Random evolution of population subject to competition

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Projet ANR MANEGE

collaborations avec Mamadou Ba, Vi Le, Anton Wakolbinger

Contents

Finite population

- Consider a continuous–time population model, where each individual gives birth at rate λ, and dies at an exponential time with parameter μ.
- We superimpose a death rate due to interaction equal to $f^-(k)$ (resp. a birth rate due to interaction equal to $f^+(k)$) while the total population size is k.
- In fact since we want to couple the models for all possible initial population sizes, we need to introduce a pecking order (e.g. from left to right) on our ancestors at time 0, which is passed on to the descendants, and so that any daughter is placed on the right of her mother.
- In all what follows, we assume that $f \in C(\mathbb{R}_+; \mathbb{R})$, $f(0) = 0$ and for some fixed $a > 0$, $f(x + y) - f(x) \le ay$, for all $x, y \ge 0$.

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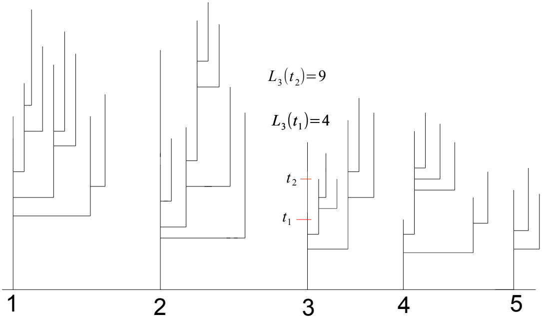
$L_3(t_2)=9$
$L_3(t_1)=4$
t_2
t_1
1
2
3
4
5

- We want that the individual i interacts only with those individuals who sit on the left of her. Let $\mathcal{L}_i(t)$ denote the number of individual alive at time t who sit on the left of i.
- Then we decide that i gives birth at rate $\lambda + [f(\mathcal{L}_i(t)) - f(\mathcal{L}_i(t) - 1)]^+$, and dies at rate $\mu + [f(\mathcal{L}_i(t)) - f(\mathcal{L}_i(t) - 1)]^-$.
- Summing up, we conclude that the size of the population X_t^m, starting from $X_0^m = m$, jumps

$$\text{from } k \quad \text{to } \begin{cases} k+1, & \text{at rate } \lambda k + \sum_{\ell=1}^{k} [f(\ell) - f(\ell-1)]^+ \\ k-1, & \text{at rate } \mu k + \sum_{\ell=1}^{k} [f(\ell) - f(\ell-1)]^- \end{cases}$$

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- In case f linear, we have a branching process, and for each $t > 0$, $\{X_t^m,\ m \geq 1\}$ has independent increments.
- In the general case, we don't expect that for fixed t, $\{X_t^m,\ m \geq 1\}$ is a Markov chain.
- However, $\{X_t^m,\ t \geq 0\}_{m \geq 1}$ is a path–valued Markov chain. We can specify the transitions as follows.
- For $1 \leq m < n$, the law of $\{X_t^n - X_t^m, t \geq 0\}$, given $\{X_t^\ell,\ t \geq 0,\ 1 \leq \ell \leq m\}$ and given that $X_t^m = x(t), t \geq 0$, is that of the time–inhomogeneous jump Markov process whose rate matrix $\{Q_{k,\ell}(t),\ k, \ell \in \mathbb{Z}_+\}$ satisfies

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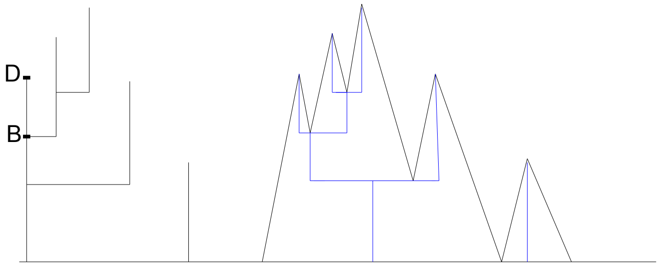
Exploration process of the forest of genealogical trees



- Call $\{H^m_s,\ s \geq 0\}$ the zigzag curve in the above picture (with slope ± 2), and define the local time accumulated by H^m at level t up to time s by

$$L^m_s(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{t \leq H^m_r < t+\varepsilon} dr.$$

- H^m is piecewise linear, with slopes ± 1. While the slope is 2, the rate of appearance of a maximum is

$$\mu + [f(\lfloor L^m_s(H^m_s) \rfloor + 1) - f(\lfloor L^m_s(H^m_s) \rfloor)]^-,$$

and the rate of appearance of a minimum while the slope is -2 is

$$\lambda + [f(\lfloor L^m_s(H^m_s) \rfloor + 1) - f(\lfloor L^m_s(H^m_s) \rfloor)]^+.$$

- Let $S^m = \inf\{s > 0,\ L^m_s(0) \geq m\}$ the time needed for H^m_s to explore the genealogical trees of m ancestors. If we assume that the population goes extinct in finite time, we have the Ray–Knight type result (see next figure)

$$\{X^m_t,\ t \geq 0,\ m \geq 1\} \equiv \{L^m_{S^m}(t),\ t \geq 0, m \geq 1\}.$$

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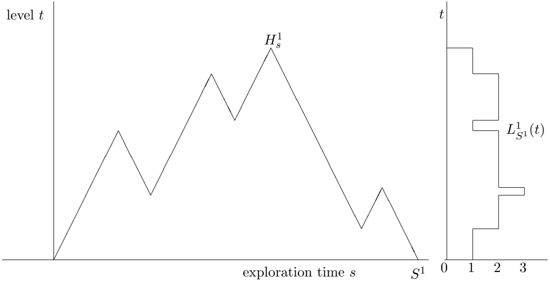
How to recover X^m from H^m ?



Renormalization

- Let $N \geq 1$. Suppose that for some $x > 0$, $m = \lfloor Nx \rfloor$, $\lambda = 2N$, $\mu = 2N$, replace f by $f_N = Nf(\cdot/N)$. We define $Z_t^{N,x} = N^{-1} X_t^{\lfloor Nx \rfloor}$.
- We have

Theorem

As $N \to \infty$,

$$\{Z_t^{N,x},\ t \geq 0, x \geq 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x \geq 0\}$$

in $D([0,\infty); D([0,\infty); \mathbb{R}_+))$ equipped with the Skorohod topology of the space of càlàg functions of x, with values in the Polish space $D([0,\infty); \mathbb{R}_+)$, equipped with the adequate metric.

- $\{Z_t^x,\ t \geq 0, x \geq 0\}$ solves for each $x > 0$ the Dawson–Li type SDE

$$Z_t^x = x + \int_0^t f(Z_s^x) ds + 2 \int_0^t \int_0^{Z_s^x} W(ds, du),$$

where $W(ds, du)$ is a space–time white noise.

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How to check tightness ?

- Our assumptions on f are pretty minimal. In order to check tightness for x fixed, we establish the two bounds

$$\sup_{N\ge1}\sup_{0\le t\le T}\mathbb{E}\left(Z_t^{N,x}\right)^2<\infty,\ \sup_{N\ge1}\sup_{0\le t\le T}\mathbb{E}\left(-\int_0^t Z_s^{N,x}f(Z_s^{N,x})ds\right)<\infty,$$

and exploit Aldous' criterion.

- Concerning the tightness "in the x direction", we establish the following bound : for any $0\le x<y<z$ with $y-x\le1$, $z-y\le1$,

$$\mathbb{E}\left[\sup_{0\le t\le T}|Z_t^{N,y}-Z_t^{N,x}|^2\times\sup_{0\le t\le T}|Z_t^{N,z}-Z_t^{N,y}|^2\right]\le C|z-x|^2.$$

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Continuous population models

- For each fixed $x > 0$, there exists a standard BM B_t such that

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \sqrt{Z_s^x}dB_s.$$

However, B depends upon x in a non obvious way, and the good way of coupling the evolution of Z^x for various x's, which is compatible with the above coupling in the discrete case, is to use the Dawson–Li formulation

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Subcriticality

- We will say that Z^x is subcritical if

$$T_0^x = \inf\{t > 0;\ Z_t^x = 0\} < \infty \ \text{ a.s.}$$

$$\text{Let } \Lambda(f) = \int_1^\infty \exp\left(-\frac{1}{2}\int_1^u \frac{f(r)}{r}dr\right) du.$$

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A generalized Ray–Knight theorem

- We assume now that $f \in C^1(\mathbb{R}_+;\mathbb{R})$, and there exists $a > 0$ such that $f'(x) \leq a$, for all $x \geq 0$. Suppose that we are in the subcritical case. We consider the SDE
- $$H_s = B_s + \frac{1}{2}\int_0^s f'(L_r^z(H_r))dr + \frac{1}{2}L_s(0),$$
  where $L_s(0)$ denotes the local time accumulated by the process H at level 0 up to time s. We define $S_x = \inf\{s > 0,\ L_s(0) > x\}$.
- We have

Theorem

The laws of the two random fields $\{L_{S_x}(t);\ t \geq 0, x \geq 0\}$ *and* $\{Z_t^x;\ t \geq 0, x \geq 0\}$ *coincide.*

The proof exploits ideas from Norris, Rogers, Williams (1987) who prove the other Ray–Knight theorem in a similar context.

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The proof exploits ideas from Norris, Rogers, Williams (1987) who prove the other Ray–Knight theorem in a similar context.

Effect of the competition on the height and length of the forest of genealogical trees

The finite population case

- We assume again that $f \in C(\mathbb{R}_+;\mathbb{R})$, $f(0) = 0$ and for some fixed $a > 0$, $f(x+y) - f(x) \leq ay$, for all $x, y \geq 0$. We assume in addition that for some $b > 0$, $f(x) < 0$ for all $x \geq b$. Define $H^m = \inf\{t > 0,\ X^m_t = 0\}$, $L^m = \int_0^{H^m} X^m_t dt$.
- We have

Theorem

1. *If* $\int_b^\infty |f(x)|^{-1}dx = \infty$*, then* $\sup_m H^m = \infty$ *a.s.*
2. *If* $\int_b^\infty |f(x)|^{-1}dx < \infty$*, then* $\sup_m \mathbb{E}(e^{cH^m}) < \infty$ *for some* $c > 0$*.*

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Assume in addition that $g(x) = f(x)/x$ *satisfies* $g(x+y) - g(x) \leq ay$*.*

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The case of continuous state space

- Same assumptions as in the discrete case. We define $T^x = \inf\{t > 0,\ Z_t^x = 0\}$, $S^x = \int_0^{T^x} Z_s^x ds$.
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Intuitive idea

- The reason why the above works is essentially because, if $g : \mathbb{R}_+ \to \mathbb{R}_+$ satifies

$$\int_0^\infty \frac{1}{g(x)} dx < \infty$$

then the solution of the ODE

$$\dot{x}(t) = g(x), \quad x(0) = x > 0$$

explodes in finite time.

- Similarly the ODE

$$\dot{x}(t) = -g(x), \quad x(0) = +\infty$$

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The path–valued Markov process

Restriction of our general model

- For the rest of this talk, we restrict ourselves to the case $f(x) = -\gamma x^2$, with $\gamma > 0$. We will only consider the continuous state–space case.
- This means that we consider the solution Z_t^x of the SDE

$$Z_t^x = x - \gamma \int_0^t (Z_s^x)^2 ds + 2 \int_0^t \int_0^{Z_s^x} W(ds, du).$$

- Let us associate to this the solution of the same SDE with $\gamma = 0$, that is the critical Feller branching diffusion

$$Y_t^x = x + 2 \int_0^t \int_0^{Y_s^x} W(ds, du).$$

If we consider those two SDEs with the same W, we obtain a coupling of Y and Z which satisfies $Z_t^x \leq Y_t^x$ a.s. for all $t \geq 0, x \geq 0$.

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A better coupling

- For each $k, n \geq 1$, let $x_n^k = k2^{-n}$, and $Y_t^{n,k} = Y_t^{x_n^k}$. For each $n \geq 1$, we now define recursively $\{Z_t^{n,k},\ t \geq 0\}$ for $k = 1, 2, \ldots$.
- We set $Z_t^{n,0} \equiv 0$ and define $Z_t^{n,1}$ to be the solution of the SDE

$$Z_t^{n,1} = 2^{-n} + \theta \int_0^t Z_s^{n,1} ds - \gamma \int_0^t (Z_s^{n,1})^2 ds + 2 \int_0^t \int_0^{Z_s^{n,1}} W(ds, du).$$

And for $k \geq 2$, we let $Z_t^{n,k} = Z_t^{n,1} + V_t^{n,2} + \cdots + V_t^{n,k}$, where

$$V_t^{n,k} = 2^{-n} + \theta \int_0^t V_s^{n,k} ds - \gamma \int_0^t \left[2 Z_s^{n,k-1} V_s^{n,k} + (V_s^{n,k})^2 \right] ds + 2 \int_0^t \int_{Y_s^{n,k-1}}^{Y_s^{n,k-1} + V_s^{n,k}} W(ds, du).$$

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- It is plain that for all $k \geq 1$,

$$Z_t^{n,k} - Z_t^{n,k-1} = V_t^{n,k} \leq Y_t^{n,k} - Y_t^{n,k-1} \text{ a.s. for all } t \geq 0,$$

  and that the law of $\{Z_t^{n,k},\, k \geq 1, t \geq 0\}$ is the right one.

- Recall that for each $t > 0$, $x \to Y_t^x$ has finitely many jumps on any compact interval, and is constant between its jumps, and if $0 < s < t$,

$$\{x,\, Y_t^x \neq Y_t^{x-}\} \subset \{x,\, Y_s^x \neq Y_s^{x-}\}.$$

- The above construction allows to show that the same is true for a properly defined $\{Z_t^x,\, t \geq 0, x > 0\}$, and moreover for all $t > 0$,

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More precisely, we can write Y^x as the solution of the SDE (E stands for the space of excursions away from 0)

$$Y^x_\cdot = \int_{[0,x]\times E} uN(dy, du),$$

where N is a Poisson random measure on $\mathbb{R}_+ \times E$ with mean measure $dy \times \mathbb{Q}(du)$, where $\mathbb{Q}$ is the excursion measure of the Feller diffusion.

- We have similarly that $x \to Z^x$ is a sum of excursions. Call $N(dy, du)$ the corresponding point process, which is such that for all $x > 0$,

$$Z^x = \int_{[0,x]\times E} uN(dy, du).$$

- The predictable intensity of N is

$$L(Z^y, u)\mathbb{Q}(du)dy,$$

where (with $\zeta = \inf\{t,\ U_t = 0\}$ the lifetime of U)

$$L(Z, U) = \exp\left(-\frac{\gamma}{4}\int_0^\zeta (2Z_t + U_t)dU_t - \frac{\gamma^2}{8}\int_0^\zeta (2Z_t + U_t)^2 U_t dt\right).$$

- This follows readily from the statement

$$Z^x = \int_{[0,x]\times E} L(Z^y, u)u\mathbb{Q}(du)dy + M_x,$$

where M^x is an E–valued $\mathcal{F}^x$–martingale.

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- The last identity is proved as follows. We want to establish that for any $t > 0$,

$$Z^x(t) = \int_{[0,x]\times E} L^\gamma(Z^y, u)u(t)\mathbb{Q}(du)dy + M_x(t).$$

- Clearly if x is a dyadic number, then for n large enough

$$Z^x(t) = \sum_{k=1}^{x2^n} 2^{-n}\mathbb{E}\left(Z^{x_{k+1}} - Z^{x_k}\Big|Z^{x_k}\right) + M_n^x(t),$$

where $\{M_n^x(t),\ x > 0\}$ is a martingale.

- Now

$$\mathbb{E}\left(Z^{x+y}(t) - Z^x(t)\Big|Z^x\right) = \mathbb{E}\left(L^\gamma(Z^x, U^y)U_t^y\Big|Z^x\right),$$

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where under $\mathbb{Q}_{y,t}$

$$U_r = y + 4t \wedge r + 2\int_0^t \sqrt{U_s} dB_s.$$

- Finally we can take the limit as $y \to 0$ in the last identity, yielding

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It just remain to verify that

$$\mathbb{E}_{\mathbb{Q}_{0,t}}\left(L^\gamma(Z^x, U) \middle| Z^x\right) = \int_E L^\gamma(Z^x, u)u(t)\mathbb{Q}(du),$$

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