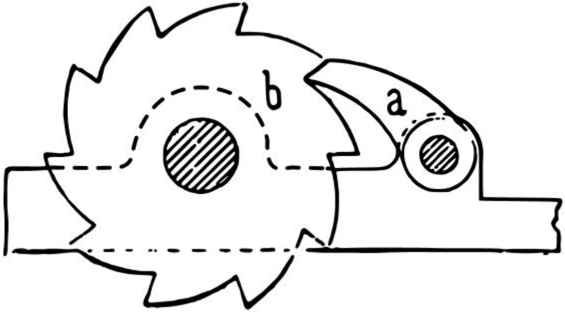
Properties of an infinite dimensional EDS system : the Muller's ratchet

LATP

June 5, 2011

A ratchet



source : wikipedia

Plan

1. Introduction : The model of Haigh
2. Model, first properties and theorem
3. Proof
   - Basic results
   - First properties on M_1
   - Ω_1 and Ω_2
   - Recurrence on M_1
   - Reaching Ω_2 starting from the recurrence
   - $E(T_0) < +\infty$

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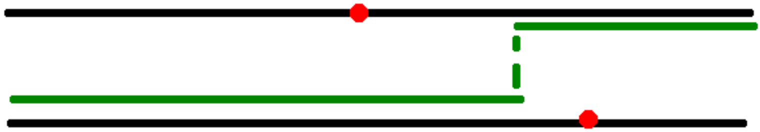

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- Note that deleterious mutations can't be lost.
- If at any given time, the best class is empty, it shall remain empty forever. That is to say, the minimal number of deleterious in this population has increased. It can't go back (like a ratchet).

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$$\frac{(1-\alpha)^{Y_k}}{\sum_{l=0}^{N}(1-\alpha)^{Y_l}}$$

- Each individual has a number of mutations equals to the number of his parent $+ P(\lambda)$.

In this model, at each generation the ratchet happens with a probability $\geq \left(\lambda e^{-\lambda}\right)^N$. (everyone mutates)

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- So the ratchet will happens infinitely many times a.s.
- This means that the fitness of the population $\rightarrow -\infty$.

Now the model studied here is the following one, with $X_k(t)$ the proportion of individual with k deleterious mutations at time t.
$\forall k \geq 0$

$$\begin{cases} dX_k = \left[\alpha(\sum_{l=0}^{\infty} lX_l - k)X_k + \lambda(X_{k-1} - X_k)\right] dt + \sum_{l\in\mathbb{N}} \sqrt{\frac{X_k X_l}{N}} dB_k, \\ X_k(0) = X_k^0 \end{cases}$$

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- where $B_{k,\ell}$ are independent Brownian motions, except for $B_{k,\ell} = -B_{\ell,k}$,
- and $\sum_k X_k(0) = 1$, $X_k(0) \geq 0 \ \forall k \geq 0$

Note that these equations are equivalent to

$$\begin{cases} dX_k = [\alpha(M_1 - k)X_k + \lambda(X_{k-1} - X_k)]\, dt + \sqrt{\dfrac{X_k(1-X_k)}{N}} dB_k, \\ X_k(0) = X_k^0 \end{cases}$$

with $M_1(t) = \sum_k kX_k(t)$ the median number of mutations in the population at time t,
and dB_k standard brownian motions, with $\forall k \neq l$
$\langle dX_k, dX_l \rangle (t) = -X_k X_l dt$

Why this model ? Because it is some kind of limit for the Haigh's one, with an infinite population which stay finite at the same time (N). If you look at the equation, you can see :

$$dX_k = \left[\underbrace{\alpha(M_1 - k)X_k} + \lambda(X_{k-1} - X_k)\right] dt + \sqrt{\frac{X_k(1-X_k)}{N}} dB_k,$$

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Difficulties of this model :

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- Each equations use all the X_k both in M_1 and in their stochastic term, they can't be separated.
- The diffusion coefficient isn't lipschiztian around 0 and 1.

We can calculate the equation of M_1 :

$$dM_1(t) = (\lambda - \alpha M_2(t))dt + \sqrt{\frac{M_2(t)}{N}}dB_t,$$

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- But if we calculate the equation of M_2, M_3 and M_4 appears, and so on. The equation of M_k use up to M_{2k}. This system can't be solved. (M_k is the k-th centered moment).

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- But if we calculate the equation of M_2, M_3 and M_4 appears, and so on. The equation of M_k use up to M_{2k}. This system can't be solved. (M_k is the k-th centered moment).
- Moreover, we have $\langle dX_k, dM_1 \rangle = -X_k M_1 dt$, they aren't independant.

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- F.Yu, A. Etheridge and C.Cuthbertson have shown that our problem is well posed.
- From this we deduce that our problem has the Markov property.
- And.. That's all.

What we want to show :

Theorem

Let $T_0 = \{\inf t \geq 0, X_0(t) = 0\}$ *Then for any initial condition* $(X_k(0))_{k\in\mathbb{N}}$, $P(T_0 < +\infty) = 1$*. In other words, the ratchet will click a.s.*

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- A.Etheridge, A. Wakolbinger, P.Pfaffelhuber have shown that in the determinist case ($N = +\infty$), the system can be explicitely solved using cumulants on the M_k. They obtain that the system will converge at exponential rate to the Poisson distribution with parameter $\frac{\lambda}{\alpha}$

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- A.Etheridge, A. Wakolbinger, P.Pfaffelhuber have shown that in the determinist case ($N = +\infty$), the system can be explicitely solved using cumulants on the M_k. They obtain that the system will converge at exponential rate to the Poisson distribution with parameter $\frac{\lambda}{\alpha}$
- But then it never clicks. The ratchet only happens due to the randomness. Moreover the cumulant system in the stochastic case has no known solution.

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The proof will use quite a few intermediate results. We will first present a comparison theorem we will use a lot :
Our processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ which is such that for each $k, \ell \geq 0$ $\{B_{k,\ell}(t),\ t \geq 0\}$ is a $\mathcal{F}_t$–Brownian motion. We denote by $\mathcal{P}$ the corresponding σ-algebra of predictable subsets of $\mathbb{R}_+ \times \Omega$.

Lemma

Let B_t be a standard $\mathcal{F}_t$–Brownian motion, T a stopping time, σ be a 1/2 Hölder function, $b_1 : \mathbb{R} \to \mathbb{R}$ a Lipschitz function and $b_2 : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P} \otimes B(\mathbb{R})$ measurable function. Consider the two SDEs

$$\begin{cases} dY_1(t) = b_1(Y_1(t))dt + \sigma(Y_1(t))dB_t, \\ Y_1(0) = y_1; \end{cases} \tag{3.1}$$

$$\begin{cases} dY_2(t) = b_2(t, Y_2(t))dt + \sigma(Y_2(t))dB_t, \\ Y_2(0) = y_2. \end{cases} \tag{3.2}$$

Lemma

Let Y_1 (resp Y_2) be a solution of (3.1) (resp (3.2)). If $y_1 \leq y_2$ (resp $y_2 \leq y_1$) and outside a measurable subset of Ω of probability zero, $\forall t \in [0, T]$, $\forall x \in \mathbb{R}$, $b_1(x) \leq b_2(t, x)$ (resp $b_1(x) \geq b_2(t, x)$), then a. s. $\forall t \in [0, T]$, $Y_1(t) \leq Y_2(t)$ (resp $Y_1(t) \geq Y_2(t)$).

And, last but not least, a remark :

Remark

let A, $B \subset \Omega$. Then $P(A \cap B) \geq P(A) + P(B) - 1$.

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Now we will compare the equation of M_1 to a more friendly one.

$$
\begin{aligned}
&(\lambda - \alpha M_2(t))dt + \sqrt{\frac{M_2(t)}{N}}dB_t \\
&= (\lambda - \frac{1}{2}\alpha M_2(t))dt + \sqrt{\frac{M_2(t)}{N}}dB_t - \frac{1}{2}\alpha M_2(t)dt \\
&\leq (\lambda - \frac{1}{2}\alpha X_0 M_1^2)dt + \sqrt{\frac{M_2(t)}{N}}dB_t - \frac{1}{2}\alpha M_2(t)dt
\end{aligned}
$$

We will show that M_1 can't grow too fast :

Lemma

$\forall c > 0$, $\forall t > 0$,

$$P(\sup_{0\le r\le t_2''} M_1(r+t) - M_1(t) \le \lambda t_2'' + c) \ge 1 - exp(-2\alpha Nc) > 0$$

PROOF : We use $Z'_{s,t} = \int_t^{t+s} \sqrt{\frac{M_2(r)}{N}} dB_r - \alpha \int_t^{s+t} M_2(r)dr,$.
We note that, at set t, $exp(2\alpha N Z'_{u,t})$ is both a local martingale and a surmartingale. We also have

$$\sup_{0\le s\le t'} M_1(s+t) - M_1(t) \le \sup_{0\le s\le t'} Z'_{s,t'} + \lambda t',$$

using the comparison theorem and the previous remark.

And $\forall c > 0$

$$P(\sup_{0\leq u\leq t'} Z'_{u,t} \geq c) \leq P(\sup_{0\leq u\leq t'} exp(2\alpha N Z'_{u,t}) \geq exp(2\alpha N c))$$
$$\leq exp(-2\alpha N c) < 1$$

Where we have taken advantage of that $exp(2\alpha N Z'_{u,t})$ is a local martingale and Doob inequality, hence $P(\sup_{u\geq 0} Z'_{u,t} \geq c) \leq exp(-2\alpha N c) < 1$ Then

$$P(\sup_{0\leq r\leq t'} M_1(r+t) - M_1(t) \leq \lambda t' + c) \geq 1 - exp(-2\alpha N c) > 0$$

$\diamondsuit$

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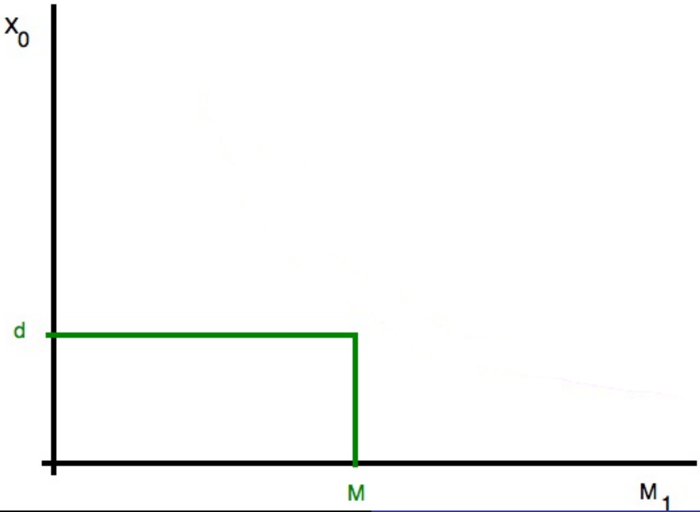

X0
d
M
M 1

What is Ω_1 ? For this we use a simpler model.
Let X be the solution of the following system :

$$\begin{cases} X(0) = \delta \\ X(t) = dt + 2\sqrt{X(t)}dB_0 \end{cases}$$

If $X_0(0) \leq \delta$, $M_1(0) \leq M$, as long as $X_0 M_1 < 2\varepsilon$ we can compare X_0 and X.

Lemma

Let $T_{min} = \inf\{t > 0, X_0(t)M_1(t) \geq 2\varepsilon$ *or* $X_0(t) \geq \delta + \mu\}$. *Then* $\forall t \in [0, T_{min}]$, *we have* $X_0(t) \leq X(A(t))$, *where* $A(t) = \frac{1}{4}\int_0^t \frac{1-X_0(s)}{N} ds$ *and* $\sigma(t) = \inf\{u > 0, A(u) \geq t\}$.

Note that $\frac{t}{5N} \leq A(t) \leq \frac{t}{4N}$ because $\frac{4}{5} \leq 1 - X_0 \leq 1$ (thanks to the choices of μ and δ), and T was chosen such as $A(T) \leq \frac{\varepsilon}{12\lambda}$.

PROOF : We note $\tilde{X}_0(t) = X_0(\sigma(t))$ (resp $\tilde{M}_1(t) = M_1(\sigma(t)))$

$$d\tilde{X}_0(t) = (\alpha\tilde{M}_1(t) - \lambda)\tilde{X}_0(t)\frac{4N}{1-\tilde{X}_0(t)}dt + 2\sqrt{\tilde{X}_0(t)}dW_t$$

Since $\tilde{M}_1(t)\tilde{X}_0(t) \leq 2\varepsilon$ and $1-\delta \geq \frac{1}{2}$,

$$(\alpha\tilde{M}_1(t) - \lambda)\tilde{X}_0(t)\frac{4N}{1-\tilde{X}_0(t)} \leq 1$$

Then, using the comparaison theorem, we obtain the conclusion.

$\diamondsuit$

Lemma

Let X(t) be the solution of the previous system. and $T_0' = \inf\{t > 0, X(t) = 0\}$. *Then* $\forall T' > 0$, $\forall p_2 < 1$ $\exists \delta > 0$ *such as*

$$P(T_0' \leq T') > p_2$$

PROOF : Let

$$Y(t) = \delta exp\left(-t + 2B_0(t)\right)$$

$$D(t) = \int_0^t Y(s)ds$$

$$\sigma(t) = inf\{s > 0, D(s) > t\}$$

We have

$$X_t = Y_{\sigma(t)}$$

$$D(\infty) < \infty$$

Then $\forall T' > 0$,

$$P(\int_0^\infty Y(t)dt \leq T') = P(\int_0^\infty exp\left(-t + 2B_0(t)\right)dt \leq \frac{T'}{\delta})$$

But $\forall T' > 0$, $\lim_{\delta\to 0} \frac{T'}{\delta} = \infty$ a. s. $\diamondsuit$

Corollary

Let $X(t)$ be the solution of (3.1), $T'_0 = \inf\{t > 0, X(t) = 0\}$, $T'_\mu = \inf\{t > 0, X(t) = \delta + \mu\}$. *Then* $\forall T' > 0$, $\forall p_2 < 1$, $\forall \mu > 0$, $\exists \delta > 0$ *such as*

$$P(T'_0 \le T' \wedge T'_\mu) > p_2$$

PROOF : We use the same proof as before, noticing that

$$\begin{aligned} &P(T'_0 \le T' \wedge T'_\mu) \\ &\ge P(\{\int_0^\infty exp(-t + 2B_0(t))dt \le \frac{T'}{\delta}\} \\ &\cap \{\sup_{t \ge 0} exp(-t + 2B_0(t)) \le \frac{\delta + \mu}{\delta}\}) \end{aligned}$$

Now we chose a value $M > 0$ for $M_1(0)$, and we set

$$T' = \frac{\varepsilon N}{3\lambda},$$

$$M_{max} = M + \lambda A(T') + \varepsilon,$$

where A is defined below,

$$p_2 = exp(-\alpha N\frac{\varepsilon}{3}),$$

$$\mu = \frac{\varepsilon}{6M_{max}} \wedge \frac{\varepsilon}{4} \wedge \frac{1}{10}.$$

Now δ' is given in terms T', M_{max}, p_2 and μ by the previous corollary, and we let

$$\delta = \delta' \wedge \frac{1}{10} \wedge \frac{\varepsilon}{M}$$

Now we need to control T_min. We need to prove is

$$X_0(t)M_1(t) \leq 2\varepsilon \; \forall 0 \leq t \leq A(T') \wedge A(T'_\mu)$$

with probability $p_3 > 1 - p_2$.

Lemma

$\exists p_3 > 1 - p_2$ such as
$P(\sup_{0 \leq t \leq A(T') \wedge A(T'_\mu)} X_0(t)M_1(t) \leq 2\varepsilon) \geq p_3$

PROOF :

$$P\Big(\sup_{0\le t\le A(T')\wedge A(T'_\mu)} M_1(t) \le M_1(0) + \lambda A(T') + \frac{\varepsilon}{6}\Big)$$

$$\ge P\Big(\sup_{0\le t\le A(T')} M_1(t) \le M_1(0) + \lambda A(T') + \frac{\varepsilon}{6}\Big)$$

$$\ge 1 - exp(-\alpha N\frac{\varepsilon}{6})$$

Then

$$
\begin{aligned}
\sup_{0\le t\le A(T')\wedge A(T'_\mu)} X_0(t)M_1(t) &\le (\delta+\mu)(M_1(0)+\lambda A(T')+\frac{\varepsilon}{6})\\
&\le \delta M_1(0)+\mu M_1(0)+\lambda A(T')+\frac{\varepsilon}{6}\\
&\le \varepsilon+\frac{\varepsilon}{6}+\frac{\varepsilon}{12}+\frac{\varepsilon}{6}\\
&\le 2\varepsilon
\end{aligned}
$$

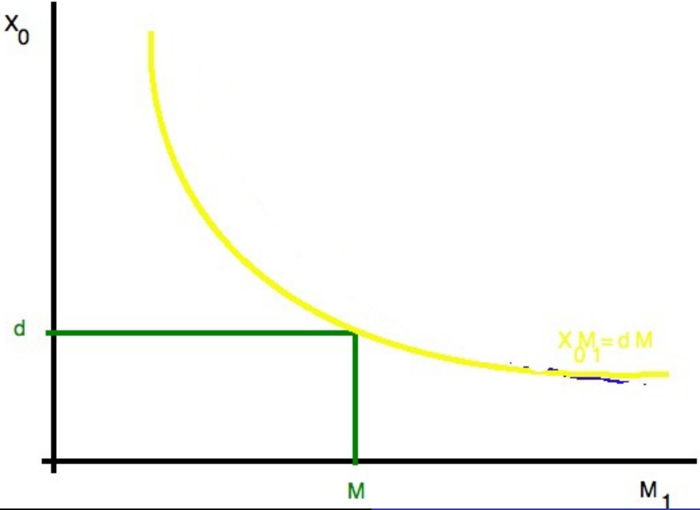
$\diamondsuit$

So, to sum up, with probability $\geq p_{fin} = p_2 + p_3 - 1$, we have :

$$T_{min} \geq A(T'_\mu) \wedge A(T'),$$

$$T'_\mu \geq T',$$

hence on the interval $\left[0, A(T') \wedge A(T'_\mu)\right] = [0, A(T')]$, we have both $X_0(t) \leq X(A(t))$ and $X(A(t))$ reaches 0.Hence the result.


X0
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X0M1=d M

Now for the set

$$\Omega_2 = \{(X, M_1),\ X \leq \delta \text{ and } XM_1 \leq \delta M (\leq \varepsilon)\}$$

We will show that $X_0'(0), M_1'(0)$ has a better probability to reach 0 before time T.
Let $C = \frac{M_1'}{M} \geq 1$ (because of the definition of X, M). Then we have $X_0' \leq \frac{\varepsilon}{C}$.
The probability to reach zero for $X(t)$ is decreasing in δ, we increase this probability by starting from $X(0) = X_0'(0) \leq \frac{\delta}{C}$.
Moreover, we start with $X_0'(0)M_1'(0) \leq \varepsilon$. The only thing which is worse than the original case is $M_1'(0)$ which is greater than $M_1(0)$, hence a greater M_{max}.

But this only appears in one place : when we defined μ. Note that if we define $M'_{1,max} = M'_1(0) + \lambda T + \frac{\varepsilon}{6}$, the maximum reached by M'_1, we have :

$$M'_{1,max} \leq CM_{max}$$

$$P(M'_{1,max} \geq \sup_{0 \leq t \leq T} M'_1(t)) \geq 1 - exp(-\alpha N \frac{\varepsilon}{6})$$

By definition of μ, if we define μ' with $M'_{1,max}$ we have $\mu' \geq \frac{\mu}{C}$.

But if we look at the demonstration , we have , since $\frac{T'}{\delta'} \geq \frac{CT'}{\delta} \geq \frac{T'}{\delta}$ and $\frac{\delta'+\mu'}{\delta'} = 1 + \frac{\mu'}{\delta'} \geq 1 + \frac{\mu}{\delta}$

$$\begin{aligned}
&P(T_0' \leq T' \wedge T_{\mu'}') \\
&\geq P\left(\{\int_0^\infty exp(-t + 2B_1(t))dt \leq \frac{T'}{\delta'}\}\right. \\
&\left.\cap\{\sup_{t\geq 0} exp(-t + 2B_1(t)) \leq \frac{\delta' + \mu'}{\delta'}\}\right) \\
&\geq P\left(\{\int_0^\infty exp(-t + 2B_1(t))dt \leq \frac{T'}{\delta}\}\right. \\
&\left.\cap\{\sup_{t\geq 0} exp(-t + 2B_1(t)) \leq \frac{\delta + \mu}{\delta}\}\right)
\end{aligned}$$

We sum up the result in the following proposition, where $\varepsilon = XM$ (it's a different ε, smaller than the previous one, but we keep the notation).

Proposition

Let $(X_k(t))_{k\in\mathbb{N}}$ be the solution of the initial model, and M_1 its mean as defined in the beginning. Then $\exists p \geq p_{fin} > 0$ such as if $\exists t > 0$ such as $X_0(t) \leq \delta$ and $X_0(t)M_1(t) \leq \varepsilon$,

$$P(T_0 < t + T) \geq p > 0$$

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First an inequality

Lemma

Let μ be a probability on $\mathbb{N}$, $x_k = \mu(k)\ \forall k \geq 0$,
$M_1 = \sum_{k\in\mathbb{N}} kx_k$, and $M_2 = \sum_{k\in\mathbb{N}} (k - M_1)^2 x_k$
Then $M_2 \geq 1 - x_0 M_2 \geq x_0 M_1^2$.

PROOF : By Jensen we have

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- $$\left(\sum_{k\geq 1}\frac{X_k}{1-X_0}k\right)^2 \leq \sum_{k\geq 1}\frac{X_k}{1-X_0}k^2$$

  with equality if and only if there exists only one $k \geq 1$ such as $X_k > 0$. Then :

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  with equality if and only if there exists only one $k \geq 1$ such as $X_k > 0$. Then :

- $$M_1^2 \leq (1-X_0) \sum_{k\geq 1} X_k k^2, \text{ hence}$$

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- $$M_1^2 \leq (1-X_0) \sum_{k\geq 1} X_k k^2, \text{ hence}$$

- $X_0 M_1^2 \leq (1-X_0) \sum_{k\geq 1} X_k k^2 - (1-X_0) M_1^2$

PROOF : By Jensen we have

- $$\left(\sum_{k\geq 1}\frac{X_k}{1-X_0}k\right)^2 \leq \sum_{k\geq 1}\frac{X_k}{1-X_0}k^2$$

  with equality if and only if there exists only one $k \geq 1$ such as $X_k > 0$. Then :
- $$M_1^2 \leq (1-X_0)\sum_{k\geq 1} X_k k^2, \text{ hence}$$
- $X_0 M_1^2 \leq (1-X_0)\sum_{k\geq 1} X_k k^2 - (1-X_0)M_1^2$
- $X_0 M_1^2 \leq (1-X_0)\left(\sum_{k\geq 1} X_k k^2 - M_1^2\right)$ $\diamondsuit$

Now we can obtain some kind of recurrence for M_1.

Lemma

Let $S_\lambda^T := \inf\{t \geq T,\ X_0(t)M_1(t)^2 \leq 2^{\frac{\lambda+1}{\alpha}}\}$. *Then for any* $T > 0$, *we have* $S_\lambda^T < +\infty$

Proof :

PROOF :

- On the interval $[0, S_\lambda]$,we have

$$-\frac{\alpha}{2}M_2 \leq -\frac{\alpha}{2}X_0 M_1^2 \leq -(\lambda+1),$$

PROOF :

- On the interval $[0, S_\lambda]$,we have

$$-\frac{\alpha}{2}M_2 \leq -\frac{\alpha}{2}X_0 M_1^2 \leq -(\lambda+1),$$

- and then the process M_1 is bounded from above by the process Y, solution of the SDE

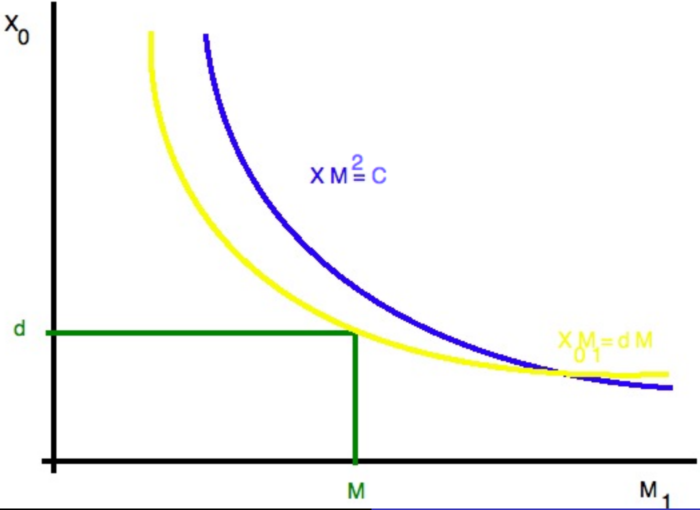
$$\begin{cases} dY_t = -dt - \frac{\alpha}{2}M_2(t)dt + \sqrt{\frac{M_2(t)}{N}}dB_t, \\ Y_0 = M_1(0). \end{cases}$$

Since M_1 cannot become negative, it now suffice to show that

$$Z_t := \int_0^t \sqrt{\frac{M_2(r)}{N}} dB_r - \frac{\alpha}{2} \int_0^t M_2(r) dr$$

is bounded from above a.s. If we define $C(t) = \frac{1}{N} \int_0^t M_2(s) ds$, we have $Z_t = W(C(t)) - \frac{N}{2} \alpha C(t)$ where W is a standard Brownian motion.

Now, if $C(\infty) = \infty$ then $\lim_{t\to\infty} Z_t = -\infty$, hence Z_t is bounded from above. Or else $C(\infty) < \infty$, and we have $\sup_{t>0} \|Z_t\| = \sup_{0<s<C(\infty)} \|W(s) - \frac{N}{2}\alpha s\| < \infty$ a.s. $\diamondsuit$


X_0
$XM^2 = C$
d
$X_0 M_1 = d M$
M
M_1

As a consequence of the previous results :

Lemma

Let $(X_k(t))_{k\in\mathbb{N}}$ be the solution of the Muller's ratchet. Then we have

$$\{\lim_{t\to+\infty} X_0(t) = 0\} \subseteq \{T_0 < +\infty\}$$

PROOF : If $\lim_{t\to+\infty} X_0(t) = 0$ then $\exists T_0' > 0$ such as $\forall t > T_0'$, we have $X_0(t) \leq \delta \wedge \frac{\varepsilon^2\alpha}{4(\lambda+1)}$.
Then we have a $T_1' > T_0'$ such as

$$X_0(t)M_1^2(t) \leq 2\frac{\lambda+1}{\alpha}.$$

Let us suppose that $X_0 M_1 > \varepsilon$. Then

$$M_1 = \frac{X_0 M_1}{X_0} > 4\frac{\lambda+1}{\alpha\varepsilon}$$
$$X_0 M_1^2 = X_0 M_1 \ M_1 \geq 4\frac{\lambda+1}{\alpha}$$

which is absurd. Then we have $X_0 M_1 \leq \varepsilon$.

Then we have $P(T_0 < T_1' + T) > p$. This situation presents itself infinitely many time, as long as the ratchet doesn't click, we obtain the conclusion, and since the system $(X_k)_{k\in\mathbb{N}}$ has the markov property. $\diamondsuit$

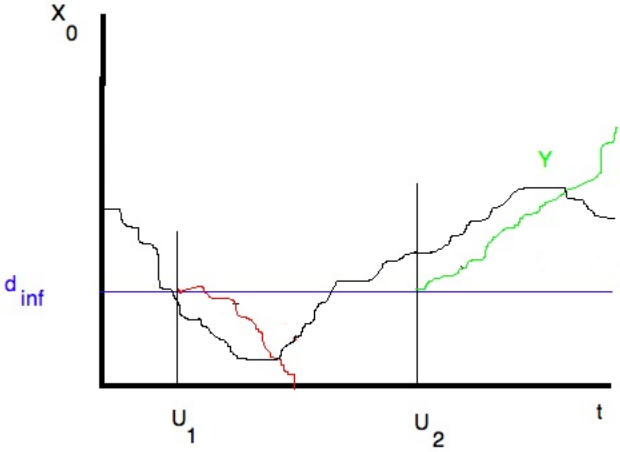
Now the recurrence on M_1 Let

$$S_t^\beta = inf\{t' > t \geq 0, M_1(t') \leq \beta\}.$$

Then we will prove the following lemma :

Lemma

$\forall t > 0,$ *we have* $P(T_0 \wedge S_t^\beta < \infty) = 1$


X_0
d_{inf}
Y
U_1
U_2
t

PROOF :

First, we note $\delta_{inf} = \delta \wedge \frac{\varepsilon^2\alpha}{4(\lambda+1)}$.

Now we introduce the process $Y_t^{t'}$, defined $\forall t' \geq 0$, $\forall t \geq t'$ which is the solution of the following system :

$$\begin{cases} dY_t^{t'} = \sqrt{\dfrac{Y_t^{t'}(1 - Y_t^{t'})}{N}}dB_0 \\ Y_{t'}^{t'} = \delta_{inf} \end{cases} \tag{3.3}$$

We note that $Y_t^{t'}$ is an instance of the processes studied in the addendum, because here we have $-2 < f(t) = 0 < 2$. Then using

$$R_u^{t'} = inf\{t \geq t', Y_t^{t'} = u\},$$

We have $E(R_0^{t'} \wedge R_1^{t'}) < +\infty$ and $P(R_1^{t'} < R_0^{t'}) > 0$. From this we deduce that $\exists K > 0, p > 0$ such as $P(R_1^{t'} \leq K \wedge R_0^{t'}) \geq p > 0)$. In particular $P(R_1^{t'} \leq K) \geq p > 0$.
We use $L = K \vee T$. (T from the step 3). Now we construct the following sequence : $U_0 = U_n =$ the first time where $x_0 M_1^2 \leq 2^{\frac{\lambda+1}{\alpha}}$, and $\forall n \geq 1$ $U_n =$ the first time after $U_{n-1} + L$ where $x_0 M_1^2 \leq 2^{\frac{\lambda+1}{\alpha}}$. This time exists a.s. and is $< +\infty$ thanks to prop 1.2.

Now, at U_0 : Either $X_0(U_0) \leq \delta_i nf$, then using the same reasoning as 4.1, we have $P(T_0 \leq U_0 + T) = p_{fin} > 0$.
Or $X_0(U_0) > \delta_i nf$. And in that case there is 2 case : Either $\inf_{U_0 \leq t \leq U_0+K} M_1(t) \geq \beta$. In that case we have $(\alpha M_1 - \lambda)x_0 \geq 0$, and then we can use the comparaison theorem and we get $X_0(t) \geq Y^t_{U_0}$. Then $P(T_1 \leq U_0 + K) \geq p > 0$. But if $X_0(t) = 1$, then $M_1(t) = 0$. Hence $P(S_\beta \leq U_0 + K) \geq p > 0$. In the other case $\inf_{U_0 \leq t \leq U_0+K} M_1(t) < \beta$, hence $S_\beta \leq U_0 + K$.

To conclude, if we use $q' = p \wedge p_{fin}$, we have

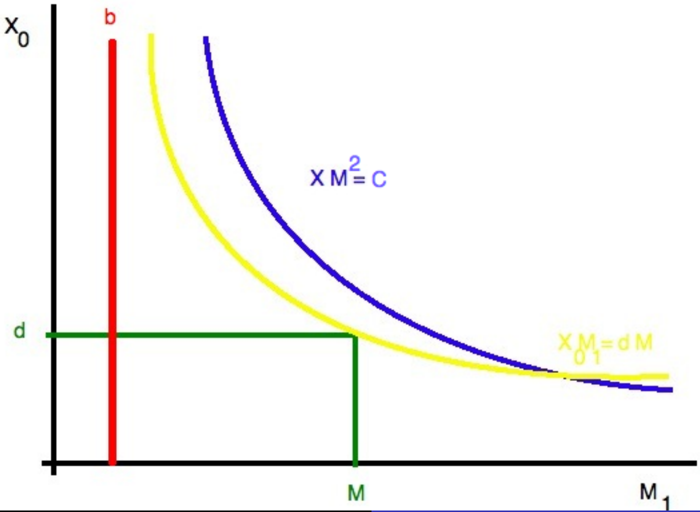
$$P(T_0 \wedge S_t^{\beta} = +\infty) \leq P(T_0 \wedge S_t^{\beta} \geq U_0 + L) \leq 1 - q'$$

.

Now since the processes X_0, M_1 are markovian, by iterating we obtain

$$P(T_0 \wedge S_t^{\beta} = +\infty) = 0$$

$\diamondsuit$


X0
b
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M1

Plan

1. Introduction : The model of Haigh
2. Model, first properties and theorem
3. Proof
   - Basic results
   - First properties on M_1
   - Ω_1 and Ω_2
   - Recurrence on M_1
   - Reaching Ω_2 starting from the recurrence
   - $E(T_0) < +\infty$

Now, starting from $((X_k)_{k\in\mathbb{N}}, M_1)$ with $M_1 \leq \beta$ we want to study a path that will reach zero.
To simplify notations we reset the time.
One of the main problem here is that the quadratic variation of X_0 is $\frac{X_0(1-X_0)}{N}$, which is a difficulty around 1 and 0. We need to study three separate cases :
$X_0 \in [\delta_1; 1]$
$X_0 \geq \delta$ or $X_0 M_1 > \varepsilon$
$X_0 \leq \delta and X_0 M_1 \leq \varepsilon$

The following lemma will show that if X_0 starts too close from 1, it will quickly goes under δ_1 :

Lemma

Let $t_1' = \frac{8}{\lambda^2}$ *and*

$$\delta_1 = max\{\frac{9}{10}, \frac{3\lambda + 5\alpha}{5(\lambda + \alpha)}, 1 - \frac{2}{\lambda}\}.$$

Then, $\forall t > 0$, *if* $X_0(t) > \delta_1$, *then* $\exists q > 0$ *such as*

$$P(\inf\{s > t, X_0(s) \leq \delta_1\} < t + t_1') \geq 1 - exp(-N) > 0$$

PROOF : During the interval $[t, T_{\delta_1}(t)]$, we have , since $X_1 \leq 1 - X_0$,

$$X_1(t) \leq \frac{2\lambda}{5(\lambda+\alpha)}.$$

$$\begin{aligned}\alpha M_1 X_1 + \lambda X_0 - (\lambda+\alpha)X_1 \geq & \lambda X_0 - (\lambda+\alpha)\frac{2\lambda}{5(\lambda+\alpha)} \\ \geq & \lambda X_0 - \frac{2\lambda}{5} \\ \geq & \frac{\lambda}{2}\end{aligned}$$

since $X_0(s) > 0, 9$ as well.

Hence $X_1 \geq Y_1$ when $s \in [t, T_{\delta_1}(t)]$, where Y_1 is the solution of the following system

$$\begin{cases} Y_1(t) = X_1(t) \\ dY_1(s) = \dfrac{\lambda}{2} ds + \sqrt{\dfrac{Y_1(1 - Y_1)}{N}} dB_1 \end{cases} \tag{3.4}$$

Noticing that $Y_1(1-Y_1) \leq \frac{1}{4}$, we have (with $C = \frac{2}{\lambda}$).

$$P\left(\left\{\int_t^{t+t_1'} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 < -C\right\} \cap \left\{T_{\delta_1} > t + t_1'\right\}\right)$$
$$= P\left(-\int_t^{t+t_1'} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 > -C\right)$$

$$= P\left(exp\left(-p\int_t^{t+t_1'}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 - \frac{p^2\int_t^{t+t_1'}Y_1(1-Y_1)}{2N}ds\right)\right.$$

$$\left.> exp\left(pC - \frac{p^2\int_t^{t+t_1'}Y_1(1-Y_1)}{2N}ds\right)\right)$$

$$\leq P\left(exp\left(-p\int_t^{t+t_1'}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 - \int_t^{t+t_1'}\frac{p^2Y_1(1-Y_1)}{2N}ds\right)\right.$$

$$\left.> exp\left(pC - \frac{p^2}{8N}t_1'\right)\right)$$

$$\leq exp\left(-pC + \frac{p^2}{8N}t_1'\right)$$

Where $p = \frac{4CN}{t}$ minimizes the quantity.

$$P\left(\int_t^{t+t_1'} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 \geq -C\right) \geq 1 - exp\left(-N\right) > 0$$

Now, since

$$\int_t^{t+t_1'} \frac{\lambda}{2} ds = \frac{4}{\lambda} = 2C$$

We have

$$\left\{\int_t^{t+t_1'} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 \geq -C\right\} \subset \{T_{\delta_1} < t + t_1'\}$$

Which implies that

$$P\left(T_{\delta_1} < t + t_1'\right) \geq 1 - exp(-N)$$

$\diamondsuit$

Now we add a control on M_1

Lemma

Let $\delta_1 = max\{\frac{9}{10}, \frac{3\lambda+5\alpha}{5(\lambda+\alpha)}, 1-\frac{2}{\lambda}\}$, $t'_1 = \frac{8}{\lambda^2}$, $\varepsilon_0 = \frac{1}{\alpha N} ln\left(\frac{2}{1-\exp(-N)}\right)$ *and* $\beta' = \beta + \lambda t'_1 + \varepsilon_0$. *Then,* $\forall t > 0$, *if* $X_0(t) > \delta_1$ *and* $M_1(t) < \beta$, *then*

$$P\left(\{T_{\delta_1} \leq t + t'_1\} \wedge \{M_1(T_{\delta_1}) \leq \beta'\}\right) = p_{init} > 0$$

PROOF :

Here

$$P(T_{\delta_1} \leq t_1') \geq q$$

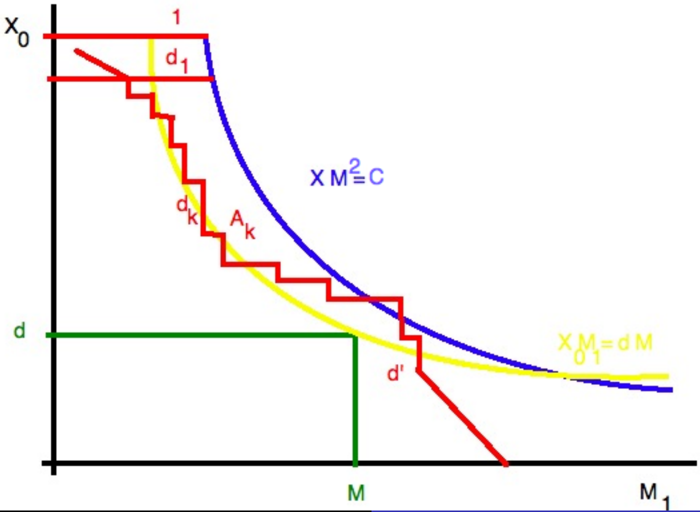
,

$$P(M_1(T_{\delta_1}) \leq \beta') \geq 1 - exp(-sN\varepsilon_0)$$

Then,

$$\begin{aligned} &P\left(\{T_{\delta_1} \leq t_1'\} \wedge \{M_1(T_{\delta_1}) \leq \beta'\}\right) = p_{init} \\ &\geq 1 - exp(-N) - exp(-\alpha N\varepsilon_0) \\ &\geq \frac{1 - exp(-N)}{2} > 0 \end{aligned}$$

$\diamondsuit$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Now the second part : $X_0 \leq \delta_1$ but either $X_0 > \delta$ or $X_0 M_1 > \varepsilon$. First some inequalities :

Lemma

Let $\{V_t,\ t \geq 0\}$ be a standard Brownian motion, and $c > 0$ a constant. The for any $t > 0$, $\tilde{\delta} > 0$,

$$\mathbb{P}\left(\inf_{0 \leq s \leq t}\{cs + B_s\} \leq -\tilde{\delta}\right) \geq 1 - \sqrt{\frac{2}{\pi}}\left(\frac{\tilde{\delta}}{\sqrt{t}} + c\sqrt{t}\right).$$

PROOF : We have, with Z denoting a $N(0,1)$ random variable,

$$\begin{aligned}
\mathbb{P}\left(\inf_{0\leq s\leq t}\{cs+V_s\}\leq -\tilde{\delta}\right) &\geq \mathbb{P}\left(\inf_{0\leq s\leq t} V_s \leq -\tilde{\delta}-ct\right)\\
&\geq \mathbb{P}\left(\sup_{0\leq s\leq t} V_s \geq \tilde{\delta}+ct\right)\\
&\geq 2\mathbb{P}(V_t \geq \tilde{\delta}+ct)\\
&\geq 1-\mathbb{P}(|Z|\leq \frac{\tilde{\delta}}{\sqrt{t}}+c\sqrt{t}),
\end{aligned}$$

from which the result clearly follows. $\diamondsuit$

From this we deduce :

Lemma

Let $\{V_t,\ t \geq 0\}$ be a standard Brownian motion, and $c > 0$ a constant. The for any $t > 0$, $\tilde{\delta} > 0$, $\tilde{\mu} > 0$,

$$\mathbb{P}\left(\inf_{0\leq s\leq t}\{cs + V_s\} \leq -\tilde{\delta},\ \sup_{0\leq s\leq t}\{cs + V_s\} \leq \tilde{\mu}\right)$$
$$\geq 1 - \sqrt{\frac{2}{\pi}}\left(\frac{\tilde{\delta}}{\sqrt{t}} + c\sqrt{t}\right) - 2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right)^2\right].$$

PROOF :

$$\mathbb{P}\left(\sup_{0\leq s\leq t}(cs+V_s)\leq\tilde{\mu}\right)\geq\mathbb{P}\left(\sup_{0\leq s\leq t}V_s\leq\tilde{\mu}-ct\right)$$
$$=1-2\mathbb{P}(V_t\geq\tilde{\mu}-ct).$$

Now, Z denoting a $N(0,1)$ random variable, for all $p>0$,

$$\mathbb{P}(B_t\geq\tilde{\mu}-ct)=\mathbb{P}\left(Z\geq\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right)$$
$$=\mathbb{P}\left(\exp(pZ-p^2/2)\geq\exp\left(p\left[\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right]-\frac{p^2}{2}\right.\right.$$
$$\leq\exp\left(-p\left[\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right]+\frac{p^2}{2}\right)$$

Choosing $p = \tilde{\mu}/\sqrt{t} - c\sqrt{t}$, we conclude from the above computations that

$$\mathbb{P}\left(\sup_{0\leq s\leq t}(cs + V_s) \leq \tilde{\mu}\right) \geq 1 - 2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right)^2\right].$$

$\diamondsuit$

We set :

$$\tilde{\varepsilon} = \frac{\log(4)}{\alpha N}, \quad \text{so that } e^{-N\alpha\tilde{\varepsilon}} = \frac{1}{4}$$
$$\tilde{\mu} = \frac{1-\delta_1}{2}$$

We will use another time change, and we use A and σ again for the time change (but they are not the same) and aim at proving that X_0 will go down to δ' in a finite number of steps, while staying below $X + \tilde{\mu}$ (so that $1 - X_0(t) \geq a := 1 - (X + \tilde{\mu})$), and while M_1 does not go too far on the right, all that with positive probability.

We have :

$$dX_0(t) = (\alpha M_1(t) - \lambda)X_0(t)dt + \sqrt{\frac{X_0(t)[1 - X_0(t)]}{N}}dB_0.$$

Let

$$\begin{aligned} A_t &:= \int_0^t \frac{X_0(s)[1 - X_0(s)]}{N} ds, \quad \text{and} \\ \sigma_t &:= \inf\{s > 0,\ A_s > t\}, \\ \tilde{X}_0(t) &:= X_0(\sigma_t), \\ \tilde{M}_1(t) &:= M_1(\sigma_t), \end{aligned}$$

We deduce

$$\sigma_t = \int_0^t \frac{N}{\tilde{X}_0(s)(1-\tilde{X}_0(s))} ds,$$

$$\tilde{X}_0(t) = X + N \int_0^t \frac{(\alpha \tilde{M}_1(s) - \lambda)}{1-\tilde{X}_0(s)} ds + B_t,$$

where B_t is a new standard Brownian motion.
At the k–th step of our iterative procedure, we let $\tilde{X}_0$ start from $X - \sum_{j=1}^{k-1} \delta_j$, and we stop the process $\tilde{X}_0$ at the first time that it reaches the level $X - \sum_{j=1}^{k} \delta_j$.

We will choose δ_k and t_k such as for each $1 \leq k \leq K$ (K to be defined below),

$$\mathbb{P}\left(\inf_{0\leq s\leq t_k}\{A_k s + B_s\} \leq -\delta_k, \sup_{0\leq s\leq t_k}\{A_k s + B_s\} \leq \tilde{\mu}\right) > \frac{2}{3},$$

where, with $T_k := \sigma_{t_k}$,

$$A_k = \frac{N\alpha}{a}\left(\beta' + k\tilde{\varepsilon} + \lambda\sum_{j=1}^{k} T_j\right),$$

so that we have from Lemma 4 and our choice of $\tilde{\varepsilon}$ that

$$\mathbb{P}(\sup_{0\leq s\leq T_k} M_1(s) \leq A_k \Big| M_1(0) \leq A_{k-1}) \geq 1/3.$$

We show that we can choose the two sequences δ_k and t_k for $k \geq 1$ in such a way that not only the two previous inequalities hold, but also that there exists $K < \infty$ such that

$$X - \sum_{k=1}^{K} \delta_k \leq \delta'.$$

Since during the k–th step we are considering the event that $1 - X_0(t) \geq a$, and also $X_0(t) \geq X - \sum_{j=1}^{k} \delta_j$, we have that

$$T_k \leq \frac{N}{a(X - \sum_{j=1}^{k} \delta_j)} t_k,$$

Then we choose

$$A_k := N\frac{\alpha}{a}\left[\beta' + k\tilde{\varepsilon} + N\frac{\lambda}{a}\sum_{j=1}^{k}\frac{t_j}{X - \sum_{i=1}^{j}\delta_i}\right].$$

First we want to insure

$$\frac{\delta_k}{\sqrt{t_k}} + A_k\sqrt{t_k} \leq 0.4,$$

which we achieve by requesting both that

$$\delta_k = 0.2\sqrt{t_k} \tag{3.5}$$

and

$$A_k\sqrt{t_k} \leq 0.2 \Leftrightarrow t_k \leq \left(\frac{0.2}{A_k}\right)^2. \tag{3.6}$$

On the other hand, we shall also request that for each $k \geq 1$,

$$\frac{\delta_k}{X - \sum_1^k \delta_j} \leq 1 \Leftrightarrow \delta_k \leq \frac{1}{2}(X - \sum_{j=1}^{k-1} \delta_j).$$

It follows from $A_k \geq N\frac{\alpha\beta'}{a}$ that with

$$C_N = N\frac{\alpha\beta'}{a} \quad \text{and } D_N = N\frac{\alpha}{a}\left(\tilde{\varepsilon} + \frac{\lambda}{\alpha\beta'}\right),$$

$$\begin{aligned} A_k &\leq C_N + 25\frac{N^2\lambda\alpha}{a^2}\left(\sup_{1\leq j\leq k} \delta_j\right) k + k\varepsilon\frac{N\alpha}{a} \\ &\leq C_N + D_N k \end{aligned}$$

Since we have $\forall j \geq 0$

$$\delta_j \leq 0.2\sqrt{t_j} \leq \frac{(0.2)^2}{A_j} \leq \frac{1}{25}\frac{a}{N\alpha\beta'}$$

Finally this leads to choosing, with $\kappa \leq \frac{1}{25}$ to be chosen below

$$\delta_k = \inf\left(\frac{\kappa}{(C_N + D_N k)}, \frac{1}{2}(x - \sum_{j=1}^{k-1}\delta_j)\right)$$
$$t_k = 25\delta_k^2.$$

Then from a certain rank on we have

Lemma

$\exists K > 0, \forall k > K,$

$$\delta_k = \frac{1}{2}(X - \sum_{j=1}^{k-1} \delta_j).$$

PROOF : Since $\sum_{k\in\mathbb{N}} \frac{1}{25(C_N+D_N k)} = +\infty$, $\exists K' > 0$ such as $\sum_{k=2}^{K'} \frac{1}{25(C_N+D_N k)} > 1$. Then $\exists 2 \leq K \leq K'$ such as $\inf\left(\frac{1}{25(C_N+D_N K)}, \frac{1}{2}(X - \sum_{j=1}^{K-1}\delta_j)\right) = \frac{1}{2}(X - \sum_{j=1}^{K-1}\delta_j)$. Then by recurrence, if we have the previous equality at rank k, for the rank $k+1$ we have

$$\begin{aligned}\frac{\frac{1}{25(C_N+D_N(k+1))}}{\delta_k} &\geq \frac{\frac{1}{25(C_N+D_N(k+1))}}{\frac{1}{25(C_N+D_N k)}} \\ &\geq \frac{1}{2} \text{ since } k \geq 2\end{aligned}$$

That is to say

$$\begin{aligned}\frac{1}{25(C_N + D_N(k+1))} &\geq \frac{1}{2}\delta_k \\ &\geq \frac{1}{4}(X - \sum_{j=1}^{k-1}\delta_j) \\ &\geq \frac{1}{2}(X - \sum_{j=1}^{k}\delta_j)\end{aligned}$$

Hence

$\delta_{k+1} = \inf\left(\frac{1}{25(C_N+D_N(k+1))}, \frac{1}{2}(X - \sum_{j=1}^{k}\delta_j)\right) = \frac{1}{2}(X - \sum_{j=1}^{k}\delta_j)$

$\diamondsuit$

Then for each $k > K$, $\tilde{X}_0$ progresses by a step equal to half the remaining distance to zero. Consequently $\exists c > 0$ $X_k = X - \sum_{j=1}^{k} \delta_j \leq c2^{-k}$. We are looking for the smallest integer $\overline{k}$ such that $c2^{-\overline{k}} \leq \delta'$, which implies that

$$\overline{k} - 1 \leq \left[\frac{\log(c) - \log(\delta')}{\log(2)} \right].$$

Since moreover $A_{\overline{k}} \leq (25\delta_{\overline{k}})^{-1}$, there exists a constant c' such that $\delta' \times A_{\overline{k}} \leq c'\delta' \log(\frac{1}{\delta'})$. Hence there exists a $\delta' \leq \delta$ (which depends only upon C_N, D_N, c' which are constants) such that at the end of the $\overline{k}$–th step, both $X_0 \leq \delta$ and $X_0 M_1 \leq \varepsilon$.

. We just need to check that the probability of the previous pathes is $= p_{trans} > 0$.
Given the choice that we have made for $\tilde{\varepsilon}$, it suffices to make sure that

$$\sqrt{\frac{2}{\pi}}\left(\frac{\delta_k}{\sqrt{t_k}} + A_k\sqrt{t_k}\right) < 1/3, \ \forall k \geq 1,$$

which is a consequence of $3^{-1}\sqrt{\pi/2} > 0.4$, and

$$2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{t_k}} - A_k\sqrt{t_k}\right)^2\right] < 1/3, \ \forall k \geq 1.$$

This is equivalent to

$$\left(\frac{\tilde{\mu}}{\sqrt{t_k}} - A_k\sqrt{t_k}\right)^2 > 2\log 6,$$

which follows from $\kappa \leq \frac{\sqrt{2\log 6+4C_N\tilde{\mu}}-\sqrt{2\log 6}}{10}$. We therefore choose

$$\kappa = \frac{1}{25} \wedge \frac{\sqrt{2\log 6 + 4C_N\tilde{\mu}} - \sqrt{2\log 6}}{10}.$$

Then $P($ the k-th step happens $) \geq 1 - \frac{1}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{12}$, hence, using the markovian property, we have $p_{trans} \geq \inf_{\overline{k}} \left(\frac{1}{12}\right)^{\overline{k}}$, but $\overline{k} < \infty$ for any initial condition X, and increasing in X, so the worst $\overline{k}$, noted $\overline{k}_{max}$ is reached for $X = \delta_1$, and then

$$p_{trans} \geq \inf_{\overline{k}} \left(\frac{1}{12}\right)^{\overline{k}_{max}} > 0$$

Note that the time (in the initial scale of time) in which we reach δ' is bounded by $t_2' = 100N\overline{k}_{max}$.
We've finally reached third situation, which leads to the conclusion.

$X_0 \leq \delta$ and $X_0 M_1 \leq \varepsilon$
Here we can immediatly apply the step 3 of this document, and we have a probability p_{fin} to reach 0 before T elapsed.
So to sump up, using again the markovian properties of the system,

Lemma

$\forall t > 0$, if $M_1(t) \leq \beta$, then

$$P(T_0 < t + t_1' + t_2' + T) \geq p_{fin} p_{trans} p_{ini} > 0$$

Plan

1. Introduction : The model of Haigh
2. Model, first properties and theorem
3. Proof
   - Basic results
   - First properties on M_1
   - Ω_1 and Ω_2
   - Recurrence on M_1
   - Reaching Ω_2 starting from the recurrence
   - $E(T_0) < +\infty$

Here we will prove the following stronger theorem

Theorem

For any choice of initial condition, let $(X_k(t))_{k\in\mathcal{Z}_+}$ the solution of (8). Then $\mathbb{E}(T_0) < \infty$.

We first note that the reasoning of section 5 can be done with any initial value ρ for M_1, instead of β. That is to say, with $S^t_\rho = inf\ \{s > t, M_1(s) \leq \rho\}$ (and $S_\rho = S^0_\rho$),

Lemma

$\exists\ t_1^\rho, t_2^\rho, t_3^\rho < \infty$, *and* $p^\rho_{ini}, p^\rho_{trans}, p^\rho_{fin} > 0$ *such that*

$$\mathbb{P}(T_0 < S^t_\rho + t_1^\rho + t_2^\rho + t_3^\rho) \geq p^\rho_{ini} p^\rho_{trans} p^\rho_{fin}$$

Now let us choose $\rho = \frac{\varepsilon}{\delta} \vee \frac{2\lambda}{\alpha}$. We have :

Lemma

Let $K = L + t_3$ (L to be defined below). Then $\exists \tilde{p} > 0$, such that for any initial condition,

$$\mathbb{P}(T_0 \wedge S_\rho \leq K) \geq \tilde{p}$$

PROOF : We are going to argue like in the recurrence for M_1. We introduce the process Y_s, defined $\forall s \geq 0$ which is the solution of the following system :

$$\begin{cases} dY_s = \frac{\alpha\varepsilon}{2} ds + \sqrt{\frac{Y_s(1-Y_s)}{N}} dB_0(s) \\ Y_0 = 0 \end{cases} \tag{3.7}$$

We define for any $0 \leq u \leq 1$

$$R_u = inf\{s \geq 0, Y_s = u\}.$$

Since $\frac{\alpha\varepsilon}{2} > 0$ we deduce that $\exists L > 0, p > 0$ such as $\mathbb{P}(R_1 \leq L) \geq p > 0$. We use $K = L + t_3$. (t_3 from Proposition 3.2).
Now there are several possibilities :
Either $\inf_{0 \leq s \leq L} M_1(s) \leq \rho$, then $S_\rho < L < K$.
Or else $\inf_{0 \leq s \leq L} M_1(s) \geq \rho$. Then either $\inf_{0 \leq s \leq L} X_0(s) M_1(s) \leq \varepsilon$, then $\exists t < L$ such as $X_0(t) M_1(t) \leq \varepsilon$ (which implies $X_0(t) \leq \delta$, because$M_1(t) \geq \rho \geq \frac{\varepsilon}{\delta}$). In that case we can use Proposition 3.2, and we have $\mathbb{P}(T_0 \leq K) = p_{fin} > 0$, which implies $\mathbb{P}(T_0 \wedge S_\rho \leq K) = p_{fin} > 0$,

Or else we have both $\inf_{0\leq s\leq L} M_1(s) \geq \rho$ and $\inf_{0\leq s\leq L} X_0(s)M_1(s) \geq \varepsilon$. In that last sub–case we have (since $X_0 \geq \frac{\varepsilon}{M_1}$, and $\alpha M_1 - \lambda \geq \lambda > 0$)

$$\inf_{0\leq s\leq L}(\alpha M_1(s) - \lambda)X_0(s) \geq \inf_{0\leq s\leq L} \varepsilon(\alpha - \frac{\lambda}{M_1(s)}) \geq \frac{\alpha\varepsilon}{2},$$

and consequently we can use the comparison theorem (Lemma 2), which implies that $\forall s \in [0, L]$, $X_0(s) \geq Y_s$. Then $\mathbb{P}(T_1 \leq L) \geq p > 0$. But when X_0 hits 1, M_1 hits 0. Hence $\mathbb{P}(S_\rho \leq L) \geq p > 0$.

$\diamondsuit$

We can now conclude.

We deduce from the previous results and the strong Markov property that $\exists \overline{K}, \overline{p} > 0$ such as for all $n \geq 0$, $\mathbb{P}(T_0 > n\overline{K}) \leq (1 - \overline{p})^n$. Consequently

$$\begin{aligned}\mathbb{E}(T_0) &= \sum_{n=0}^{\infty} \int_{nK}^{(n+1)K} \mathbb{P}(T > t)dt \\ &\leq \sum_{n=0}^{\infty} K\mathbb{P}(T > nK) \\ &= \frac{K}{\overline{p}}\end{aligned}$$

$\diamondsuit$

THANK YOU FOR YOUR ATTENTION !