Modelling adaptive dynamics for structured populations with function-valued traits

Ankit Gupta

Centre de Mathématiquees Appliquées, Ecole Polytechnique

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Joint work with V.C. Tran and J.A.J. Metz.

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- We want to understand mathematically how this evolutionary cycle works.

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  - Allows us to work with population density.
  - Makes models more tractable.

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- We get a jump process over the space of equilibriums. This is called the TSS.

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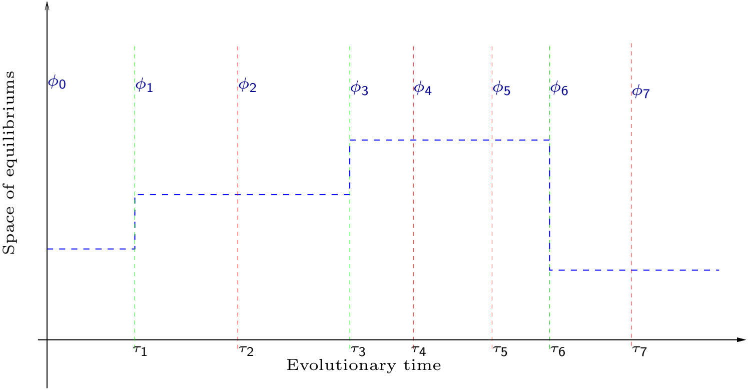

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- The process that gives the surviving trait value at any time is a jump process in the trait space.
- We can scale time and mutation step sizes to obtain an ordinary differential equation in the trait space.
- This differential equation describes the evolution of advantageous trait values.

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- In our case we consider infinite dimensional function-valued traits.
- We use averaging techniques to construct the TSS. This technique works equally well when the selection dynamics leads to coexistence or cyclic/chaotic attractors.

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- Phenotypic plasticity : Function-valued trait is the reaction norm of an organism that describes the phenotypic response to a particular set of environmental conditions (temperature, salinity etc.)
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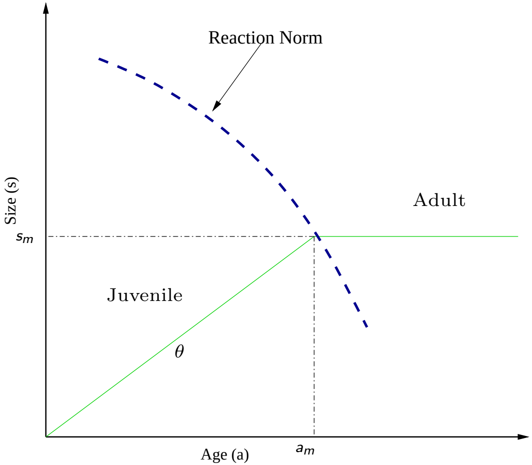
Apart from function-valued traits, the individuals in our model are also structured by :

- Physical age.
- Noise parameter that accounts for randomness in the environment.


Reaction Norm
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- The functions b, U and V are bounded while

$$\sup_{x \in E} d(x,u) \leq \bar{d}(1+u).$$

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The process X^K characterized by its generator L^K. For $F_f(X) = F(\langle f, X\rangle)$ we define

$$L^K F_f(X)$$
$$= F'(\langle f, X\rangle)\left\langle \frac{\partial f}{\partial a}, X\right\rangle + K\int_E d(x, U * X(x))\left(F_f\left(X - \frac{1}{K}\delta_x\right) - F_f(X)\right)X(dx)$$
$$+ K\int_E b(x, V * X(x))\left[\int_{\mathcal{H}\times\mathbb{R}}\left(F_f\left(X + \frac{1}{K}\delta_{\bar{x}(h,\nu)}\right) - F_f(X)\right)\Lambda^K(x, dh, d\nu)\right]X(dx).$$

Theorem (Deterministic Approximation)

Suppose that $u_K \to 0$ as $K \to \infty$. Also assume that $X^K(0) \Rightarrow \xi(0)$. Then the sequence $\{X^K : K \geq 1\}$ converges weakly in $\mathbb{D}([0,\infty), \mathcal{M}_F(E))$ to the deterministic continuous $\mathcal{M}_F(E)$-valued process ξ which is characterized by the following equation. For all nice functions $f : E \to \mathbb{R}$

$$\langle f, \xi(t) \rangle = \langle f, \xi(0) \rangle + \int_0^t \left\langle \frac{\partial f}{\partial a}, \xi(s) \right\rangle ds - \int_0^t \int_E d\left(x, (U * \xi(s))(x)\right) f(x) \xi(s, dx) ds$$

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- The mutation term does not appear in the limit.
- To prove the result we only have to worry about compact containment.
- One can show that the total weight of *mutants* produced in any time interval $[0, T]$ is $O(u_k)$.

Evolutionary time scale

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- The generator of Z^K is

$$\begin{aligned}
\mathbb{L}^K F_f(Z) &= \left(\frac{1}{Ku_K}\right) L^K F_f(Z) \\
&= \int_E p(x) b(x, V * Z(x)) \left[\int_{\mathbb{R}} \int_{\mathcal{H}} \left(F_f\left(Z + \frac{1}{K}\delta_{\bar{x}(h,\nu)}\right) - F_f(Z)\right) \Xi(x, dh)\Theta(d\nu)\right] Z(dx) \\
&+ \frac{1}{Ku_K}\left[F'\left(\langle f, Z\rangle\right)\left\langle \frac{\partial f}{\partial a}, Z\right\rangle\right. \\
&+ K\int_E (1 - u_K p(x)) b(x, V * Z(x)) \left(\int_{\mathbb{R}} \left(F_f\left(Z + \frac{1}{K}\delta_{\bar{x}(0,\nu)}\right) - F_f(Z)\right)\Theta(d\nu)\right) Z(dx) \\
&\left. + K\int_E d(x, U * Z(x)) \left(F_f\left(Z - \frac{1}{K}\delta_x\right) - F_f(Z)\right) Z(dx)\right] \\
&= \text{Mutation} + \left(\frac{1}{Ku_K}\right)\text{Population Dynamics.}
\end{aligned}$$

- Let χ^K be the $\mathcal{M}_P(\mathcal{H})$-valued process such that if

$$\chi^K(t) = \sum_{i=1}^{n} \delta_{\phi_i}$$

then n mutant trait values $\{\phi_1, \cdots, \phi_n\}$ have appeared in the population until time t.

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- Let Γ^K be the occupation measure process of Z^K. For any $t \geq 0$ and $A \in \mathcal{B}(\mathcal{M}_F(E))$

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- We will use the averaging techniques developed by Kurtz in 1992 to study the behaviour of $\{(\chi^K, \Gamma^K)\}$ as $K \to \infty$.

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- Let χ^K be the $\mathcal{M}_P(\mathcal{H})$-valued process such that if

$$\chi^K(t) = \sum_{i=1}^{n} \delta_{\phi_i}$$

then n mutant trait values $\{\phi_1, \cdots, \phi_n\}$ have appeared in the population until time t.

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- When we have convergence we get the TSS.

More assumptions

Remark (Metivier)

A subset $A \subset \mathcal{H}$ is relatively compact if and only if it is bounded and for every $\epsilon > 0$, there exists a finite dimensional vector space G_ϵ such that

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- *There exists a positive constant $\underline{d} > 0$ such that for any $(\phi, a, \theta) \in E$ and any measure $Z \in \mathcal{M}_F(E)$ we have*

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Averaging : Tightness

Under the above assumptions we have the following.

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- The sequence of random measures $\{\Gamma^K\}$ is relatively compact.

- For any continuous bounded function $G : \mathcal{M}_P(\mathcal{H}) \to \mathbb{R}$,

$$M_G^{\chi,K}(t) = G(\chi^K(t)) - G(\chi^K(0)) - \int_0^t \int_E b\left(x, (V * Z^K(s))(x)\right) p(x)$$
$$\int_{\mathcal{H}} \left(G(\chi^K(s) + \delta_{\phi+h}) - G(\chi^K(s))\right) \Xi(x, dh) Z^K(s, dx) ds$$

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- Define the *population dynamics* operator $\mathbb{B}$ by

$$\mathbb{B}F_f(\mu) = F'(\langle f, \mu\rangle)\left[\left\langle \frac{\partial f}{\partial a}, \mu \right\rangle + \int_E \left(b(x, V * \mu(x)) \int_{\mathbb{R}} f(\bar{x}(0,\nu))\Theta(d\nu) - d(x, U * \mu(x)) f(x) \right) \mu(dx) \right].$$

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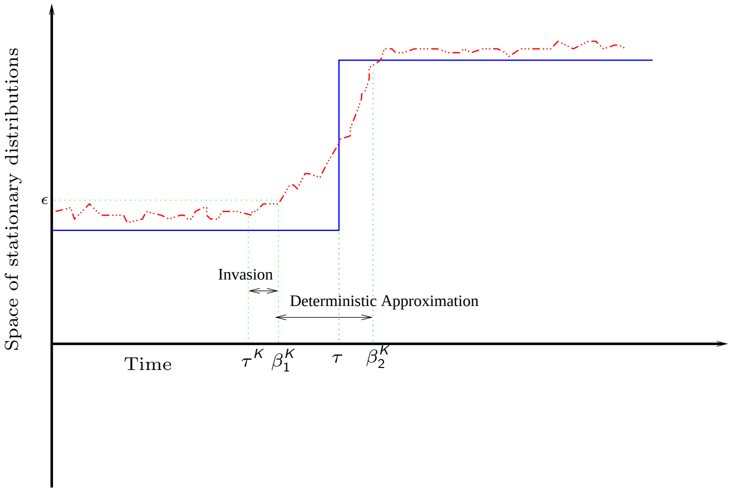
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ϵ
Invasion
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- We assume that there exists a set $A' \subset A \cup \{\phi\}$ and a distribution $\pi_{A'} \in \mathcal{P}(\mathcal{M}_F(E))$ such that for any $C \in \mathcal{B}(\mathcal{M}_F(E))$ we have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}_C(\xi(s))ds = \pi_{A'}(C).$$

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- This would be the TSS in our setting.
- If for all $s \geq 0$, $\gamma_s = \delta_{\hat{m}_s}$, then we recover the TSS obtained in earlier literature. In this case convergence is in the sense of finite dimensional distributions.