Long-range percolation on the hierarchical lattice

Pieter Trapman

Department of Mathematics, Stockholm University

Paris, 30 January 2013

joint work with
Slavik Koval (Utrecht University)
Ronald Meester (VU University Amsterdam)

Electronic Journal of Probability **17**.57

Epidemiological "justification"

- Consider a well organized population, where everybody lives in a household of N individuals,
- Every family lives in a N-floors apartment building, at each floor are N apartments,
- N of those apartment buildings are served by one supermarket
- etc.
- The frequency that people in the same household meet is higher than the frequency that people on the same floor, but not in the same household meet,
  which in turn is higher than the frequency at which people from the same building but from different floors meet etc.

Epidemiological "justification"

- Consider a well organized population, where everybody lives in a household of N individuals,
- Every family lives in a N-floors apartment building, at each floor are N apartments,
- N of those apartment buildings are served by one supermarket
- etc.
- The frequency that people in the same household meet is higher than the frequency that people on the same floor, but not in the same household meet,
  which in turn is higher than the frequency at which people from the same building but from different floors meet etc.

Model

The vertices are "the leaves of an infinite regular N-tree", and the distance between vertices is the distance to their "most recent common ancestor".

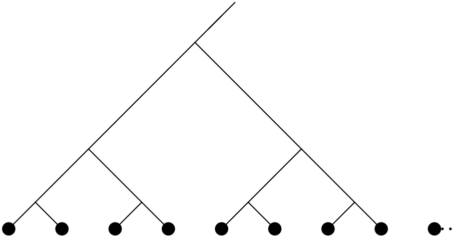


Figure: Hierarchical lattice of order 2 (the ultimate points) with the metric generating tree attached.

- Formal definition of hierarchical lattice of order N:

$$\Omega_N := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_i \in \{0, 1, \dots, N-1\}, \sum_{i=1}^{\infty} x_i < \infty \right\}$$

- Labeling by non-negative integers: $f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} x_i N^{i-1}$

- Distance on Ω_N

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\} & \text{if } \mathbf{x} \neq \mathbf{y} \end{cases}$$

- Formal definition of hierarchical lattice of order N:

$$\Omega_N := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_i \in \{0, 1, \dots, N-1\}, \sum_{i=1}^{\infty} x_i < \infty \right\}$$

- Labeling by non-negative integers: $f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} x_i N^{i-1}$
- Distance on Ω_N

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\} & \text{if } \mathbf{x} \neq \mathbf{y} \end{cases}$$

For $x \in \Omega_N$, define $\mathcal{B}_r(x)$ to be the ball of radius r around x.
Some properties:

1. (Ω_N, d) is ultrametric: It satisfies the strengthened version of the triangle inequality
$$d(x,y) \leq \max(d(x,z), d(z,y))$$
for any triple $x, y, z \in \Omega_N$
2. $\mathcal{B}_r(x)$ contains N^r vertices for any x
3. For every $x \in \Omega_N$ there are $(N-1)N^{k-1}$ vertices at distance k
4. If $y \in \mathcal{B}_r(x)$ then $\mathcal{B}_r(x) = \mathcal{B}_r(y)$
5. Either $\mathcal{B}_r(x) = \mathcal{B}_r(y)$ or $\mathcal{B}_r(x) \cap \mathcal{B}_r(y) = \emptyset$

- SIR epidemic with fixed infectious period of length 1 on this network
- After infectious period infectious individual recovers and stay immune forever
- If an infectious person meets a susceptible person, the susceptible one becomes infectious: They share an edge in the *infection graph*
- Individuals at distance k meet according to Poisson process with intensity $\lambda(k) = \alpha/\beta^k$.
- Cluster of ultimately infected individuals is distributed as the cluster around the initial infected individual (the origin) in "long-range percolation" with $p(k) = 1 - e^{-\lambda(k)}$.

- SIR epidemic with fixed infectious period of length 1 on this network
- After infectious period infectious individual recovers and stay immune forever
- If an infectious person meets a susceptible person, the susceptible one becomes infectious: They share an edge in the *infection graph*
- Individuals at distance k meet according to Poisson process with intensity $\lambda(k) = \alpha/\beta^k$.
- Cluster of ultimately infected individuals is distributed as the cluster around the initial infected individual (the origin) in "long-range percolation" with $p(k) = 1 - e^{-\lambda(k)}$.

- Hierarchical lattice of order N, denoted by Ω_N
- Presence of absence of (undirected) edges between different pairs of vertices are independent
- Connection probability of vertices at distance k is $p_k = 1 - \exp[-\alpha/\beta^k]$ ($\approx \alpha/\beta^k$ for large k)
- $\mathcal{C}(x)$ is the cluster (connected component) of vertex x and $|\mathcal{C}(x)|$ is its size
- All vertices are "the same", so we may consider $\mathcal{C} = \mathcal{C}(0)$, without loss of generality
- $\mathbb{P}_{\alpha,\beta}$ is the probability measure corresponding to long-range percolation with parameters α and β
- $\theta(\alpha, \beta) := \mathbb{P}_{\alpha,\beta}(|\mathcal{C}| = \infty)$
- For $\mathcal{S}_1, \mathcal{S}_2 \subset \Omega_N$, $\mathcal{S}_1 \leftrightarrow \mathcal{S}_2$ denotes the presence of an edge between the two sets

Remark: If we represent the vertices by the non-negative integers and if the (Euclidean) distance between x and y is r, then the distance in the hierarchical tree is at least $\log[r]/\log[N]$
The probability that two vertices at (Euclidean) distance r are connected is therefore at most

$$\hat{p}_r = 1 - \exp[-\alpha\beta^{-\log[r]/\log[N]}] = 1 - \exp[-\alpha r^{-\log[\beta]/\log[N]}]$$

Behaviour of the largest cluster of long-range percolation on hierarchical lattice with exponentially decaying connection function is expected to be comparable with behaviour of the largest cluster of ordinary long-range percolation on (half-)line with polynomially decaying connection function

Remark: If we represent the vertices by the non-negative integers and if the (Euclidean) distance between x and y is r, then the distance in the hierarchical tree is at least $\log[r]/\log[N]$
The probability that two vertices at (Euclidean) distance r are connected is therefore at most

$$\hat{p}_r = 1 - \exp[-\alpha\beta^{-\log[r]/\log[N]}] = 1 - \exp[-\alpha r^{-\log[\beta]/\log[N]}]$$

Behaviour of the largest cluster of long-range percolation on hierarchical lattice with exponentially decaying connection function is expected to be comparable with behaviour of the largest cluster of ordinary long-range percolation on (half-)line with polynomially decaying connection function

Regime $\beta \leq N$

Theorem

If $\beta \leq N$, then $\theta(\alpha, \beta) = 1$.

Almost trivial: By

$$\sum_{k=1}^{\infty}(N-1)N^{k-1}(1-\exp[-\alpha/\beta^k]) = \infty$$

the origin is almost surely connected to infinitely many vertices

Regime $\beta \leq N$

Theorem

If $\beta \leq N$, then $\theta(\alpha, \beta) = 1$.

Almost trivial: By

$$\sum_{k=1}^{\infty} (N-1) N^{k-1} (1 - \exp[-\alpha/\beta^k]) = \infty$$

the origin is almost surely connected to infinitely many vertices

Regime $\beta \geq N^2$

Theorem

If $\beta \geq N^2$, then $\theta(\alpha, \beta) = 0$.

Proof for $\beta = N^2$: Proof relies on the fact that for each k, the probability that "ball of diameter k around 0 is not connected to its complement" is bounded away from 0.

$$\mathbb{P}(\mathcal{B}_k(0) \not\leftrightarrow \overline{\mathcal{B}_k(0)}) = \exp\left(-\alpha N^k \sum_{j=k+1}^{\infty} \frac{(N-1)N^{j-1}}{N^{2j}}\right)$$

$$= \exp\left(-\alpha \frac{(N-1)}{N^2} \sum_{j=1}^{\infty} N^{-(j-1)}\right) = \exp\left(-\frac{\alpha}{N}\right) > 0,$$

So, this event will eventually happen and therefore $\theta(\alpha, \beta) = 0$

Regime $\beta \geq N^2$

Theorem

If $\beta \geq N^2$, then $\theta(\alpha, \beta) = 0$.

Proof for $\beta = N^2$: Proof relies on the fact that for each k, the probability that "ball of diameter k around 0 is not connected to its complement" is bounded away from 0.

$$\mathbb{P}(\mathcal{B}_k(0) \not\leftrightarrow \overline{\mathcal{B}_k(0)}) = \exp\left(-\alpha N^k \sum_{j=k+1}^{\infty} \frac{(N-1)N^{j-1}}{N^{2j}}\right)$$

$$= \exp\left(-\alpha \frac{(N-1)}{N^2} \sum_{j=1}^{\infty} N^{-(j-1)}\right) = \exp\left(-\frac{\alpha}{N}\right) > 0,$$

So, this event will eventually happen and therefore $\theta(\alpha, \beta) = 0$

Regime $N < \beta < N^2$

Theorem

If $N < \beta < N^2$, then $0 < \alpha_c(\beta) := \inf\{\alpha; \theta(\alpha, \beta) > 0\} < \infty$

Lower bound follows by coupling with branching process
Upper bound: brute force and renormalization:

Regime $N < \beta < N^2$

Theorem

If $N < \beta < N^2$, then $0 < \alpha_c(\beta) := \inf\{\alpha; \theta(\alpha, \beta) > 0\} < \infty$

Lower bound follows by coupling with branching process

Upper bound: brute force and renormalization:

Regime $N < \beta < N^2$

Theorem

If $N < \beta < N^2$, then $0 < \alpha_c(\beta) := \inf\{\alpha; \theta(\alpha, \beta) > 0\} < \infty$

Lower bound follows by coupling with branching process
Upper bound: brute force and renormalization:

Upper bound: brute force and renormalization:

- Choose η and K such that $\sqrt{\beta} < \eta \leq (N^K - 1)^{1/K}$ this is possible since $\sqrt{\beta} < N$
- A ball of radius nK is *good* if the largest connected component contained in it (say $\mathcal{C}^m_{nK}$) has size at least η^{nK}

$$s_n := \mathbb{P}\left(|\mathcal{C}^m_{nK}| \geq \eta^{nK}\right)$$

- Probability that two good clusters of radius nK at distance $(n+1)K$ share an edge is at least

$$1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)$$

Upper bound: brute force and renormalization:

- Choose η and K such that $\sqrt{\beta} < \eta \leq (N^K - 1)^{1/K}$ this is possible since $\sqrt{\beta} < N$
- A ball of radius nK is *good* if the largest connected component contained in it (say $\mathcal{C}^m_{nK}$) has size at least η^{nK}

$$s_n := \mathbb{P}\left(|\mathcal{C}^m_{nK}| \geq \eta^{nK}\right)$$

- Probability that two good clusters of radius nK at distance $(n+1)K$ share an edge is at least

$$1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)$$

Upper bound: brute force and renormalization:

- Choose η and K such that $\sqrt{\beta} < \eta \leq (N^K - 1)^{1/K}$ this is possible since $\sqrt{\beta} < N$
- A ball of radius nK is *good* if the largest connected component contained in it (say $\mathcal{C}^m_{nK}$) has size at least η^{nK}

$$s_n := \mathbb{P}\left(|\mathcal{C}^m_{nK}| \geq \eta^{nK}\right)$$

- Probability that two good clusters of radius nK at distance $(n+1)K$ share an edge is at least

$$1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)$$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters

  So, s_{n+1} is at least

$$\mathbb{P}\left[Bin\left(N^k, s_n\left[1-\exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right]\right) \geq N^K - 1\right]$$

- If $X \sim Bin(n,p)$, then $\mathbb{P}(X \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2$.Therefore, s_{n+1} is at least

$$1 - \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1 - s_{n+1} < \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters
So, s_{n+1} is at least

$$\mathbb{P}\left[Bin\left(N^k, s_n\left[1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right]\right) \geq N^K - 1\right]$$

- If $X \sim Bin(n,p)$, then
$\mathbb{P}(X \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2$.Therefore, s_{n+1} is at least

$$1 - \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1 - s_{n+1} < \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters So, s_{n+1} is at least

$$\mathbb{P}\left[Bin\left(N^k, s_n\left[1-\exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right]\right) \geq N^K - 1\right]$$

- If $X \sim Bin(n,p)$, then $\mathbb{P}(X \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2$. Therefore, s_{n+1} is at least

$$1-\binom{N^K}{2}\left(1-s_n+\exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1-s_{n+1} < \binom{N^K}{2}\left(1-s_n+\exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters
  So, s_{n+1} is at least

$$\mathbb{P}\left[Bin\left(N^k, s_n\left[1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right]\right) \geq N^K - 1\right]$$

- If $X \sim Bin(n, p)$, then
  $\mathbb{P}(X \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2$.Therefore, s_{n+1} is at least

$$1 - \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1 - s_{n+1} < \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters So, s_{n+1} is at least

$$\mathbb{P}\left[Bin\left(N^k, s_n\left[1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right]\right) \geq N^K - 1\right]$$

- If $X \sim Bin(n, p)$, then $\mathbb{P}(X \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2$. Therefore, s_{n+1} is at least

$$1 - \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1 - s_{n+1} < \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters
  So, s_{n+1} is at least

$$\mathbb{P}\left[Bin\left(N^k, s_n\left[1-\exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right]\right) \geq N^K - 1\right]$$

- If $X \sim Bin(n, p)$, then
  $\mathbb{P}(X \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2$.Therefore, s_{n+1} is at least

$$1 - \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1 - s_{n+1} < \binom{N^K}{2}\left(1 - s_n + \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$

- By induction we can show that for α large enough and all $n > 1$, $1 - s_n < \gamma^{n+1}$, for $0 < \gamma$ arbitrary small
- This implies that a large ball is with high probability good
- Proving that with positive probability the origin is contained in a large good cluster for all large enough n can be done along the same lines

- By induction we can show that for α large enough and all $n > 1$, $1 - s_n < \gamma^{n+1}$, for $0 < \gamma$ arbitrary small
- This implies that a large ball is with high probability good
- Proving that with positive probability the origin is contained in a large good cluster for all large enough n can be done along the same lines

Uniqueness of infinite component

Theorem

The infinite component for supercritical long-range percolation on the hierarchical lattice is almost surely unique.

Use

Theorem (Gandolfi, Keane and Newman (1992))

If a supercritical long-range percolation measure on $\mathbb{Z}^d$ is translation invariant and satisfies a finite energy condition, then the infinite component is almost surely unique.

The finite energy condition is that the configuration of edges on $\Omega_N \times \Omega_N \setminus e$, does almost surely not determine whether edge e is present or absent.

Problem: We do not consider percolation on $\mathbb{Z}^d$.

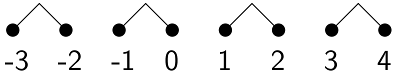
Idea of solution: Construct random projection of "distance generating tree" in $\mathbb{Z}$, which is translation invariant (even ergodic)

Problem: We do not consider percolation on $\mathbb{Z}^d$.

Idea of solution: Construct random projection of "distance generating tree" in $\mathbb{Z}$, which is translation invariant (even ergodic)

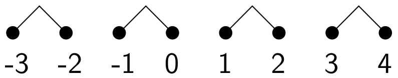
Construction for $N = 2$:

Step 1: ($n \in \mathbb{Z}$) Flip a fair coin: if heads, then $2n$ has distance 1 to $2n+1$, if tails, then $2n$ has distance 1 to $2n-1$.

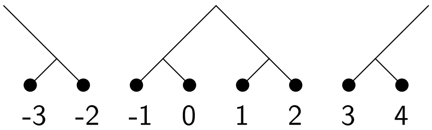


Step k Flip a fair coin: if heads, then $2^k n$ has distance k to $2^{k-1}(2n+1)$, if tails, then $2^k n$ has distance k to $2^{k-1}(2n-1)$.

Construction for $N = 2$:
Step 1: ($n \in \mathbb{Z}$) Flip a fair coin: if heads, then $2n$ has distance 1 to $2n + 1$, if tails, then $2n$ has distance 1 to $2n - 1$.



Step k Flip a fair coin: if heads, then $2^k n$ has distance k to $2^{k-1}(2n + 1)$, if tails, then $2^k n$ has distance k to $2^{k-1}(2n - 1)$.



Finite energy condition is satisfied and by construction the percolation measure is translation invariant

For ergodicity some extra work has to be done.

Finite energy condition is satisfied and by construction the percolation measure is translation invariant
For ergodicity some extra work has to be done.

Continuity of $\theta(\alpha, \beta)$

Theorem

The percolation probability $\theta(\alpha, \beta)$ is continuous for $\alpha, \beta > 0$.

Proof of continuity from the right (resp. left) in α (resp. β) is standard.

Proof of continuity from the left (resp. right) in α (resp. β) is involved. The ideas of the proof are similar to ideas used by Noam Berger (2002).

We need an intermediate lemma.

Continuity of $\theta(\alpha, \beta)$

Theorem

The percolation probability $\theta(\alpha, \beta)$ is continuous for $\alpha, \beta > 0$.

Proof of continuity from the right (resp. left) in α (resp. β) is standard.

Proof of continuity from the left (resp. right) in α (resp. β) is involved. The ideas of the proof are similar to ideas used by Noam Berger (2002).

We need an intermediate lemma.

Continuity of $\theta(\alpha, \beta)$

Theorem

The percolation probability $\theta(\alpha, \beta)$ is continuous for $\alpha, \beta > 0$.

Proof of continuity from the right (resp. left) in α (resp. β) is standard.

Proof of continuity from the left (resp. right) in α (resp. β) is involved. The ideas of the proof are similar to ideas used by Noam Berger (2002).
We need an intermediate lemma.

Lemma

The fraction of vertices in the largest component of long-range percolation graph restricted to $\mathcal{B}_k$ is for large k close to θ, with high probability.

Idea of proof:

1. For every constant $K > 0$ the indicator function of the event that both $|\mathcal{C}(0)| = \infty$ and $|\mathcal{C}_n(0)| < K(\beta/N)^n$ converges a.s. to 0 as $n \to \infty$. (straightforward computation)
2. The fraction of the vertices in $\mathcal{B}_n(0)$ which are in a cluster of size at least $K(\beta/N)^n$, converges a.s. to θ as $n \to \infty$.(ergodicity)
3. Combine the previous two steps: The large clusters at level n, are with high probability all in the same cluster at level $n+1$

Lemma

The fraction of vertices in the largest component of long-range percolation graph restricted to $\mathcal{B}_k$ is for large k close to θ, with high probability.

Idea of proof:

1. For every constant $K > 0$ the indicator function of the event that both $|\mathcal{C}(0)| = \infty$ and $|\mathcal{C}_n(0)| < K(\beta/N)^n$ converges a.s. to 0 as $n \to \infty$. (straightforward computation)
2. The fraction of the vertices in $\mathcal{B}_n(0)$ which are in a cluster of size at least $K(\beta/N)^n$, converges a.s. to θ as $n \to \infty$.(ergodicity)
3. Combine the previous two steps: The large clusters at level n, are with high probability all in the same cluster at level $n+1$

Lemma

The fraction of vertices in the largest component of long-range percolation graph restricted to $\mathcal{B}_k$ is for large k close to θ, with high probability.

Idea of proof:

1. For every constant $K > 0$ the indicator function of the event that both $|\mathcal{C}(0)| = \infty$ and $|\mathcal{C}_n(0)| < K(\beta/N)^n$ converges a.s. to 0 as $n \to \infty$. (straightforward computation)
2. The fraction of the vertices in $\mathcal{B}_n(0)$ which are in a cluster of size at least $K(\beta/N)^n$, converges a.s. to θ as $n \to \infty$.(ergodicity)
3. Combine the previous two steps: The large clusters at level n, are with high probability all in the same cluster at level $n + 1$

Theorem

$\theta(\alpha_c(\beta), \beta) = 0$ *for* $N < \beta < N^2$.

Idea of the proof:

- Assume $\theta := \theta(\alpha, \beta) > 0$: The density of the largest component of random graph restricted to large sub-ball is close to θ, with high probability
- Since subballs are finite, the density of largest cluster in large subball using $\alpha-$ and $\beta+$ is also close to θ
- Rescaled process at level K has parameters $\beta+$ and $\alpha \approx (\alpha-)\theta^2(N^2/\beta)^K$, which can be taken arbitrary large, by choosing K large enough
- So, there is also percolation for $\alpha-$ and $\beta+$, with density arbitrary close to θ

Theorem

$\theta(\alpha_c(\beta), \beta) = 0$ for $N < \beta < N^2$.

Idea of the proof:

- Assume $\theta := \theta(\alpha, \beta) > 0$: The density of the largest component of random graph restricted to large sub-ball is close to θ, with high probability
- Since subballs are finite, the density of largest cluster in large subball using $\alpha-$ and $\beta+$ is also close to θ
- Rescaled process at level K has parameters $\beta+$ and $\alpha \approx (\alpha-)\theta^2(N^2/\beta)^K$, which can be taken arbitrary large, by choosing K large enough
- So, there is also percolation for $\alpha-$ and $\beta+$, with density arbitrary close to θ

Theorem

$\theta(\alpha_c(\beta), \beta) = 0$ *for* $N < \beta < N^2$.

Idea of the proof:

- Assume $\theta := \theta(\alpha, \beta) > 0$: The density of the largest component of random graph restricted to large sub-ball is close to θ, with high probability
- Since subballs are finite, the density of largest cluster in large subball using $\alpha-$ and $\beta+$ is also close to θ
- Rescaled process at level K has parameters $\beta+$ and $\alpha \approx (\alpha-)\theta^2(N^2/\beta)^K$, which can be taken arbitrary large, by choosing K large enough
- So, there is also percolation for $\alpha-$ and $\beta+$, with density arbitrary close to θ

Theorem

$\theta(\alpha_c(\beta), \beta) = 0$ *for* $N < \beta < N^2$.

Idea of the proof:

- Assume $\theta := \theta(\alpha, \beta) > 0$: The density of the largest component of random graph restricted to large sub-ball is close to θ, with high probability
- Since subballs are finite, the density of largest cluster in large subball using $\alpha-$ and $\beta+$ is also close to θ
- Rescaled process at level K has parameters $\beta+$ and $\alpha \approx (\alpha-)\theta^2(N^2/\beta)^K$, which can be taken arbitrary large, by choosing K large enough
- So, there is also percolation for $\alpha-$ and $\beta+$, with density arbitrary close to θ

Continuity of $\alpha_c(\beta)$

Theorem

$\alpha_c(\beta)$ is continuous on $\beta \in (0, N^2)$ and strictly increasing on $\beta \in [N, N^2)$. Furthermore, $\alpha_c(\beta) \nearrow \infty$ for $\beta \nearrow N^2$.

The proof relies on the result by Aizenman and Barsky (1987) that for independent long-range percolation on $\mathbb{Z}^d$:

$$\inf\{\alpha : \theta(\alpha, \beta) > 0\} = \sup\{\alpha : \mathbb{E}_{\alpha,\beta}(|\mathcal{C}(0)|) < \infty\}$$

Close inspection of their proof shows that this result also holds for independent long-range percolation on the hierarchical lattice.

Continuity of $\alpha_c(\beta)$

Theorem

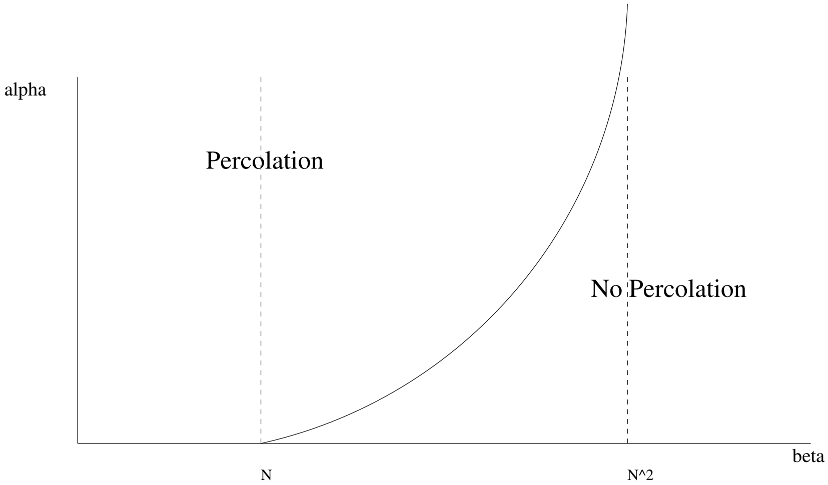
$\alpha_c(\beta)$ is continuous on $\beta \in (0, N^2)$ and strictly increasing on $\beta \in [N, N^2)$. Furthermore, $\alpha_c(\beta) \nearrow \infty$ for $\beta \nearrow N^2$.

The proof relies on the result by Aizenman and Barsky (1987) that for independent long-range percolation on $\mathbb{Z}^d$:

$$\inf\{\alpha : \theta(\alpha, \beta) > 0\} = \sup\{\alpha : \mathbb{E}_{\alpha,\beta}(|\mathcal{C}(0)|) < \infty\}$$

Close inspection of their proof shows that this result also holds for independent long-range percolation on the hierarchical lattice.

- By continuity of θ: If $\theta(\alpha, \beta) > 0$, then $\theta(\alpha-, \beta+) > 0$. This implies continuity from the right of $\alpha_c(\beta)$
- If $\mathbb{E}_{\alpha,\beta}(|\mathcal{C}(0)|) < \infty$, then (after some work:) $\mathbb{E}_{\alpha+,\beta-}(|\mathcal{C}(0)|) < \infty$
  This implies continuity from the left of $\alpha_c(\beta)$


alpha
Percolation
No Percolation
beta
N
N^2

Related work

- In independent work, Dawson and Gorostiza also studied percolation on the hierarchical lattice. They obtained additional results on percolation around $\beta = N^2$
  In particular they studied
  $$\lambda(k) = \alpha k^{\gamma} N^{-2k}$$
  and for given constants C, K, a, b, with $k_n = \lfloor Kn \log[n] \rfloor$,
  $$\lambda(k_n) = C + a \log[n] n^{b \log[N]} N^{-2k_n}$$

- Athreya and Swart studied the contact process on the hierarchical lattice and derrived conditions for survival. In particular if the contact rate is exponentially decaying in the distance, there is survival for large enough recovery rate

Related work

- In independent work, Dawson and Gorostiza also studied percolation on the hierarchical lattice. They obtained additional results on percolation around $\beta = N^2$
  In particular they studied
  $$\lambda(k) = \alpha k^{\gamma} N^{-2k}$$
  and for given constants C, K, a, b, with $k_n = \lfloor Kn \log[n] \rfloor$,
  $$\lambda(k_n) = C + a \log[n] n^{b \log[N]} N^{-2k_n}$$

- Athreya and Swart studied the contact process on the hierarchical lattice and derrived conditions for survival. In particular if the contact rate is exponentially decaying in the distance, there is survival for large enough recovery rate

- Biskup studied the graph diameter for long-range percolation on $\mathbb{Z}^d$, with polynomial decay. It is expected that his results also hold for percolation on the Hierarchical lattice with $N < \beta < N^2$
- T. investigated how the volume of the random “graph-ball” of radius k grows in k for long-range percolation on $\mathbb{Z}^d$. The results imply that for long-range percolation on the hierarchical lattice this growth is sub-exponential for $\beta > N$. If $\lambda(k) = f(k)N^{-k}$, where $f(k)$ is polynomial, this growth might be exponential

- Biskup studied the graph diameter for long-range percolation on $\mathbb{Z}^d$, with polynomial decay. It is expected that his results also hold for percolation on the Hierarchical lattice with $N < \beta < N^2$
- T. investigated how the volume of the random "graph-ball" of radius k grows in k for long-range percolation on $\mathbb{Z}^d$. The results imply that for long-range percolation on the hierarchical lattice this growth is sub-exponential for $\beta > N$. If $\lambda(k) = f(k)N^{-k}$, where $f(k)$ is polynomial, this growth might be exponential

Open problems

- How to deal with random "metric generating" trees and how to construct them?
- Is it possible to give useful bounds for $\alpha_c(\beta)$?
- How about FK-models on the hierarchical lattice?

Open problems

- How to deal with random "metric generating" trees and how to construct them?
- Is it possible to give useful bounds for $\alpha_c(\beta)$?
- How about FK-models on the hierarchical lattice?

Open problems

- How to deal with random "metric generating" trees and how to construct them?
- Is it possible to give useful bounds for $\alpha_c(\beta)$?
- How about FK-models on the hierarchical lattice?

Open problems

- How to deal with random “metric generating” trees and how to construct them?
- Is it possible to give useful bounds for $\alpha_c(\beta)$?
- How about FK-models on the hierarchical lattice?

Bibliography

- M. Aizenman and D.J. Barsky, *Comm. Math. Phys.*, 108(3):489–526, 1987.
- S.R. Athreya and J.M. Swart, *Probab. Theory Relat. Fields* 147:529563, 2010.
- N. Berger, *Comm. Math. Phys.*, 226(3):531–558, 2002.
- M. Biskup, to appear in *Random Structures & Algorithms*
- D. Dawson and L. Gorostiza, arXiv:1006.4400
- A. Gandolfi, M.S. Keane, and C.M. Newman, *Probab. Theory Related Fields*, 92(4):511–527, 1992.
- P. Trapman, *Ann. Probab.*, 38(4): 1583–1608, 2010.