

MyBEM, a fast boundary element solver by Sparse Cardinal Sine Decomposition

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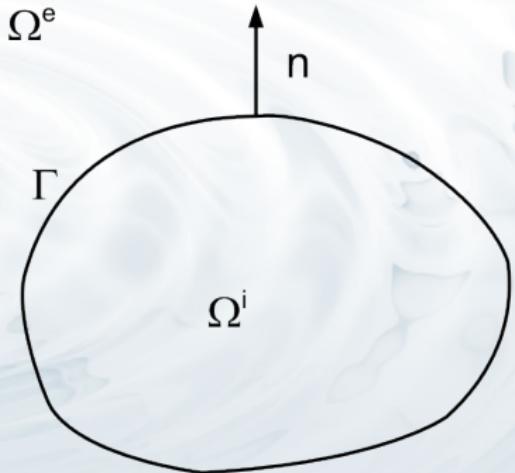
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Acoustic scattering problem

Notations :

Let a solution u defined in the full space $\mathbb{R}^3 \setminus \Gamma$ with :

- Γ smooth and oriented,
- $u^i = u|_{\Omega_i}$ and $u^e = u|_{\Omega_e}$,
- $\mu = [u] = u^e - u^i$ and $\lambda = [\partial_n u] = \partial_n u^e - \partial_n u^i$.



Theorem of integral representation

If u satisfies Helmholtz equation and Sommerfeld radiation condition :

$$\begin{cases} -(\Delta u^i + k^2 u^i) = 0 & \in \Omega^i, \\ -(\Delta u^e + k^2 u^e) = 0 & \in \Omega^e, \\ \lim_{r \rightarrow +\infty} r (\partial_r u^e + iku^e) = 0, \end{cases}$$

then u satisfies :

$$u(\mathbf{x}) = \mathcal{D}\mu(\mathbf{x}) - \mathcal{S}\lambda(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega^i \cup \Omega^e,$$

$$\frac{1}{2}(u^i(\mathbf{x}) + u^e(\mathbf{x})) = D\mu(\mathbf{x}) - S\lambda(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma,$$

$$\frac{1}{2}(\partial_n u^i(\mathbf{x}) + \partial_n u^e(\mathbf{x})) = H\mu(\mathbf{x}) - D^t \lambda(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma.$$

Boundary Element Method

- Single layer potential (idem for Double layer) :

$$\mathcal{S}\lambda(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y})\lambda(\mathbf{y})d\Gamma_y, \quad G(\mathbf{x}, \mathbf{y}) = \frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

- Boundary finite elements $(\phi_n(\mathbf{x}))_{1 \leq n \leq N_{dof}}$:

$$\lambda(\mathbf{x}) \sim \lambda_\phi(\mathbf{x}) = \sum_{n=1}^{N_{dof}} \lambda_n \phi_n(\mathbf{x}).$$

- Discretization of Γ with a quadrature $(\mathbf{y}_q, \gamma_q)_{1 \leq q \leq N_q}$:

$$\mathcal{S}\lambda(\mathbf{x}) \sim G \star \lambda_\phi(\mathbf{x}) = \sum_{q=1}^{N_q} \gamma_q G(\mathbf{x}, \mathbf{y}_q) \lambda_\phi(\mathbf{y}_q).$$

- Galerkin formulation (**dense**) :

$$[S]_{i,j} = \int_{\Gamma} \int_{\Gamma} \phi_i(\mathbf{x}) G(\mathbf{x}, \mathbf{y}) \phi_j(\mathbf{y}) d\Gamma_x d\Gamma_y.$$

Point to point interactions

With N_0 depending on computer used :

- **Direct method** : Computing and storing dense matrix ...
... $O(N^2)$ operations, impossible for $N \geq N_0$.
 - **Iterative method** : Only computing dense matrix ...
... $O(N^2)$ operations, slow for $N \geq N_0$.
 - **Fast iterative method** : Split the variables x and y in $G(x, y)$...
... $O(N)$ operations, fast for $N \geq N_0$.

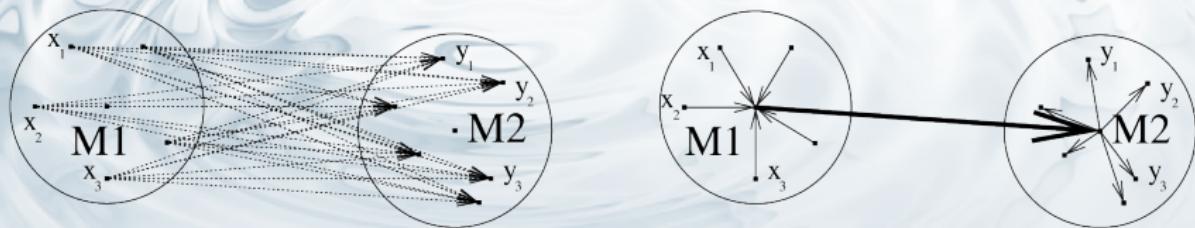
Example : If $G(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2$, then

$$\forall i, \quad v_i = \sum_{i,j} G(\mathbf{x}_i, \mathbf{x}_j) u_j = |\mathbf{x}_i|^2 \sum_j u_j - 2\mathbf{x}_i \cdot \sum_j \mathbf{x}_j u_j + \sum_j |\mathbf{x}_j|^2 u_j$$

The Fast Multipole Method (FMM)

- Helmholtz Green kernel with Gegenbauer addition theorem, for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3$:

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} = \frac{ik}{16\pi^2} \lim_{L \rightarrow +\infty} \int_{\mathbf{s} \in S^2} e^{iks \cdot \mathbf{x} M_1} T_{M_1 M_2}^L(s) e^{iks \cdot M_2 \mathbf{y}} ds.$$



- Separation of the variables \mathbf{x} et $\mathbf{y} \rightarrow$ **Compression !**
- **(A lot of) numerical and technical difficulties ...**
... Complexity $\rightarrow O(N \log N)$.

The SCSD

- **Principle :** Convolution in space \leftrightarrow Product in Fourier variables.
- Fourier transform of the cardinal sine ($S^2 = \text{unit sphere}$) :

$$\mathcal{F}\left(\frac{\sin(|\mathbf{z}|)}{|\mathbf{z}|}\right) = 2\pi^2 \delta_{S^2} \quad \forall \mathbf{z} \in \mathbb{R}^3.$$

- Integral representation :

$$\frac{\sin(|\mathbf{z}|)}{|\mathbf{z}|} = \frac{1}{4\pi} \int_{S^2} e^{i\mathbf{s} \cdot \mathbf{z}} ds \quad \forall \mathbf{z} \in \mathbb{R}^3.$$

- Imaginary part of G , setting $\mathbf{z} = k(\mathbf{x} - \mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3$:

$$\Im(G(\mathbf{x}, \mathbf{y})) = \frac{\sin(k|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} = \frac{k}{(4\pi)^2} \int_{S^2} e^{iks \cdot \mathbf{x}} e^{-iks \cdot \mathbf{y}} ds.$$

- Simple separation of the variables \mathbf{x} and \mathbf{y} .

First step towards a fast convolution

- Convolution for the imaginary part :

$$\Im(\mathcal{G}) \star \lambda_\phi(\mathbf{x}_r) = \sum_{q=1}^{N_q} \gamma_q \left[\frac{k}{(4\pi)^2} \int_{S^2} e^{i\mathbf{ks} \cdot \mathbf{x}_r} e^{-i\mathbf{ks} \cdot \mathbf{y}_q} d\mathbf{s} \right] \lambda_\phi(\mathbf{y}_q).$$

- After discretization of S^2 with a quadrature $(\mathbf{s}_p; \sigma_p)_{1 \leq p \leq N_p}$:

$$\mathfrak{Im}(G) \star \lambda_\phi(\mathbf{x}_r) \sim \frac{k}{(4\pi)^2} \sum_{p=1}^{N_p} e^{ik\mathbf{s}_p \cdot \mathbf{x}_r} \sigma_p \left[\sum_{q=1}^{N_q} e^{-ik\mathbf{s}_p \cdot \mathbf{y}_q} \gamma_q \lambda_\phi(\mathbf{y}_q) \right]_p.$$

- Fast evaluation thanks to a Non Uniform 3D Fourier Transform (**NUFFT 3D type-III**), in space x_n and frequencies ξ_p :

$$NUFFT(f)_p = \sum_{n=1}^N e^{\pm i \xi_p \cdot \mathbf{x}_n} f_n$$



- Complexity $O(N \log N)$!

Real part of the Green kernel

- **Principle :** $\Re(G)$ is expressed using $\Im(G)$
- Fourier transform of the cardinal cosine :

$$\mathcal{F}\left(\frac{\cos(|\mathbf{z}|)}{|\mathbf{z}|}\right) = \frac{4\pi}{|\xi|^2 - 1} \quad \forall (\mathbf{z}, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

- Integral representation of the Green kernel :

$$\frac{\cos(|\mathbf{z}|)}{|\mathbf{z}|} = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{1}{|\xi|^2 - 1} e^{i\xi \cdot \mathbf{z}} d\xi.$$

- Change of variable Cartesian \rightarrow Spherical :

$$\frac{\cos(|\mathbf{z}|)}{|\mathbf{z}|} = \frac{1}{2\pi^2} \int_{\mathbb{R}^+} \frac{\rho^2}{\rho^2 - 1} \left(\int_{S^2} e^{i\rho \mathbf{s} \cdot \mathbf{z}} d\mathbf{s} \right) d\rho.$$

- Expression 1D of cosc as a function of sinc :

$$\frac{\cos(|\mathbf{z}|)}{|\mathbf{z}|} = \frac{2}{\pi} \int_{\mathbb{R}^+} \frac{\rho}{\rho^2 - 1} \frac{\sin(\rho|\mathbf{z}|)}{|\mathbf{z}|} d\rho$$

Quadrature

$$\frac{\cos(|\mathbf{z}|)}{|\mathbf{z}|} = \frac{2}{\pi} \int_{\mathbb{R}^+} \frac{\rho}{\rho^2 - 1} \frac{\sin(\rho|\mathbf{z}|)}{|\mathbf{z}|} d\rho.$$

- We look for points and weights $(\rho_m; \alpha_m)_{1 \leq m \leq M}$

$$\frac{\cos(|\mathbf{z}|)}{|\mathbf{z}|} \sim \sum_{m=1}^M \alpha_m \frac{\sin(\rho_m |\mathbf{z}|)}{|\mathbf{z}|}.$$

- We solve in $(\rho_m; \alpha_m)_{1 \leq m \leq M}$ with a least square approximation, and $\rho_m = \frac{\pi}{b}(2m - 1)$

$$\forall |\mathbf{z}_i| \in [a, b] : \sum_{m=1}^M \alpha_m \sin(\rho_m |\mathbf{z}_i|) = \cos(|\mathbf{z}_i|) \Rightarrow A(\rho)\alpha = B.$$

- Fundamental result :**

$$M \propto \frac{a+b}{a} |\log(\epsilon)|$$

Final SCSD-formalism

- For $k|\mathbf{x} - \mathbf{y}| \in [a, b]$:

$$\frac{\cos(k|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} \sim \sum_{m=1}^M \alpha_m \frac{\sin(\rho_m k|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

- We append $(\rho_m; \alpha_m)$ with $\alpha_{M+1} = -i$ and $\rho_{M+1} = 1$:

$$\frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \sim \sum_{m=1}^{M+1} \frac{\alpha_m}{4\pi} \frac{\sin(\rho_m k|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} = \sum_{m=1}^{M+1} \frac{\alpha_m}{(4\pi)^2} \int_{S_{k\rho_m}^2} e^{i\mathbf{s}\cdot\mathbf{x}} e^{-i\mathbf{s}\cdot\mathbf{y}} d\mathbf{s}.$$

- Quadrature of \mathbb{R}^3 with $(\xi_p \in \cup S_{k\rho_m}^2; \omega_p)_{1 \leq p \leq N_p}$:

$$G * \lambda_\phi(\mathbf{x}) \sim \frac{1}{(4\pi)^2} \sum_{p=1}^{N_p} e^{i\xi_p \cdot \mathbf{x}} \omega_p \left[\sum_{q=1}^{N_q} e^{-i\xi_p \cdot \mathbf{y}_q} \gamma_q \lambda_\phi(\mathbf{y}_q) \right]_p.$$

- Fundamental result :** $N_p \propto \left(\frac{b}{a} |\log(\epsilon)|\right)^3$

Explanation

- Quadrature of \mathbb{R}^3 with $(\xi_p \in \cup S_{k\rho_m}^2; \omega_p)_{1 \leq p \leq N_p}$:

$$G * \lambda_\phi(\mathbf{x}) \sim \frac{1}{(4\pi)^2} \sum_{p=1}^{N_p} e^{i\xi_p \cdot \mathbf{x}} \omega_p \left[\sum_{q=1}^{N_q} e^{-i\xi_p \cdot \mathbf{y}_q} \gamma_q \lambda_\phi(\mathbf{y}_q) \right]_p.$$

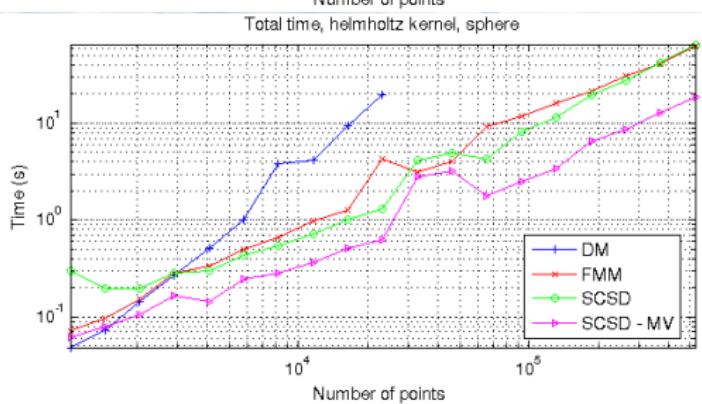
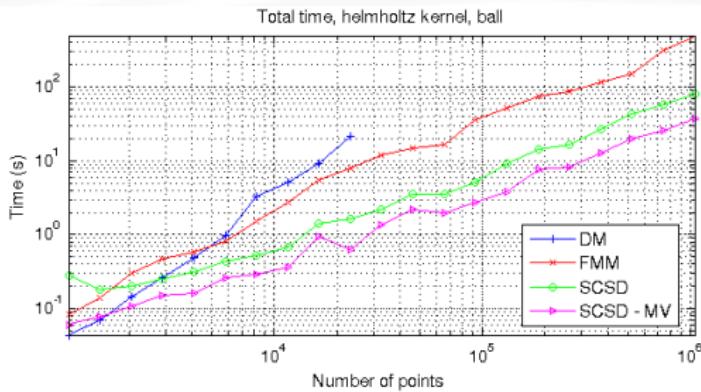
- Mimicks

$$\begin{aligned} G * \lambda_\phi(\mathbf{x}) &= \mathcal{F}^{-1} \left(\hat{G}(\xi) \mathcal{F}(\lambda_\phi \delta_\Gamma) \right) \\ &= \int_{\mathbb{R}^3} e^{i\xi \cdot \mathbf{x}} \hat{G}(\xi) \left[\int_\Gamma e^{-i\xi \cdot \mathbf{y}} \lambda_\phi(\mathbf{y}) d\mathbf{y} \right] d\xi \end{aligned}$$

Algorithm

- ➊ SCSD quadrature for **far** interactions (in $[a, b]$) :
 $(\xi_p \in \mathbb{R}^3; \omega_p)_{1 \leq p \leq N_p}$.
- ➋ Type-III NUFFT $(\mathbf{y}_q)_{1 \leq q \leq N_q}$ to $(\xi_p)_{1 \leq p \leq N_p}$ on
 $(\gamma_q \lambda_\phi(\mathbf{y}_q))_{1 \leq q \leq N_q}$.
- ➌ Weighting of the result ② with $(\omega_p)_{1 \leq p \leq N_p}$.
- ➍ Type-III NUFFT $(\xi_p)_{1 \leq p \leq N_p}$ to $(\mathbf{x}_i)_{1 \leq i \leq N_i}$ on the result ③ :
 $G_{far} \star \lambda_\phi(\mathbf{x}_i)_{1 \leq i \leq N_i}$.
- ➎ **Close** interactions correction (in $[0, a]$) : $G_{corr} \star \lambda_\phi(\mathbf{x}_i)_{1 \leq i \leq N_i}$.
- ➏ $G \star \lambda_\phi(\mathbf{x}_i)_{1 \leq i \leq N_i} \sim (G_{far} + G_{corr}) \star \lambda_\phi(\mathbf{x}_i)_{1 \leq i \leq N_i}$.

Comparison FMM and SCSD



Legend :

DM (exact) on $[0, b]$.
FMM ^a on $[0, b]$.
SCSD on $[0, b]$.
SCSD-MV on $[a, b]$.

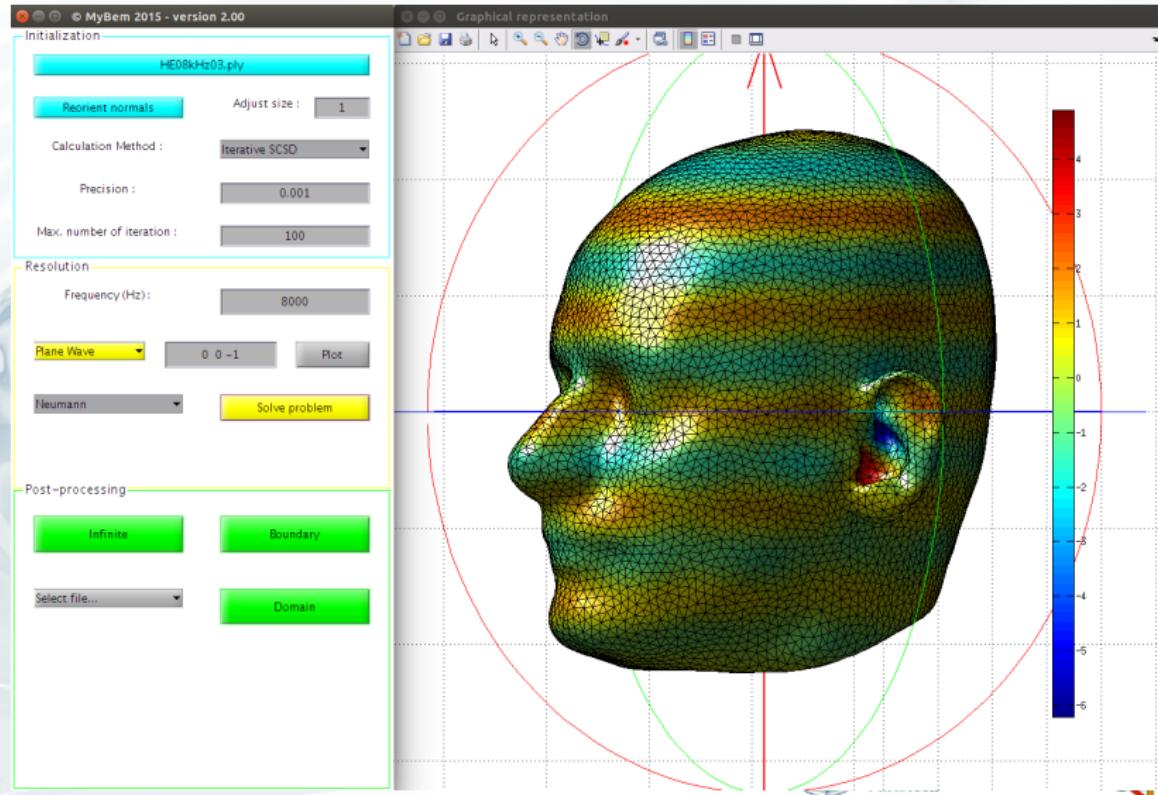
Precision : $\epsilon = 10^{-3}$.

Theoretical complexity :

Ball : $O(N \log N)$.
Sphere : $O(N^{6/5} \log N)$.

a. see www.cims.nyu.edu/cmcl

MyBEM - A Matlab fast BEM library

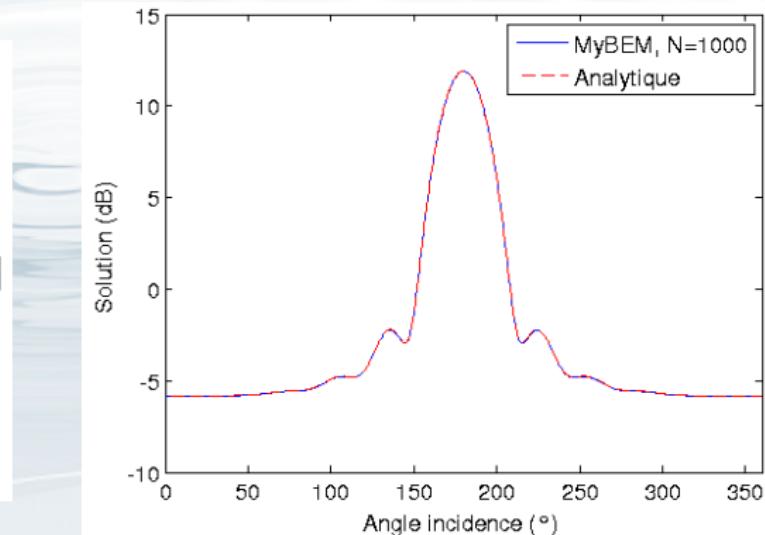
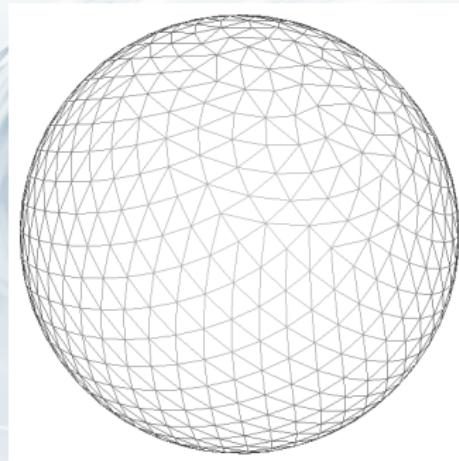


Main features

- Galerkin approximation with Finite Elements type P^1 and RT on triangles
- Semi-analytical for singular integrations in close interaction
- New fast method SCSD with NUFFT mexfile¹
- Fast Multipole Method¹ for comparisons
- Infinite radiation (RCS), on any field and on the mesh
- LU preconditioning and Brackage-Werner regularization
- Indirect jump formulations
- Object-Oriented Programming
- High-level script call or standalone GUI
- Parallel loops (*parfor*)
- Non regression tests

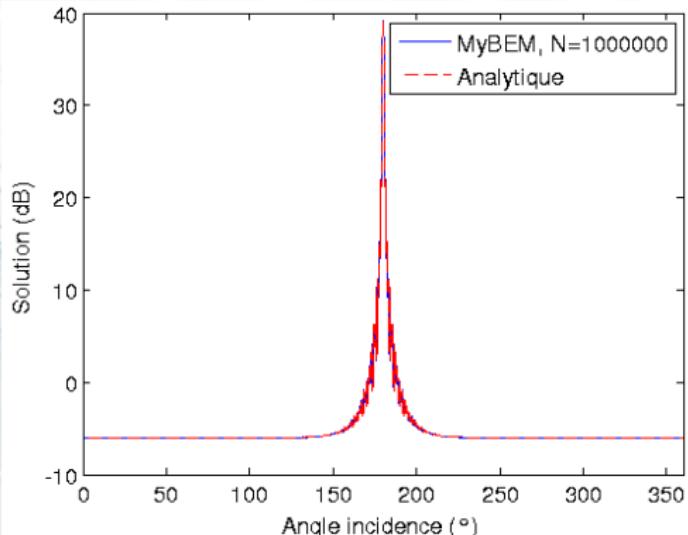
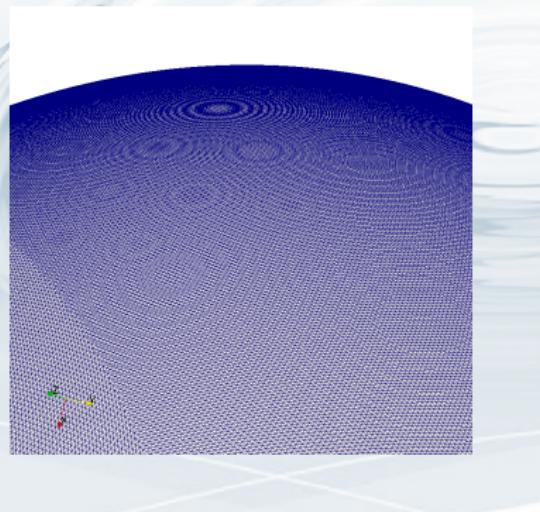
RCS of the unit sphere (1)

Helmholtz problem - Dirichlet Brackage Werner
Mesh generated with Matlab - 1 000 degrees of freedom
0.3 kHz - 5 iterations - 2 seconds



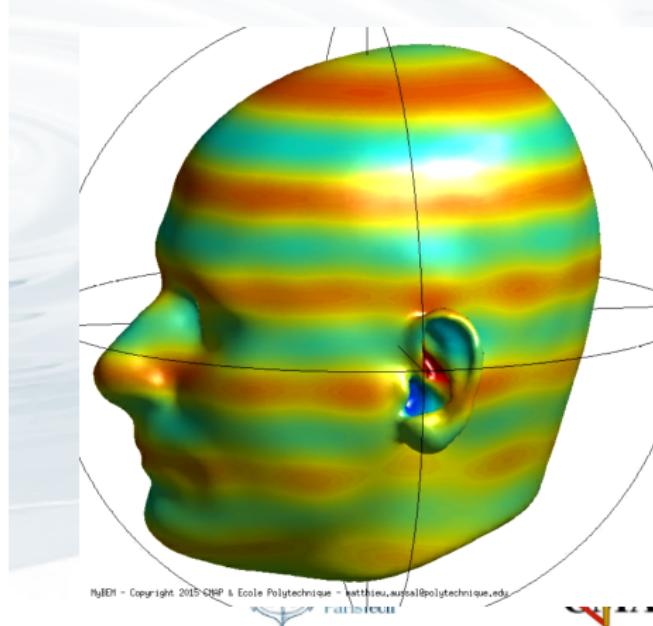
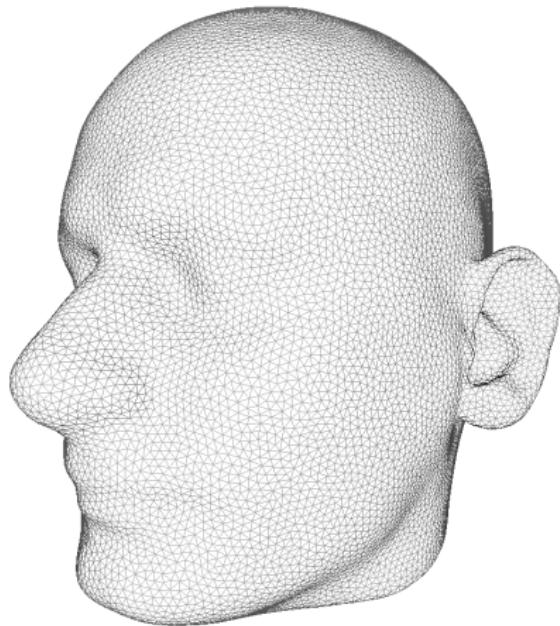
RCS of the unit sphere (2)

Helmholtz problem - Dirichlet Brackage Werner
Mesh generated with Matlab - 1 000 000 degrees of freedom
10 kHz ($kr_{max} = 369$) - 12 iterations - 20 minutes



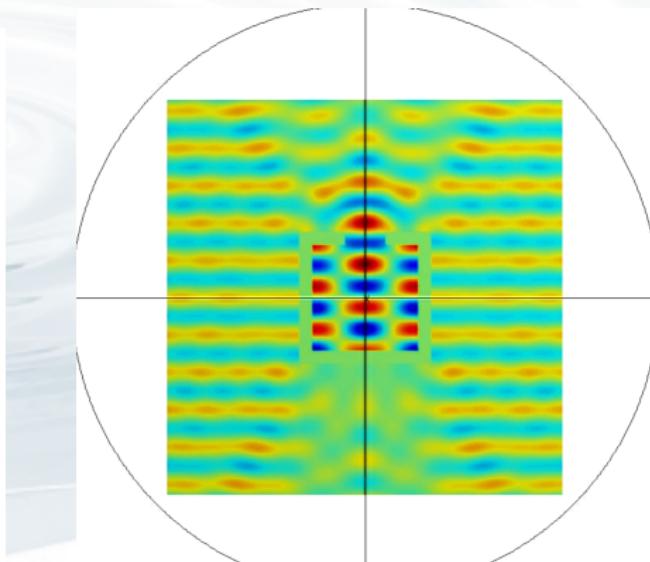
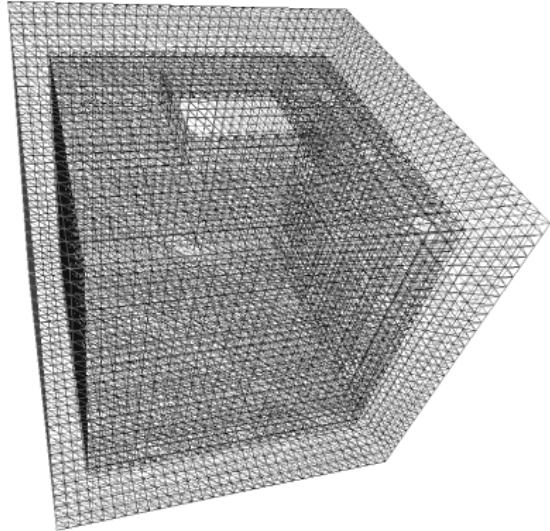
Ears modes of human head

Helmholtz Problem - Neumann Brackage Werner
Mesh from SYMARE project - 20 000 degrees of freedom
8 kHz - 17 iterations - 54 seconds



Resonance in a 3D cubic cavity

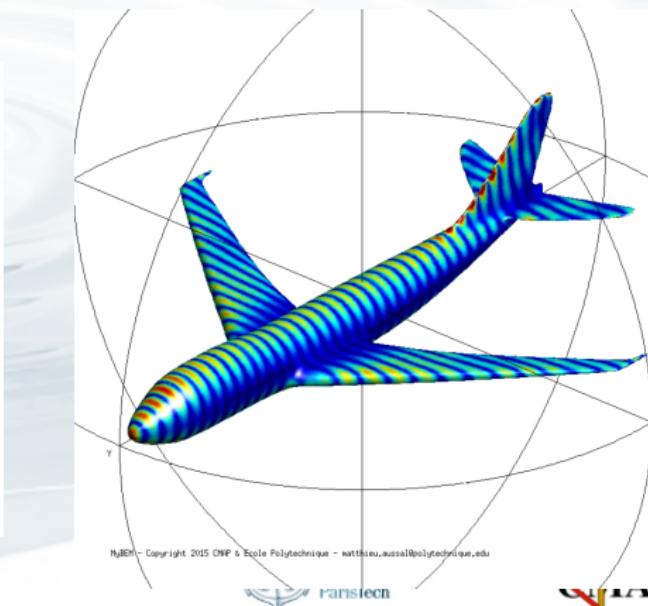
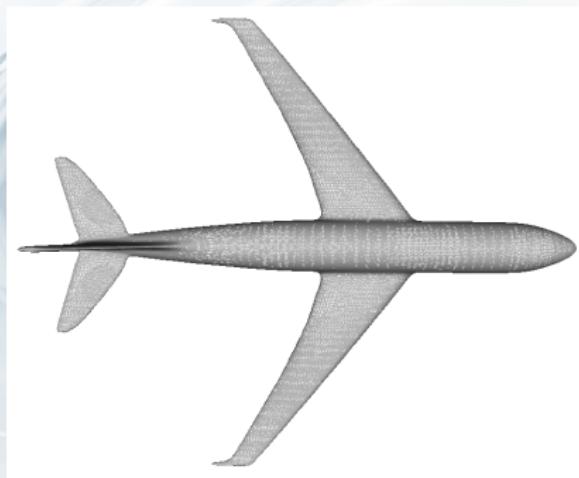
Helmholtz problem - Neumann Brackage Werner
Mesh generated with Matlab - 10 000 degrees of freedom
600 Hz - 56 iterations - 1 minute



Surface current on Boeing 747

Maxwell problem - PEC CFIE

Mesh from Gamma project - 150 000 degrees of freedom
5 GHz - Vertical polarisation - 164 iterations - 16 minutes

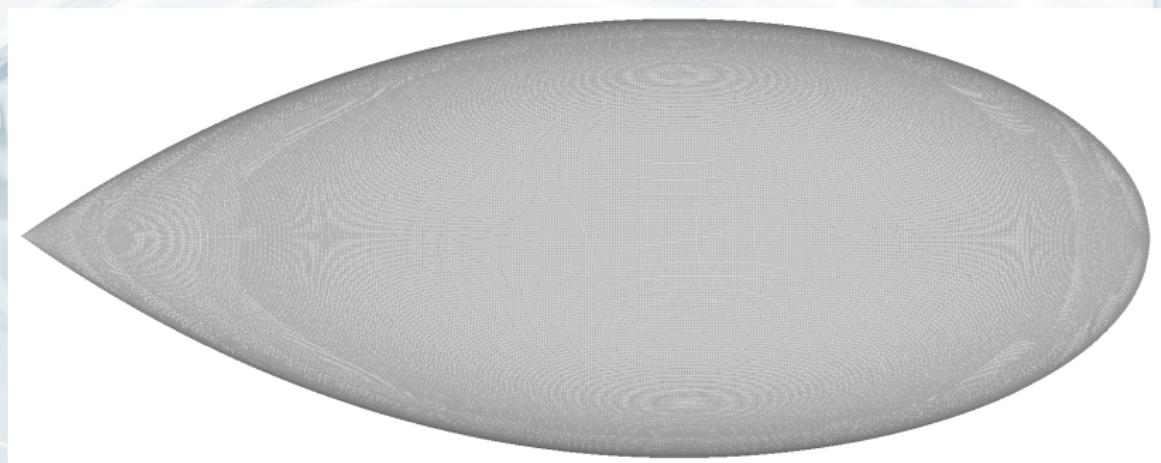


Surface current on NASA Almond

Maxwell problem - PEC CFIE

Mesh of an industrial - 1 000 000 degrees of freedom

8.5 GHz ($kr_{max} = 478$) - Vertical polarisation - 112 iterations - 2 hours

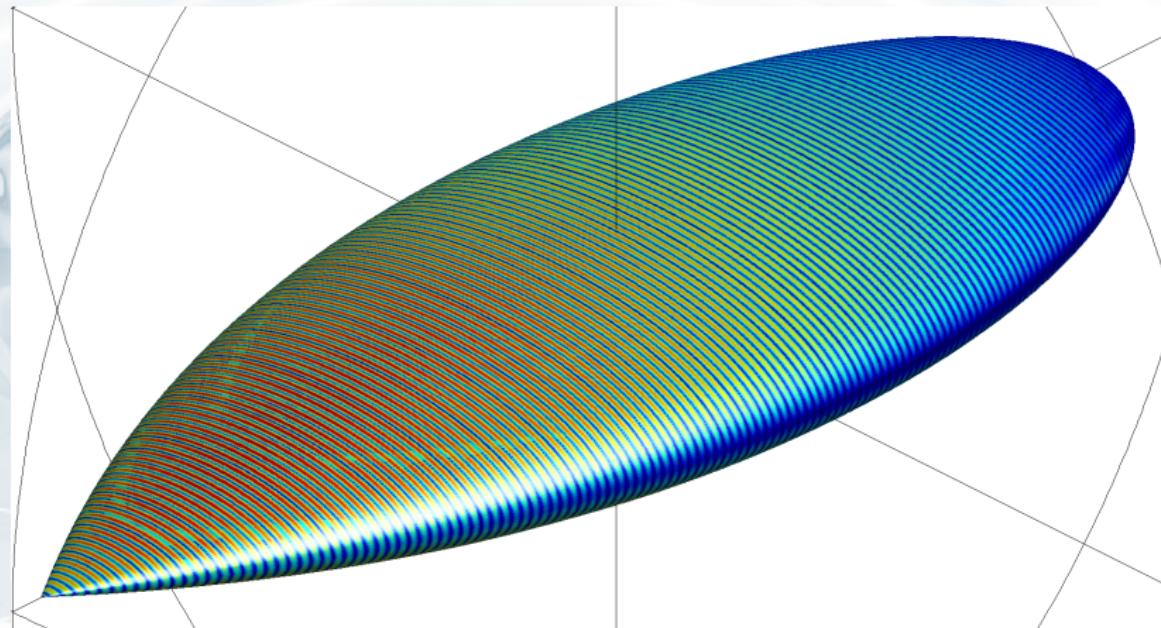


Surface current on NASA Almond

Maxwell problem - PEC CFIE

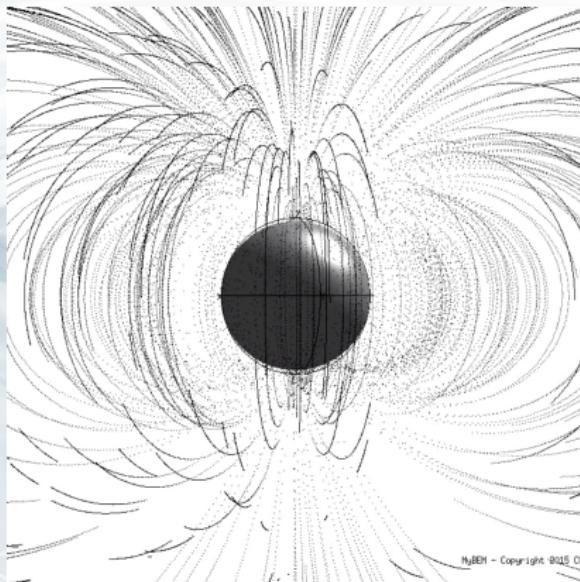
Mesh of an industrial - 1 000 000 degrees of freedom

8.5 GHz ($kr_{max} = 478$) - Vertical polarisation - 112 iterations - 2 hours



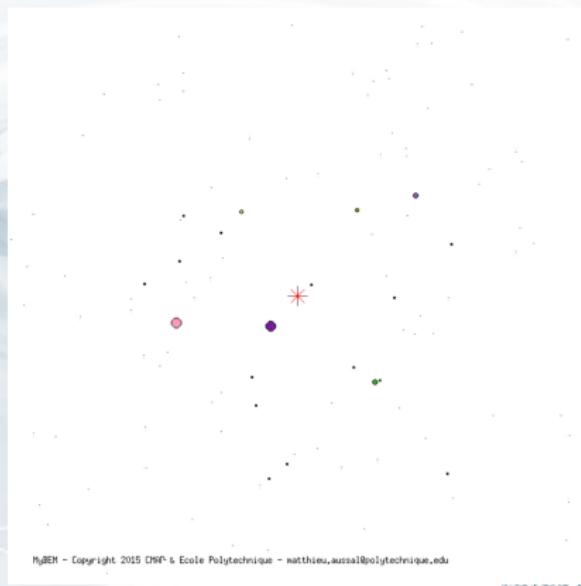
Earth's magnetic field

Laplace problem - Double layer potential
Mesh generated with Matlab - 10 000 degrees of freedom - 3 seconds



Gravitational interactions

Laplace problem - Double layer potential
10 000 degrees of freedom - Runge-Kutta 4 scheme (100 000 time step) -
10 minutes



Conclusion and future works

CONCLUSION :

- New SCSD fast convolution for many kernels (Laplace, Helmholtz, Maxwell, Stokes),
- Creation of an object based Matlab library for fast BEM,
- Analytical validation up to 10^6 degrees of freedom.

FUTURE WORKS :

- More kernels...,
- Preconditioning,
- High Performance Computing,
- Domain Decomposition Method and coupled FEM/BEM,
- Benchmarks and industrial applications.

Thanks for your attention

-  Alouges, F., Aussal, M. (2015) *The Sparse Cardinal Sine Decomposition and its application for fast numerical convolution.* Numerical Algorithms, 1-22.
-  Aussal, M. (2014) *Méthodes numériques pour la spatialisation sonore, de la simulation à la synthèse binaurale.* PhD thesis to be published.
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-  Greengard, L., & Lee, J. Y. (2004). *Accelerating the nonuniform fast Fourier transform.* SIAM review, 46(3), 443-454.
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