

Fast numerical convolution with the Sparse Cardinal Sine Decomposition

François Alouges¹, Matthieu Aussal² and Emile Parolin³

CMAP - Ecole Polytechnique, Route de Saclay, 91128, Palaiseau Cedex, France,

¹ francois.alouges@polytechnique.edu

² matthieu.aussal@polytechnique.edu

³ emile.parolin@polytechnique.edu

Abstract

Fast convolution on unstructured grids have been developed for many applications (e.g. electrostatics, magnetostatics, acoustics, linear-elasticity, etc.). The main algorithmic goal of numerical methods used in this respect is to reduce the complexity of matrix-vector products, from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$.

In this presentation, we describe a new efficient numerical method, referred to as the *Sparse Cardinal Sine Decomposition* (SCSD) method [1]. It is based on a suitable Fourier decomposition of the Green kernel, sparse quadrature formulae and Type-III Non Uniform Fast Fourier Transforms [5, 7].

This talk outlines the basic SCSD methodology and provides numerical evidence of the efficiency of the method with comparisons between SCSD and other established techniques such as \mathcal{H} -Matrix compression [6] and Fast Multipole Methods [3, 4]. We compare in addition the computational performances of two equivalent Matlab and Python implementations.

Keywords: Quadrature, Non Uniform Fast Fourier Transform, Fast convolution.

Introduction

In undersea warfare, being able to develop stealthy submarines is a long standing concern. As such, the goal is to develop efficient predictive tools for evaluating acoustic target strength of submarines.

To transpose the problem of interest in mathematical terms, we consider a surface obstacle Γ dividing the space \mathbb{R}^3 in two open interior Ω^i and exterior Ω^e domains. Given an incident acoustic field p_{inc} defined in \mathbb{R}^3 , we aim at evaluating the total acoustic field $p_{\text{tot}} = p_{\text{sca}} + p_{\text{inc}}$ resulting of the scattering by the obstacle and characterised by the scattered acoustic field p_{sca} . The exterior Neumann problem to be solved is then written as

$$-(\Delta p_e + k^2 p_e) = 0, \quad \text{in } \Omega^e, \quad (1)$$

$$\partial_n p_e = -\partial_n p_{\text{inc}}, \quad \text{on } \Gamma, \quad (2)$$

$$r(\partial_r p_e - ik p_e) \rightarrow 0, \quad \text{as } r \rightarrow \infty. \quad (3)$$

Using the Brakhage-Werner approach for $\beta \in \mathbb{C}$, the exterior problem can be recast as the following integral equation formulation on the boundary Γ ,

$$\left(H + ik\beta \left(\frac{1}{2}I - D^* \right) \right) \mu = -\partial_n p_{\text{inc}}, \quad \text{on } \Gamma, \quad (4)$$

$$\lambda = ik\beta\mu, \quad \text{on } \Gamma. \quad (5)$$

where the jumps μ and λ are defined as

$$\mu = [p] = p_i - p_e, \quad \lambda = [\partial_n p] = \partial_n p_i - \partial_n p_e, \quad \text{on } \Gamma, \quad (6)$$

and the boundary operators D^* and H are defined as

$$D^* \mu(\mathbf{x}) = \int_{\Gamma} \partial_{n_x} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \, d\Gamma_{\mathbf{y}}, \quad H \mu(\mathbf{x}) = \int_{\Gamma} \partial_{n_x n_y}^2 G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \, d\Gamma_{\mathbf{y}}, \quad \mathbf{x} \in \Gamma. \quad (7)$$

Here $G(\mathbf{x}, \mathbf{y}) = e^{-ik|\mathbf{x}-\mathbf{y}|}/(4\pi|\mathbf{x}-\mathbf{y}|)$ is the free space Green's function in \mathbb{R}^3 . Given the jumps, defined in (6), solutions of equations (4) and (5) on the boundary of the scatterer, classical integral representation theorems can then be used to compute the acoustic field in the domain of interest as well as the radiated field at infinity.

When trying to solve (4) (and (5)) using a classical Boundary Element Method (by collocation or within a Galerkin framework for instance), a discrete approximation space is introduced, and the continuous problem is recast in the form of a *dense* linear system, of size N by N , where N is the dimension of the discrete approximation space. The computation of the elements of this system mostly involves the numerical evaluation of the boundary operators D^* and H , which is typically conducted via Gauss quadrature methods (except for singularities). One is then faced with the evaluation of terms (convolution products) taking the form of

$$\tilde{K}_f(\mathbf{x}) = \sum_{\mathbf{y} \in \Gamma} K(\mathbf{x}, \mathbf{y}) f(\mathbf{y}), \quad \text{for some } \mathbf{x} \in \Gamma, \quad (8)$$

where K is a Green's kernel and f is a scalar function. In a sense, this amounts to evaluate all interactions between two point clouds in \mathbf{x} and \mathbf{y} .

The memory cost in a classical BEM for computing and storing the dense matrix of the linear system is $\mathcal{O}(N^2)$. Direct solvers then require $\mathcal{O}(N^3)$ operations making any attempt impractical for large problems. One hence turns to iterative solvers relying on matrix-vector products, at a still quite important cost of $\mathcal{O}(N^2)$ operations per iteration. There is therefore a need for fast methods that aim to reduce this complexity from quadratic to quasi-linear $\mathcal{O}(N \log N)$. A successful attempt in this direction was first achieved with the Fast Multipole Methods [3, 4] and later with hierarchical-matrix compression [6]. We present a more recent approach with a similar complexity, the *Sparse Cardinal Sine Decomposition* (SCSD) method, introduced in [1, 2].

Sparse Cardinal Sine Decomposition

The building block of the SCSD is the integral representation of the cardinal sine on the unit sphere S , namely

$$\frac{\sin(k|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} = \frac{k}{4\pi} \int_S e^{i\mathbf{k}\mathbf{s}\cdot(\mathbf{x}-\mathbf{y})} dS(\mathbf{s}). \quad (9)$$

A suitable decomposition of the radial Green's kernel $K(\mathbf{x}, \mathbf{y})$ as a cardinal sine series is then computed. We look for two series $(\alpha_n)_{n \in [0, N_K]}$ and $(\rho_n)_{n \in [0, N_K]}$ in order to approximate the Green's kernel considered as

$$K(\mathbf{x}, \mathbf{y}) \approx \sum_{n=0}^{N_K} \alpha_n \frac{\sin(\rho_n k |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|}, \quad \text{with } |\mathbf{x} - \mathbf{y}| \geq R_{\min}. \quad (10)$$

Note that for this decomposition to be valid, we impose $|\mathbf{x} - \mathbf{y}|$ to be larger than a given R_{\min} , to eliminate the difficulties arising at $|\mathbf{x} - \mathbf{y}| \approx 0$. A quadrature of the cardinal sine integral representation (9) on the unit sphere is then constructed as a set of nodes $(\mathbf{s}_m)_{m \in [0, N_s]}$ in \mathbb{R}^3 and weights $(\omega_n)_{m \in [0, N_s]}$ in \mathbb{R} such that

$$\frac{\sin(k|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \approx \frac{k}{4\pi} \sum_{m=0}^{N_s} \omega_m e^{i\mathbf{k}\mathbf{s}_m \cdot (\mathbf{x} - \mathbf{y})}. \quad (11)$$

A Gauss-Legendre quadrature is used for this purpose in our numerical experimentations but other choices are possible. Let $N_l = (N_K + 1)(N_s + 1)$, for $l \in [1, N_l]$, we write

$$\omega_l = \omega_m \alpha_n \frac{k \rho_n}{4\pi}, \quad \mathbf{s}_l = k \rho_n \mathbf{s}_m, \quad m \in [0, N_s] \text{ and } n \in [0, N_K]. \quad (12)$$

The final discrete Fourier quadrature for the Green's kernel is then written as

$$K(\mathbf{x}, \mathbf{y}) \approx \sum_{l=1}^{N_l} \omega_l e^{i\mathbf{s}_l \cdot (\mathbf{x} - \mathbf{y})}, \quad \text{with } |\mathbf{x} - \mathbf{y}| \geq R_{\min}. \quad (13)$$

Terms like (8) are now approximated by

$$\tilde{K}_f(\mathbf{x}) \approx \sum_{l=1}^{N_l} e^{i\mathbf{s}_l \cdot \mathbf{x}} \omega_l \left(\sum_{\mathbf{y} \in \Gamma} e^{-i\mathbf{s}_l \cdot \mathbf{y}} f(\mathbf{y}) \right), \quad \text{for } \mathbf{x} \in \Gamma, \quad (14)$$

which can then be computed efficiently using (direct and inverse) Type-III Non Uniform Fast Fourier Transforms [5, 7] for the computation of sums taking the form of

$$\sum_{n=1}^N e^{\pm i\mathbf{a}_n \cdot \mathbf{b}_n} g(\mathbf{b}_n), \quad \text{for } \mathbf{a}_n, \mathbf{b}_n \in \mathbb{R}^3. \quad (15)$$

NUFFT algorithms with a quasi-linear complexity in $\mathcal{O}(N \log N)$ are available, which yields a final complexity of the method at $\mathcal{O}(N^{6/5} \log N)$ [1].

	BEM	SCSD	FMM	\mathcal{H} -Matrix
$N = 10^3$	11	8	26	11
$N = 10^4$	1200	107	448	158
$N = 10^5$	X	1980	6780	5640

Table 1: Comparison of execution times (in seconds) of several methods.

	<i>MyBEM</i>	<i>PyBEM</i>
BEM ($N = 10^4$)	1200	2880
SCSD ($N = 10^5$)	1980	6260

Table 2: Comparison of execution times (in seconds) of MATLAB and PYTHON codes.

Since the SCSD quadrature is only valid for far interactions $|\mathbf{x} - \mathbf{y}| \geq R_{\min}$, the close interaction terms can be straightforwardly computed as usual. The performance of the method then relies on an equilibrated balance between close and far interaction computations, which can be controlled by R_{\min} .

The SCSD method has been applied with success to several problems with radial Green’s kernel, and not only to the exterior Neumann problem which is the focus of this paper.

Numerical results

We now compare numerically the performance of the SCSD with respect to a classical Boundary Element Method and other established techniques: \mathcal{H} -Matrix compression [6] and Fast Multipole Methods (FMM) [3, 4]. For the purpose of this comparison we use the simple test case of the unit sphere as the obstacle, which permits at the same time to validate the code analytically. The classical BEM, the SCSD and the \mathcal{H} -Matrix methods are implemented in native MATLAB within the *MyBEM* code, which is available in sequential or parallel versions. The FMM and the NUFFT are FORTRAN routines from L. Greengard [10]. The computing times of the different methods are reported in Table 1 and highlight the good performance of the SCSD method. This benchmark was conducted with the sequential versions available, the FORTRAN routines where recompiled to run on a single thread. The classical BEM and the SCSD are also implemented in PYTHON, within the *PyBEM* code, available in sequential mode for the moment. The computing times are reported in Table 2. It is interesting to note that the two implementations, allowing fast high-level developments, perform similarly.

We finally consider the case of a submarine model from the Benchmark Target Strength Simulation (BeTSSi) international workshop [9]. We consider a plane wave as incident field $p_{\text{inc}}(\mathbf{x}) = e^{-i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = (-\cos \alpha, -\sin \alpha, 0)$ where α is the aspect angle. The mesh used for the computations contains 100,001 vertices, allowing for computations up to 1 kHz. The acoustic pressure magnitude $p_{\text{tot}} = p_{\text{sca}} + p_{\text{inc}}$ on the boundary of the submarine and in the Oxy plan is given respectively in Figure 1 and 2. The

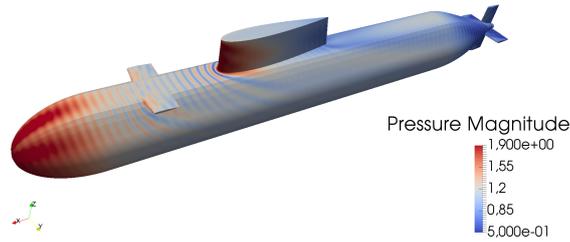


Figure 1: Acoustic pressure magnitude $p_{\text{tot}} = p_{\text{sca}} + p_{\text{inc}}$ on the boundary at 1kHz ($\alpha = 0^\circ$).

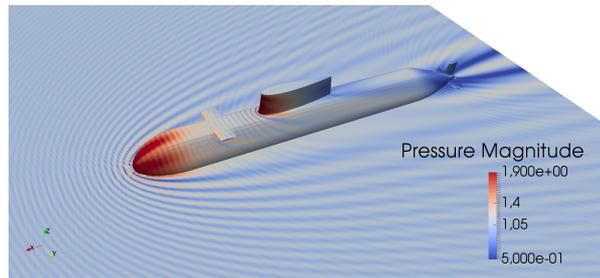


Figure 2: Acoustic pressure magnitude $p_{\text{tot}} = p_{\text{sca}} + p_{\text{inc}}$ in the Oxy plan at 1kHz ($\alpha = 0^\circ$).

target strength $TS(\mathbf{r})$ at range \mathbf{r} is the quantity of interest for evaluating the stealth capabilities of submarines. It is defined as

$$TS(\mathbf{r}) = 10 \log \frac{|\mathbf{r} - \mathbf{r}_0|^2 p_{\text{sca}}^2(\mathbf{r})}{p_{\text{inc}}^2(\mathbf{r}_0)}, \quad (16)$$

where \mathbf{r}_0 is the geometric centre of the obstacle. In Figure 3, we plot the monostatic target strength $TS(\alpha)$ at 200 Hz evaluated in the far field at elevation angle $\beta = 0^\circ$, with respect to the source and receiver aspect angle α running from 0° to 360° with 1° steps. The result is qualitatively in agreement with other results, see for example [8].

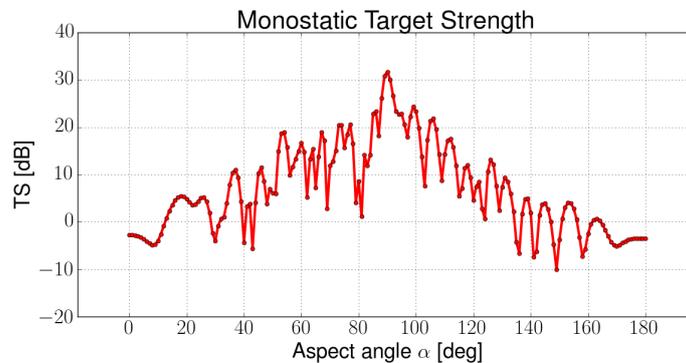


Figure 3: Monostatic target strength $TS(\alpha)$ at 200 Hz in the Oxy plan (elevation $\beta = 0^\circ$).

Conclusion

We presented a recently developed method for fast numerical convolution in the context of integral equations within Boundary Element Methods. It is based on a suitable Fourier decomposition of the Green's kernel as a sum of cardinal sine. The final algorithm relies on Non-Uniform Fast Fourier Transforms for fast evaluation and exhibits a similar complexity as other established techniques. The performance of the method allows to tackle real-life size problems. In particular, we computed the monostatic target strength at 200 Hz of the BeTSSi submarine model.

Acknowledgements

We gratefully thank Nicolas Zerbib and Lassen Mebarek from ESI Group for having provided us with the BeTSSi submarine mesh.

References

- [1] F. ALOUGES AND M. AUSSAL, *The Sparse Cardinal Sine Decomposition and its application for fast numerical convolution*, Numerical Algorithms, 1-22, 2014.
- [2] M. AUSSAL, *Méthodes numériques pour la spatialisation sonore, de la simulation à la synthèse binaurale*, PhD Thesis, 2014.
- [3] L. GREENGARD AND V. ROKHLIN, *A fast algorithm for particle simulations*, Journal of Computational Physics, **73**(2):325–348, 1987.
- [4] L. GREENGARD, *The rapid evaluation of potential fields in particle systems*, MIT press, 1988.
- [5] L. GREENGARD AND J. Y. LEE, *Accelerating the non-uniform Fast Fourier Transform*, SIAM review, **46**(3):443–455, 2004.
- [6] W. HACKBUSCH, *A sparse matrix arithmetic based on \mathcal{H} -matrices. Part I: introduction to \mathcal{H} -matrices*, Computing, **62**(2):89–108, 1999.
- [7] J. Y. LEE AND L. GREENGARD, *The type 3 non-uniform FFT and its applications*, Journal of Computational Physics, **206**(1):1–7, 2005.
- [8] C. W. NELL AND L. E. GILROY, *An improved BASIS model for the BeTSSi submarine*, DRDC Atlantic TR, 199, 2003.
- [9] H. G. SCHNEIDER, R. BERG, L. GILROY, I. KARASALO, I. MACGILLIVRAY, M. T. MORSHUIZEN AND A. VOLKER, *Acoustic scattering by a submarine: results from a benchmark target strength simulation workshop*, ICSV10, 2475–2482, 2003.
- [10] See <http://www.cims.nyu.edu/cmcl>