Concentration of quadratic forms under a Bernstein moment assumption

Pierre C. Bellec

ENSAE,
3 avenue Pierre Larousse,
92245 Malakoff Cedex, France.

Abstract: A concentration result for quadratic form of independent subgaussian random variables is derived. If the moments of the random variables satisfy a "Bernstein condition", then the variance term of the Hanson-Wright inequality can be improved.

1. Concentration of a quadratic form of subgaussian random variables

Throughout this note, $A \in \mathbb{R}^{n \times n}$ is a real matrix, and $\xi = (\xi_1, ..., \xi_n)^T$ is a centered random vector with independent components. We are interested in the concentration behavior of the random variable

$$\xi^T A \xi - E[\xi^T A \xi].$$

Let $\sigma_i^2 = E[\xi_i^2]$ for all $i = 1, ..., n$ and define $D_\sigma = \text{diag}(\sigma_1, ..., \sigma_n)$. If the random variables $\xi_1, ..., \xi_n$ are Gaussian, we have the following concentration inequality.

**Proposition 1** (Gaussian chaos of order 2). Let $\xi_1, ..., \xi_n$ be independent zero-mean normal random variables with for all $i = 1, ..., n$, $E[\xi_i^2] = \sigma_i^2$. Let $A$ be any $n \times n$ real matrix. Then for any $x > 0$,

$$P \left( \xi^T A \xi - E[\xi^T A \xi] > 2 \|D_\sigma A D_\sigma\|_F \sqrt{x} + 2 \|D_\sigma A D_\sigma\|_2^2 x \right) \leq \exp(-x).$$

A proof of this concentration result can be found in \cite[Example 2.12]{3}. We will refer to the term $2 \|D_\sigma A D_\sigma\|_F \sqrt{x}$ as the variance term, since if $A$ is diagonal-free, the random variable $\xi^T A \xi$ is centered with variance

$$\|D_\sigma A D_\sigma\|_F^2.$$

A similar concentration result is available for subgaussian random variables. It is known as the Hanson-Wright inequality and is given in Proposition 2 below. First versions of this inequality can be found in Hanson and Wright \cite{5} and Wright \cite{9}, although with a weaker statement than Proposition 2 below since these results involve $\|\{|a_{ij}\}|\|_2$ instead of $\|A\|_2$. Recent proofs of this concentration inequality with $\|A\|_2$ instead of $\|\{|a_{ij}\}|\|_2$ can be found in Rudelson and Vershynin \cite{6} or Barthe and Milman \cite[Theorem A.5]{2}.

**Proposition 2** (Hanson-Wright inequality \cite{6}). There exist an absolute constant $c > 0$ such that the following holds. Let $n \geq 1$ and $\xi_1, ..., \xi_n$ be independent zero-mean

---

\* This work was supported by GENES and by the grant Investissements d'Avenir (ANR-11-IDEX-0003/ Labex Ecodec/ ANR-11-LABX-0047).
subgaussian random variables with $\max_{i=1,...,n} \|\xi_i\|_{\psi_2} \leq K$ for some real number $K > 0$. Let $A$ be any $n \times n$ real matrix. Then for all $t > 0$,
\[ P \left( \xi^T A \xi - E[\xi^T A \xi] > t \right) \leq \exp \left( -c \min \left( \frac{t^2}{K^4 \|A\|_F^2}, \frac{t}{K^2 \|A\|_2^2} \right) \right) \]
(1)
where $\xi = (\xi_1, \ldots, \xi_n)^T$. Furthermore, for any $x > 0$, with probability greater than $1 - \exp(-x)$,
\[ \xi^T A \xi - E[\xi^T A \xi] \leq cK^2 \|A\|_2 x + cK^2 \|A\|_F \sqrt{x}. \]

For some random variables $\xi_1, \ldots, \xi_n$, the “variance term” $K^2 \|A\|_F \sqrt{x}$ is far from the variance of the random variable $\xi^T A \xi$. The goal of the present paper is to show that under a mild assumption on the moments of $\xi_1, \ldots, \xi_n$, it is possible to substantially reduce the variance term. This assumption is the following.

**Assumption 1** (Bernstein condition on $\xi_1^2, \ldots, \xi_n^2$). Let $K > 0$ and assume that $\xi_1, \ldots, \xi_n$ are independent and satisfy
\[ E[|\xi_i|^p] \leq \frac{1}{2} p! \sigma_i^2 K^{2(p-1)}. \]
(2)

**Example 1.** Centered variables almost surely bounded by $K$ and zero-mean Gaussian random variables with variance smaller than $K^2$ satisfy (2).

**Example 2** (Log-concave random variables). In [7], the authors consider a slightly stronger condition [7, Definition 1.1]. They consider random variables $Z$ satisfying for any integer $p \geq 1$ and some constant $K$:
\[ E[|Z|^p] \leq p K E[|Z|^{p-1}], \]
(3)
and they showed in [7, Section 7] that any distribution that is log-concave satisfies (3). Thus, if $X^2$ is log-concave then our assumption (2) holds. See [1, Section 6] for a comprehensive list of the common log-concave distributions.

The next theorem provides a concentration inequality for quadratic forms of independent random variables satisfying the moment assumption (2). It is sharper than the Hanson-Wright inequality given in Proposition 2.

**Theorem 3.** Assume that the random variable $\xi = (\xi_1, \ldots, \xi_n)^T$ satisfies Assumption 1 for some $K > 0$. Let $A$ be any $n \times n$ real matrix. Then for all $t > 0$,
\[ P \left( \xi^T A \xi - E[\xi^T A \xi] > t \right) \leq \exp \left( -\min \left( \frac{t^2}{192 K^2 \|AD_\sigma\|^2_T}, \frac{t}{256 K^2 \|A\|_2^2} \right) \right) \]
(4)
where $D_\sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$. Furthermore, for any $x > 0$, with probability greater than $1 - \exp(-x)$,
\[ \xi^T A \xi - E[\xi^T A \xi] \leq 256K^2 \|A\|_2 x + 8\sqrt{3}K \|AD_\sigma\|_F \sqrt{x}. \]
(5)

The proof of this result relies on the decoupling inequality for quadratic forms [8] [4, Theorem 8.11].

If $t$ is small, the right hand side of (4) becomes
\[ \exp \left( -\frac{t^2}{192 K^2 \|AD_\sigma\|^2_T} \right). \]
whereas the right hand side of the Hanson-Wright inequality (1) becomes
\[
\exp \left( -c \frac{t^2}{K^4 \|A\|_F^2} \right),
\]
for some absolute constant \( c > 0 \). The element of the diagonal matrix \( D_\sigma \) are bounded from above by \( K \), so Theorem 3 gives a sharper bound than the Hanson-Wright inequality in this regime.

2. Proof of Theorem 3

The goal of this section is to prove Theorem 3. We start with preliminary calculations that will be useful in the proof. Let \( A \) be any \( n \times n \) real matrix. Let \( \lambda > 0 \) satisfy
\[
128 \|A\|_2 K^2 \lambda \leq 1,
\]
and define
\[
\eta = 32 K^2 \lambda^2.
\]
The inequality (6) can be rewritten in terms of \( \eta \):
\[
512 K^2 \|A\|_2^2 \eta \leq 1.
\]
Let \( A_0 \) be the matrix \( A \) with the diagonal entries set to 0. Then, using the triangle inequality with \( A_0 = A - \text{diag}(a_{11}, \ldots, a_{nn}) \) and \( |a_{ii}| \leq \|A\|_2 \) for all \( i = 1, \ldots, n \), we obtain
\[
\|A_0\|_2 \leq 2 \|A\|_2.
\]
Let \( B = A_0^T A_0 = (b_{ij})_{i,j=1,\ldots,n} \) and let \( B_0 \) be the matrix \( B \) with the diagonal entries set to 0. Then
\[
\forall i = 1, \ldots, n, \quad 0 \leq b_{ii} = \sum_{j \neq i} a_{ji}^2 \leq \|A\|_2^2.
\]
By using the decomposition \( B_0 = B - \text{diag}(b_{11}, \ldots, b_{nn}) \) and the inequality \( \|v + v'\|_2 \leq 2\|v\|_2 + 2\|v'\|_2 \), (10) and (9), we have:
\[
\|B_0 \xi\|_2^2 \leq 2\|B \xi\|_2^2 + 2 \sum_{i=1}^n b_{ii} \xi_i^2,
\]
\[
\quad \leq 2\|A_0\|_2^2 \|A_0 \xi\|_2^2 + 2\|A\|_2^2 \sum_{i=1}^n b_{ii} \xi_i^2,
\]
\[
\quad \leq 8\|A\|_2^2 \|A_0 \xi\|_2^2 + 2\|A\|_2^2 \sum_{i=1}^n b_{ii} \xi_i^2.
\]
Combining the previous display with (8), we obtain for any \( K > 0 \):
\[
16 K^2 \eta^2 \|B_0 \xi\|_2^2 \leq (512 K^2 \|A\|_2^2 \eta) \left( \frac{\eta}{4} \|A_0 \xi\|_2^2 + \frac{\eta}{16} \sum_{i=1}^n b_{ii} \xi_i^2 \right),
\]
\[
\quad \leq \frac{\eta}{4} \|A_0 \xi\|_2^2 + \frac{\eta}{16} \sum_{i=1}^n b_{ii} \xi_i^2.
\]
Proof of Theorem 3. Throughout the proof, let $\lambda > 0$ satisfy (6). The value of $\lambda$ will be specified later.

First we treat the diagonal terms by bounding the moment generating function of

$$S_{\text{diag}} := \sum_{i=1}^{n} a_{ii} \xi_i^2 - \sum_{i=1}^{n} s_{ii} \sigma_i^2.$$  

Using the independence of $\xi_1, \ldots, \xi_n$ and (17) with $s = a_{ii} \lambda$ with each $i = 1, \ldots, n$:

$$\mathbb{E} \exp(\lambda S_{\text{diag}}) \leq \exp \left( \lambda^2 \sum_{i=1}^{n} a_{ii}^2 \sigma_i^2 K^2 \right), \quad (12)$$

provided that for all $i = 1, \ldots, n$, $2|a_{ii}|\lambda K^2 \leq 1$ which is satisfied as (6) holds and $|a_{ii}| \leq \|A\|_2$.

Now we bound the moment generating function of the off-diagonal terms. Let

$$S_{\text{off-diag}} := \sum_{i,j=1,\ldots,n: i \neq j} a_{ij} \xi_i \xi_j.$$  

Let the random vector $\xi' = (\xi'_1, \ldots, \xi'_n)^T$ be independent of $\xi$ with the same distribution as $\xi$. We apply the decoupling inequality [8] (see also [4, Theorem 8.11]) to the convex function $s \to \exp(\lambda s)$:

$$\mathbb{E} \exp(\lambda S_{\text{off-diag}}) \leq \mathbb{E} \exp \left( 4\lambda \sum_{i,j=1,\ldots,n: i \neq j} a_{ij} \xi'_i \xi'_j \right).$$

Conditionally on $\xi_1, \ldots, \xi_n$, for each $i = 1, \ldots, n$, we use the independence of $\xi'_1, \ldots, \xi'_n$ and (16) applied to $\xi'_i$ with $s = 4\sum_{j=1,\ldots,n: i \neq j} a_{ij} \xi_j$:

$$\mathbb{E} \exp \left( 4\lambda \sum_{i \neq j} a_{ij} \xi'_i \xi'_j \right) \leq \mathbb{E} \exp \left( 16K^2 \lambda^2 \sum_{i=1,\ldots,n} \left( \sum_{j=1,\ldots,n: i \neq j} a_{ij} \xi_j \right)^2 \right),$$

$$\quad = \mathbb{E} \exp \left( 16K^2 \lambda^2 \|A_0 \xi\|_2^2 \right) = \mathbb{E} \exp \left( \frac{\eta}{2} \|A_0 \xi\|_2^2 \right),$$

where $\eta$ is defined in (7) and $A_0$ is the matrix $A$ with the diagonal entries set to 0. Let $B = A_0^T A_0 = (b_{ij})_{i,j=1,\ldots,n}$. Then $\|A_0 \xi\|_2^2 = \sum_{i=1}^{n} b_i \xi_i^2 + \sum_{i \neq j} b_{ij} \xi_i \xi_j$.

We use the Cauchy-Schwarz inequality to separate the diagonal terms from the off-diagonal ones:

$$\left( \mathbb{E} \exp \left( \frac{\eta}{2} \|A_0 \xi\|_2^2 \right) \right)^2 \leq \mathbb{E} \exp \left( \eta \sum_{i=1}^{n} b_{ii} \xi_i^2 \right) \mathbb{E} \exp \left( \eta \sum_{i \neq j} b_{ij} \xi_i \xi_j \right). \quad (13)$$

For the off-diagonal terms of (13), using the decoupling inequality [8] (see also [4, Theorem 8.11]) we have:

$$\mathbb{E} \exp \left( \eta \sum_{i \neq j} b_{ij} \xi_i \xi_j \right) \leq \mathbb{E} \exp \left( 4\eta \sum_{i \neq j} b_{ij} \xi'_i \xi'_j \right).$$
Again, conditionally on $\xi_1, \ldots, \xi_n$, for each $j = 1, \ldots, n$, we use (16) applied to $\xi'_j$ and the independence of $\xi'_1, \ldots, \xi'_n$:

$$
E \exp \left( 4\eta \sum_{i \neq j} b_{ij} \xi'_i \xi_j \right) \leq E \exp \left( 16K^2 \eta^2 \sum_{i=1}^n \left( \sum_{j=1, \ldots, n; i \neq j} b_{ij} \xi_j \right)^2 \right),
$$

$$
= E \exp \left( 16K^2 \eta^2 \| B_0 \xi \|_2^2 \right),
$$

$$
\leq E \exp \left( \frac{\eta}{4} \| A_0 \xi \|_2^2 + \frac{\eta}{16} \sum_{i=1}^n b_{ii} \xi_i^2 \right),
$$

where we used the preliminary calculation (11) for the last display. Finally, the Cauchy-Schwarz inequality yields

$$
E \exp \left( \frac{4\eta}{3} \sum_{i \neq j} b_{ij} \xi'_i \xi'_j \right) \leq \sqrt{E \exp \left( \frac{\eta}{2} \| A_0 \xi \|_2^2 \right) E \exp \left( \frac{\eta}{8} \sum_{i=1}^n b_{ii} \xi_i^2 \right)}.
$$

We plug this upper bound back into (13). After rearranging, we find

$$
\left( E \exp \left( \frac{\eta}{3} \sum_{i=1}^n b_{ii} \xi_i^2 \right) \right)^{3/2} \leq E \exp \left( \eta \sum_{i=1}^n b_{ii} \xi_i^2 \right) \sqrt{E \exp \left( \frac{\eta}{8} \sum_{i=1}^n b_{ii} \xi_i^2 \right)}.
$$

As $b_{ii} \geq 0$, this implies:

$$
E \exp \left( \frac{\eta}{3} \| A_0 \xi \|_2^2 \right) \leq E \exp \left( \eta \sum_{i=1}^n b_{ii} \xi_i^2 \right).
$$

For each $i = 1, \ldots, n$, we apply (18) to the variable $\xi_i$ with $s = b_{ii} \eta \geq 0$. Using the independence of $\xi_1^2, \ldots, \xi_n^2$, we obtain:

$$
E \exp \left( \eta \sum_{i=1}^n b_{ii} \xi_i^2 \right) = \prod_{i=1}^n E \exp(\eta b_{ii} \xi_i^2),
$$

$$
\leq \exp \left( \frac{3}{2} \eta \sum_{i=1}^n b_{ii} \sigma_i^2 \right) = \exp \left( \frac{3}{2} \eta \| A_0 D \sigma \|_F^2 \right),
$$

provided that for all $i = 1, \ldots, n$, $2K^2 b_{ii} \eta \leq 1$ which is satisfied thanks to (6) and (10).

We remove $\eta$ from the above displays using its definition (7):

$$
E \exp(\lambda S_{\text{off-diag}}) \leq \exp \left( 48\lambda^2 K^2 \| A_0 D \sigma \|_F^2 \right),
$$

where $A_0$ is the matrix $A$ with the diagonal entries set to 0.

Now we combine the bound on the moment generating function of $S_{\text{diag}}$ and $S_{\text{off-diag}}$, given respectively in (12) and (14). Using the Chernoff bound and the Cauchy-Schwarz
inequality: we have that for all $\lambda$ satisfying (6),
\[
P(\mathcal{S}_{\text{diag}} + \mathcal{S}_{\text{off-diag}} > t) \leq \exp(-\lambda t) \mathbb{E}[\exp(\lambda \mathcal{S}_{\text{diag}}) \exp(\lambda \mathcal{S}_{\text{off-diag}})],
\]
\[
\leq \exp(-\lambda t) \sqrt{\mathbb{E}[\exp(2\lambda \mathcal{S}_{\text{diag}})]} \sqrt{\mathbb{E}[\exp(2\lambda \mathcal{S}_{\text{off-diag}})]},
\]
\[
\leq \exp \left( -\lambda t + \lambda^2 K^2 \left( \sum_{i=1}^{n} \sigma_i^2 a_{ii}^2 + 48 \|A_0 D\sigma\|_F^2 \right) \right),
\]
\[
\leq \exp \left( -\lambda t + 48 \lambda^2 K^2 \|AD\sigma\|_F^2 \right), \tag{15}
\]
where for the last display we used the equality
\[
\|AD\sigma\|_F^2 = \sum_{i,j=1,\ldots,n} a_{ij}^2 \sigma_i^2 = \|A_0 D\sigma\|_F^2 + \sum_{i=1}^{n} a_{ii}^2 \sigma_i^2.
\]
It now remains to choose the parameter $\lambda$. The unconstrained minimum of (15) is attained at $\bar{\lambda} = t/(96 K^2 \|AD\sigma\|_F^2)$. If $\bar{\lambda}$ satisfies the constraint (6), then
\[
P(\mathcal{S}_{\text{diag}} + \mathcal{S}_{\text{off-diag}} > t) \leq \exp(-\frac{t^2}{192 K^2 \|AD\sigma\|_F^2}).
\]
On the other hand, if $\bar{\lambda}$ does not satisfy (6), then the constraint (6) is binding and the minimum of (15) is attained at $\lambda_b = 1/(128 \|A\|_2 K^2) < \bar{\lambda}$. In this case,
\[
-t \lambda_b + \frac{t^2}{2} 48 K^2 \|AD\sigma\|_F^2 \leq -t \lambda_b + \lambda_b \lambda 48 K^2 \|AD\sigma\|_F^2 = -t \lambda_b + \frac{t}{2} \lambda_b = -\frac{t}{256 K^2 \|A\|_2}.
\]
Combining the two regimes, we obtain
\[
P(\mathcal{S}_{\text{diag}} + \mathcal{S}_{\text{off-diag}} > t) \leq \exp \left( -\min \left( \frac{t^2}{192 K^2 \|AD\sigma\|_F^2}, \frac{t}{256 K^2 \|A\|_2} \right) \right).
\]
The proof of (4) is complete.

Now we prove (5). The function
\[
t \to x(t) = \min \left( \frac{t^2}{192 K^2 \|AD\sigma\|_F^2}, \frac{t}{256 K^2 \|A\|_2} \right)
\]
is increasing and bijective from the set of positive real numbers to itself. Furthermore, for all $t > 0$,
\[
t \leq 8 \sqrt{3} K \|AD\sigma\|_F \sqrt{x(t)} + 256 K^2 \|A\|_2 x(t),
\]
so the variable change $x = x(t)$ completes the proof of (5).

3. Technical lemmas: bounds on moment generating functions

The condition (2) leads to the following bounds on the moment generating functions of $X$ and $X^2$, which are crucial to prove Theorem 3.
Proposition 4. Let $K > 0$ and let $\xi_i$ be a random variable satisfying (2) with $\sigma_i^2 = E[\xi_i^2]$. Then for all $s \in \mathbb{R}$:

$$E \exp(s\xi_i) \leq \exp(s^2 K^2).$$

(16)

Furthermore, if $0 \leq 2sK^2 \leq 1$, then

$$E \exp(s\xi_i^2 - s\sigma_i^2) \leq \exp(s^2 \sigma_i^2 K^2),$$

(17)

and

$$E \exp(s\xi_i^2) \leq \exp\left(\frac{3}{2} s\sigma_i^2\right).$$

(18)

Inequality (16) shows that a random variable $X$ satisfying the moment assumption (2) is subgaussian and its $\psi_2$ norm is bounded by $K$ up to a multiplicative absolute constant. The proof of Proposition 4 is based on Taylor expansions and some algebra.

Proof of Proposition 4. To simplify the notation, let $X = \xi_i$ and $\sigma = \sigma_i$. We first prove (17). We apply the assumption on the even moments of $X$:

$$E \exp(sX^2) = 1 + s\sigma^2 + \sum_{p \geq 2} \frac{s^p E X^{2p}}{p!},$$

$$\leq 1 + s\sigma^2 + \frac{\sigma^2 s}{2} \sum_{k=1}^{\infty} (sK^2)^k = 1 + s\sigma^2 + \frac{\sigma^2 K^2 s^2}{2(1 - sK^2)},$$

and using the inequality $0 < 2sK^2 \leq 1$, we obtain:

$$E \exp(sX^2) \leq 1 + s\sigma^2 + \sigma^2 K^2 \leq \exp(s\sigma^2 + s^2 \sigma^2 K^2),$$

which completes the proof of (17). Inequality (18) is a direct consequence of (17) after applying again the inequality $2sK^2 \leq 1$.

We now prove (16). Using the Cauchy-Schwarz inequality and the assumption on the moments for $p = 2$, we get $\sigma^4 \leq E[\xi_i^4] \leq \sigma^2 K^2$, so $\sigma \leq K$. Let $p \geq 1$. For the even terms of the expansion of $E \exp(sX)$, we get:

$$\frac{s^{2p} E X^{2p}}{(2p)!} \leq \frac{1}{2} (sK)^{2p} \frac{p!}{(2p)!} \leq \frac{1}{2} (sK)^{2p} \frac{p!}{p!},$$

where for the last inequality we used $(p!)^2 \leq (2p)!$. For the odd terms, by using the Jensen inequality for $p \geq 1$:

$$\frac{s^{2p+1} E X^{2p+1}}{(2p + 1)!} \leq \frac{s^{2p+1} (E X^{2p+2})^{\frac{2p+1}{2p+2}}}{(2p + 1)!} \leq |sK|^{2p+1} \frac{(p+1)!}{(2p + 1)!},$$

$$\leq \frac{1}{2} |sK|^{2p+1} \frac{(p+1)!}{(2p + 1)!}. $$

If $|sK| > 1$, we use the inequality $(p+1)!^2 \leq (2p)!$ to obtain

$$\frac{s^{2p+1} E X^{2p+1}}{(2p + 1)!} \leq \frac{|sK|^{2(p+1)}}{2((p+1)!)}.$$
and by combining the inequality for the even and the odd terms:

$$
\mathbb{E}\exp(sX) = 1 + \sum_{p \geq 1} \frac{s^{2p}\mathbb{E}X^{2p}}{(2p)!} + \frac{s^{2p+1}\mathbb{E}X^{2p+1}}{(2p + 1)!},
$$

$$
\leq 1 + \frac{1}{2} \sum_{p \geq 1} \frac{(sK)^{2p}}{p!} + \frac{|sK|^{2(p+1)}}{(p + 1)!},
$$

$$
\leq 1 + \sum_{p \geq 1} \frac{(sK)^{2p}}{p!} = \exp(s^2 K^2).
$$

If $|sK| \leq 1$, we use the inequality $(p + 1)!p! \leq (2p + 1)!$ to obtain

$$
\frac{s^{2p+1}\mathbb{E}X^{2p+1}}{(2p + 1)!} \leq \frac{(sK)^{2p}}{2(p!)},
$$

and by combining the inequality for the even and the odd terms:

$$
\mathbb{E}\exp(sX) = 1 + \sum_{p \geq 1} \frac{s^{2p}\mathbb{E}X^{2p}}{(2p)!} + \frac{s^{2p+1}\mathbb{E}X^{2p+1}}{(2p + 1)!},
$$

$$
\leq 1 + \frac{1}{2} \sum_{p \geq 1} \frac{(sK)^{2p}}{p!} + \frac{(sK)^{2p}}{p!} = 1 + \sum_{p \geq 1} \frac{(sK)^{2p}}{p!} = \exp(s^2 K^2).
$$

References