Λ-look-down model with selection

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Séminaire ANR MANEGE

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1. Description of the model
   - Birth :
   - Death :
2. Convergence to the Λ-W-F SDE with selection
3. Fixation and non-fixation in the Λ-W-F SDE

- We consider a population of infinite size. We assume that two types of individuals coexist in the population : individuals with the wild-type allele b and individuals with the advantageous allele B. This selective advantage is modeled by a death rate α for the type b individuals.

- We consider a population of infinite size. We assume that two types of individuals coexist in the population : individuals with the wild-type allele b and individuals with the advantageous allele B. This selective advantage is modeled by a death rate α for the type b individuals.
- We assume that individuals are placed at time 0 on levels $1, 2, \cdots$, each one being, independently from the others, b with probability x, B with probability $1 - x$, for some $0 < x < 1$.

- For any $t \geq 0, i \geq 1$, let

$$\eta_t(i) = \begin{cases} 1 & \text{if the i-th individual is b at time } t \\ 0 & \text{if the i-th individual is B at time } t. \end{cases}$$

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- The evolution of the population is governed by two following mechanism.

- *Births*. Let Λ be an arbitrary finite measure on $[0,1]$ such that $\Lambda(\{0\}) = 0$. Consider a Poisson random measure on $\mathbb{R}_+ \times]0,1]$,

$$m = \sum_{k=1}^{\infty} \delta_{t_k, p_k}$$

with intensity measure $dt \otimes \nu(dp)$, where $\nu(dp) = p^{-2}\Lambda(dp)$.

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with intensity measure $dt\otimes\nu(dp)$, where $\nu(dp)=p^{-2}\Lambda(dp)$.
- Each atom (t,p) of m corresponds to a birth event.

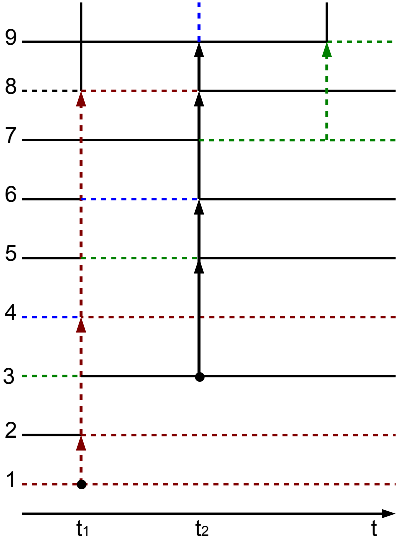
- For each level $i \geq 1$, we define $Z_i \simeq Bernoulli(p)$. Let

$$I_{t,p} = \{i \geq 1 : Z_i = 1\}$$

and

$$\ell_{t,p} = \inf\{i \in I_{t,p} : i > \min I_{t,p}\}.$$

- $I_{t,p}$ is called the set of individuals that participate to the birth event.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t_1
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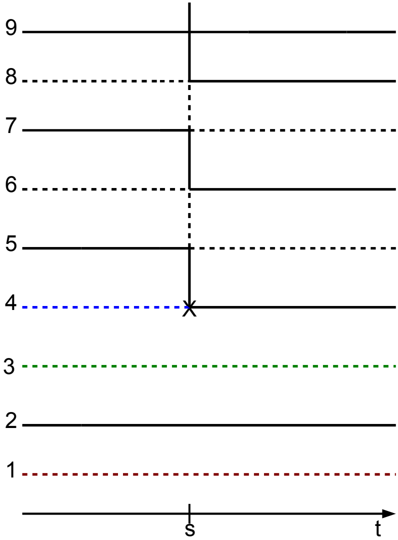
- *Deaths*. Any type b individual dies at rate α. If the level of the dying individual is i, then for all $j > i$, the individual at level j replaces instantaneously the individual at level $j-1$. In other words,

$$\eta_t(j) = \begin{cases} \eta_{t^-}(j) & \text{for } j < i \\ \eta_{t^-}(j+1) & \text{for } j \geq i \end{cases}$$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- **Case 2 :** $K_t \nrightarrow \infty, t \rightarrow \infty.$

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- Let S_N the first time where all the N first individuals are of B type.

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- Let S_N the first time where all the N first individuals are of B type.
- let $\varphi(N) = Ne^{\alpha S_N}(Ne^{\alpha S_N} + 1) + M$, where
$$M = \sup_{0 \leq t \leq T_1} K_t.$$

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- Now, let $\{\xi_t^{\varphi(N)},\ t \geq 0\}$ denote the process which describes the position at time t of the individual sitting on level $\varphi(N)$ at time 0.

Proposition

If $T_1 < \infty$, then for each $N \geq M$,

$$\widehat{\mathbb{P}}_N(\exists 0 < t \leq S_N \text{ such that } \xi_t^{\varphi(N)} \leq N) \leq \frac{2}{N^2},$$

where $\widehat{\mathbb{P}}_N[.] = \mathbb{P}(. \mid S_N)$

Proposition

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Remark :The collection of random process $\{\eta_t(i), t \geq 0\}_{i \geq 1}$ is not exchangeable. Indeed, $\eta_t(1)$ can jump from 1 to 0, but never from 0 to 1, while the other $\eta_t(i)$ do not have that property

- For $N \geq 1$ and $t \geq 0$, denote by X_t^N the proportion of type b individuals at time t among the first N individuals, i.e.

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- From the Tightness of X^N and (2), it is not hard to show there exists a process $X \in D([0, \infty))$, such that for all $t \geq 0$,

$$X_t^N \to X_t \ \ a.s \quad \text{and} \quad X^N \Rightarrow X \text{ weakly in } D([0, \infty)).$$

Theorem (B. Bah, E. Pardoux, 2012)

For all $T > 0$,

$$\sup_{0\leq t\leq T} |X_t^N - X_t| \to 0 \text{ in probability, as } N \to \infty.$$

- Let
$$M = \sum_{k=1}^{\infty} \delta_{t_k, u_k, p_k}$$
Poisson point process on $\mathbb{R}_+ \times]0,1] \times]0,1]$ with intensity $dtdup^{-2}\Lambda(dp)$.
- For every $u \in]0,1[$ and $r \in [0,1]$, we introduce the elementary function
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- $$\int_0^1 \Psi(u,r)du = 0$$

Definition

We shall call Λ-W-F SDE with selection the following Poissonian stochastic differential equation

$$X_t = x - \alpha \int_0^t X_s(1 - X_s)ds + \int_{[0,t]\times]0,1[^2} p\Psi(u, X_{s^-})\bar{M}(ds, du, dp) \tag{3}$$

where $\alpha \in \mathbb{R}$ *and* $\bar{M}$ *is the compensated measure* M*. The solution* $\{X_t, t \geq 0\}$ *is a cadlag adpted processses which takes values in the interval* $[0, 1]$*.*

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Suppose that $X_0^N \to x$ a.s, as $N \to \infty$. Then the $[0,1]-$valued process $\{X_t, t \geq 0\}$ is the (unique in law) solution of the Λ-Wright-Fisher SDE (3).

- We suppose that the measure Λ is general (i.e $\Lambda(\{0\}) > 0$).

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Suppose that $X_0^N \to x$ a.s, as $N \to \infty$. Then the $[0,1]-$valued process $\{X_t, t \geq 0\}$ is the (unique in law) solution of the stochastic differential equation

$$X_t = x - \alpha \int_0^t X_s(1-X_s)ds + \int_0^t \sqrt{\Lambda(0)X_s(1-X_s)}dB_s + \int_{[0,t]\times]0,1[^2} p(\mathbf{1}_{u \leq X_{s^-}} - X_{s^-})\bar{M}(ds,du,dp),$$

where $\bar{M}$ is the compensated measure M, and B is a standard Brownian motion.

1. Description of the model
2. Convergence to the Λ-W-F SDE with selection
3. Fixation and non-fixation in the Λ-W-F SDE
   - Λ-coalescent
   - Comes down from infinity
   - fixation and non fixation
   - The law of X_∞

Definition

Λ-coalescent is a Markov process $(\Pi_t, t \geq 0)$ with values in $\mathcal{P}_\infty$ (the set of partition of $\mathbb{N}$), characterized as follows. If $n \in \mathbb{N}$, then the restriction $(\Pi_t^n, t \geq 0)$ of $(\Pi_t, t \geq 0)$ to $[n]$ is a Markov chain, taking values in $\mathcal{P}_n$, with a following dynamics : whenever Π_t^n is a partition consisting of k blocks, the rate at which a given ℓ-tuple of its blocks merges is

$$\lambda_{k,\ell} = \int_0^1 p^{\ell-2}(1-p)^{k-\ell}\Lambda(dp).$$

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- We say the Λ-coalescent comes down from infinity ($\Lambda \in$ **CDI**) if $\mathbb{P}(\#\Pi_t < \infty) = 1$ for all $t > 0$.
- We say it stays infinite ($\Lambda \notin$ **CDI**) if $\mathbb{P}(\#\Pi_t = \infty) = 1$ for all $t > 0$.

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- We say it stays infinite ($\Lambda \notin$ **CDI**) if $\mathbb{P}(\#\Pi_t = \infty) = 1$ for all $t > 0$.
- $$\text{if } \int_0^1 p^{-1}\Lambda(dp) < \infty \text{ then } \Lambda \notin \textbf{CDI} \qquad \text{(J. Pitman (1999)).}$$
- Let
$$\varphi(n) = \int_0^1 (np - 1 + (1-p)^n)p^{-2}\Lambda(dp).$$
$$\Lambda \in \textbf{CDI} \iff \sum_{n=2}^{\infty} \frac{1}{\varphi(n)} < \infty \quad \text{(J. SCHWEINSBERG 2000)}$$

Let

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We have the following

Theorem (B. Bah, E. Pardoux, 2012)

*If $\Lambda \in$ **CDI**, then one of the two types (b or B) fixates in finite time, i.e.*

$$\exists\, \zeta < \infty \;\; a.s : X_\zeta = X_\infty \in \{0,1\}$$

*If $\Lambda \notin$ **CDI**, then*

$$\forall t \geq 0,\; 0 < X_t < 1 \; a.s.$$

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- If $\alpha = 0$, X_t is bounded martingale, so

$$\mathbb{P}(X_\infty = 1) = \mathbb{E}X_\infty = \mathbb{E}X_0 = x.$$

- If $\alpha > 0$, we have

$$\mathbb{P}(X_\infty = 1) = \mathbb{E}X_\infty < x$$

Let

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Theorem (B. Bah, E. Pardoux, 2012)

If $\mu < \alpha$, then

$$\mathbb{P}(X_\infty = 1) = 0.$$

Theorem

If $\Lambda = \mathcal{U}(0,1)$, *then*

$$\mathbb{P}(X_\infty = 1) > 0.$$

Thank you for your attention !