In the case $\mathbb{P}(Z_1 = 0) = 0$, let us compute the value of

$$\rho_k = \lim_{n\to\infty} -\frac{1}{n}\log(\mathbb{P}_k(Z_n = k)).$$

In that case Z_n is a.s. non-decreasing, so

$$\{Z_n = k\} = \{Z_0 = Z_1 = \ldots = Z_n = k\}$$

a.s. under $\mathbb{P}_k$. Thus, each individual has exactly one offspring in each generation and

$$\mathbb{P}_k(Z_n = k|\mathcal{E}) = \Pi_{i=0}^{n-1}\mathbb{P}(Z_{i+1} = k|Z_i = k, \mathcal{E}) = \Pi_{i=0}^{n-1}Q_i(1)^k,$$

where Q_i are i.i.d. environments distributed as Q and Q is the random probability law of the the number of offsprings.Then

$$\mathbb{P}_k(Z_n = k) = \mathbb{E}(Q(1)^k)^n$$

and

$$\rho_k = -\log(\mathbb{E}(Q(1)^k))$$

and more generally this provides the limit of $-\frac{1}{n}\log(\mathbb{P}_k(Z_n = j))$ for $n \to \infty$, $(j \geq k)$.

ERRATUM

$$i\log(\mathbb{E}(Q(1))) \quad \text{should be replaced by} \quad \log(\mathbb{E}(Q(1)^i))$$

in Theorem 3.1 (ii) and similarly the correct limiting value in Proposition 2.1. (i) is $\log(\mathbb{E}(Q(1)^k))$. Both coincide (a priori only) for $k = 1$ or Galton Watson processes.