consider the 2-variate point process

$$(Q_r^{1,N}, Q_r^{2,N}) = \left(\int_0^r \mathbf{1}_{\{V_{u-}^N=-1\}} dP_u^N, \int_0^r \mathbf{1}_{\{V_{u-}^N=1\}} dP_u^N\right), \quad r \geq 0,$$

Under $\tilde{\mathbb{P}}^N$,

$$\begin{aligned} Q_r^{1,N} &\quad \text{has intensity} \quad (4N^2 + 2\theta N)\mathbf{1}_{\{V_{r-}^N=-1\}} dr \\ Q_r^{2,N} &\quad \text{has intensity} \quad 4[N^2 + \gamma N L_r^N(H_r^N)]\mathbf{1}_{\{V_{r-}^N=1\}} dr. \end{aligned}$$

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$$Y^N \Rightarrow Y = \exp\left(X_s - \int_0^s \left[\frac{\theta^2}{4} + \gamma^2 L_r^2(H_r) \right] dr \right).$$

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Conclusion

- Our main theorem follows taking the limit as $N \to \infty$ in the identity

$$Z_t^{N,x} = L^N_{S_x^N}(t),$$

under the measures $\tilde{\mathbb{P}}^N$ and $\tilde{\mathbb{P}}$,

- thanks to the following elementary

Lemma

Let (ξ_N, η_N), (ξ, η) be random pairs defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with η_N, η nonnegative scalar random variables, and ξ_N, ξ taking values in some complete separable metric space $\mathcal{X}$. Assume that $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$. Write $(\tilde{\xi}_N, \tilde{\eta}_N)$ for the random pair (ξ_N, η_N) defined under the probability measure $\tilde{\mathbb{P}}^N$ which has density η_N with respect to $\mathbb{P}$, and $(\tilde{\eta}, \tilde{\xi})$ for the random pair (η, ξ) defined under the probability measure $\tilde{\mathbb{P}}$ which has density η with respect to $\mathbb{P}$. Then $(\tilde{\xi}_N, \tilde{\eta}_N)$ converges in distribution to $(\tilde{\eta}, \tilde{\xi})$, provided that (ξ_N, η_N) converges in distribution to (ξ, η).

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Appendix : Girsanov's theorem for Poisson processes 1

- Let $\{(Q_s^{(1)}, ..., Q_s^{(d)}), s \geq 0\}$ be a d-variate point process adapted to some filtration $\mathcal{F}$, and let $\{\lambda_s^{(i)}, s \geq 0\}$ be the predictable $(\mathbb{P}, \mathcal{F})$–intensity of $Q^{(i)}, 1 \leq i \leq d$. In other words, $M_r^{(i)} := Q_s^{(i)} - \int_0^s \lambda_r^{(i)} dr$ is a $(\mathbb{P}, \mathcal{F})$–martingale, $1 \leq i \leq d$.
- Assume that none of the $Q^{(i)}$, $Q^{(j)}$, $i \neq j$, jump simultaneously (so that the $M^{(i)}$'s are mutually orthogonal).
- Let $\{\mu_r^{(i)}, r \geq 0\}, 1 \leq i \leq d$, be nonnegative $\mathcal{F}$-predictable processes such that for all $s \geq 0$ and all $1 \leq i \leq d$

$$\int_0^s \mu_r^{(i)} \lambda_r^{(i)} dr < \infty \qquad \mathbb{P}\text{ -a.s.}$$

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- Assume that none of the $Q^{(i)}$, $Q^{(j)}$, $i \neq j$, jump simultaneously (so that the $M^{(i)}$'s are mutually orthogonal).
- Let $\{\mu_r^{(i)}, r \geq 0\}, 1 \leq i \leq d$, be nonnegative $\mathcal{F}$-predictable processes such that for all $s \geq 0$ and all $1 \leq i \leq d$

$$\int_0^s \mu_r^{(i)} \lambda_r^{(i)} dr < \infty \qquad \mathbb{P}\text{ -a.s.}$$

- Let $\{(Q_s^{(1)}, ..., Q_s^{(d)}), s \geq 0\}$ be a d-variate point process adapted to some filtration $\mathcal{F}$, and let $\{\lambda_s^{(i)}, s \geq 0\}$ be the predictable $(\mathbb{P}, \mathcal{F})$–intensity of $Q^{(i)}, 1 \leq i \leq d$. In other words, $M_r^{(i)} := Q_s^{(i)} - \int_0^s \lambda_r^{(i)} dr$ is a $(\mathbb{P}, \mathcal{F})$–martingale, $1 \leq i \leq d$.
- Assume that none of the $Q^{(i)}$, $Q^{(j)}$, $i \neq j$, jump simultaneously (so that the $M^{(i)}$'s are mutually orthogonal).
- Let $\{\mu_r^{(i)}, r \geq 0\}, 1 \leq i \leq d$, be nonnegative $\mathcal{F}$-predictable processes such that for all $s \geq 0$ and all $1 \leq i \leq d$

$$\int_0^s \mu_r^{(i)} \lambda_r^{(i)} dr < \infty \qquad \mathbb{P}\text{ -a.s.}$$

Appendix : Girsanov's theorem for Poisson processes 2

Theorem

$\{T_k^i, k = 1, 2 \dots\}$ *denoting the jump times of* $Q^{(i)}$*, let for* $s \geq 0$

$$Y_s^{(i)} := \Big(\prod_{k \geq 1 : T_k^i \leq s} \mu_{T_k^i}^{(i)} \Big) \exp\Big\{ \int_0^s (1 - \mu_r^{(i)}) \lambda_r^{(i)} dr \Big\} \quad \textit{and} \quad Y_s = \prod_{j=1}^{d} Y_s^{(j)}.$$

If $\mathbb{E}[Y_s] = 1$*,* $s \geq 0$*, then, for each* $1 \leq i \leq d$*, the process* $Q^{(i)}$ *has the* $(\tilde{\mathbb{P}}, \mathcal{F})$*-intensity* $\tilde{\lambda}_r^{(i)} = \mu_r^{(i)} \lambda_r^{(i)}$*,* $r \geq 0$*, where the probability measure* $\tilde{\mathbb{P}}$ *is defined by*

$$\frac{d\tilde{\mathbb{P}}|_{\mathcal{F}_s}}{d\mathbb{P}|_{\mathcal{F}_s}} = Y_s, \quad s \geq 0.$$

Note that $Y^{(i)} = \mathcal{E}(X^{(i)})$, with $X_s^{(i)} := \int_0^s (\mu_r^{(i)} - 1) dM_r^{(i)}$, $1 \leq i \leq d$, $s \geq 0$.