Cheeger sets and Optimal Packings

a Γ-convergence approach

Beniamin BOGOSEL, Dorin BUCUR, Ilaria FRAGALA
beniamin.bogosel@cmap.polytechnique.fr

Abstract

This work proposes a new approach to solve some classical optimal packing problems for balls. These problems, which are discrete in nature, are tackled using shape optimization techniques for suitable Γ-converging sequences associated to Cheeger type problems. Different optimal packing problems are found as limit of sequences of optimal clusters associated to the minimization of energies involving suitable (generalized) Cheeger constants. These theoretical results are then used in order to approximate numerically generalized Cheeger constants, their optimal clusters and, as a consequence of the asymptotic result, optimal packings. Numerical experiments are carried out in 2D and 3D. The continuous shape optimization approach gives good approximation of the optimal packings searched, circumventing all the combinatorial difficulties of the associated discrete problem.

1. Description of the Problem

Given a domain $\subset \mathbb{R}^N$, the objective is to obtain a variational approach in order to find solutions to the optimal packing problem

$$\max\{r : D_1, ..., D_n \subset D, D_i \cap D_j = \emptyset\}$$

where $D_1, ..., D_n$ are balls of radius r.

One disk: the inradius. The isoperimetric inequality implies that the disk minimizes the **isoperimetric ratio** $I(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)/|\Omega|^{(N-1)/N}$. The problem is that **any disk** minimizes $I(\Omega)$ so considering

$$\min_{\Omega\subset D} I(\Omega)$$

does not give us the inscribed disk. If, however, one considers the *modified isoperimetric ratio* $I_\alpha(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)/|\Omega|^\alpha$ for $\alpha > (N-1)/N$, then the problem

$$h_\alpha(E) := \min_{\Omega\subset E} I_\alpha(\Omega)$$

admits a solution Ω_α. Furthermore, as $\alpha \to (N-1)/N$ the solutions Ω_α converge to a disk of maximal radius contained in D.

The constant $h_\alpha(D)$ is called the α-Cheeger constant. The classical Cheeger constant is found for $\alpha = 1$.

Many disks: Cheeger clusters. When dealing with multiple phases one may consider the following optimal partitioning problems

$$\min\{\max_{i=1,...,n} h_\alpha(E_i) : (E_1, ..., E_n) \in \mathcal{P}_n(D)\} \quad \text{(M)}$$

and

$$\min\{\sum_{i=1,...,n} h_\alpha(E_i) : (E_1, ..., E_n) \in \mathcal{P}_n(D)\} \quad \text{(S)}$$

where $\mathcal{P}_n(D)$ denotes the family of partitions of D into n pairwise disjoint subsets.

Theorem: Limiting behavior

1. Problem (M) admits a solution and as $\alpha \to \left(\frac{N-1}{N}\right)_+$ it converges in L^1 to a family of balls solving: $\max\{r : r_i \geq r, B_i = B(x_i, r_i) \subset D, B_i \cap B_j = \emptyset\}$
2. Problem (S) admits a solution and as $\alpha \to \left(\frac{N-1}{N}\right)_+$ it converges in L^1 to a family of balls solving: $\max\{\prod_{i=1}^n r_i : B_i = B(x_i, r_i) \subset D, B_i \cap B_j = \emptyset\}$

Idea of the proof: Use the equality

$$\frac{\mathcal{H}^{N-1}(\partial\Omega_i)}{|\Omega_i|^\alpha} = I(\Omega_i)\frac{1}{|\Omega_i|^{\alpha-(N-1)/N}}$$

which by the isoperimetric inequality favors ball-shaped sets with large volumes.

The case $\alpha \to +\infty$. In this case, up to a subsequence, solutions of problem (M) converge in $L^1(D, \mathbb{R}^k)$ to a partition of D into k mutually disjoint sets of equal measure and this partition minimizes the product of perimeters. In the case of problem (S) it can be shown that solutions converge in L^1 to a partition of D into k sets of equal volume, but we have no conjecture by the problem solved in the limit.

2. Phase-field approximation

In [1] a phase-field approach is used to approximate numerically partitions minimizing the sum of perimeters. This approach is based on a Modica-Mortola approximation of the perimeter by Γ-convergence.

The approach presented below is of the same nature. The perimeter term in the numerator of the α-Cheeger ratio is approximated by the Modica-Mortola theorem. The measure term in the denominator is approximated by an L^q norm of the density function for q sufficiently high. It turns out that the right choice is $q = 2N/(N-1)$.

The non-overlapping condition can be introduced by adding an inequality constraint on the sum of the densities u_i or by considering a penalization term (more convenient for numerical purposes). The Γ-convergence result is stated below.

Theorem: Γ-convergence approximation

Let D be a bounded, open, Lipschitz domain in $\mathbb{R}^N$. For any fixed $\alpha > \frac{N-1}{N}$ and $p > 1$ the sequence of functionals defined on $L^1(D, \mathbb{R}^k)$ by

$$F_{p,\varepsilon}(u_1, ..., u_k) = \sum_{i=1}^k \left(\frac{\varepsilon\int_D |\nabla u_i|^2 dx + \frac{9}{\varepsilon}\int_D u_i^2(1-u_i)^2 dx}{\left(\int_D |u_i|^{\frac{2N}{N-1}} dx\right)^\alpha}\right)^p + \frac{1}{\varepsilon}\sum_{1\le i<j\le k}\int_D u_i^2 u_j^2 \quad \text{(GC)}$$

if $u_i \in H_0^1(D),\ 0 \le u_i \le 1$ and $+\infty$ if not, Γ-converges as $\varepsilon \to 0$ in $L^1(D, \mathbb{R}^k)$ to the functional

$$F_p(\Omega_1, ..., \Omega_k) = \sum_{i=1}^k \left(\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha}\right)^p.$$

Moreover, if for $p > 1$ we denote $(\Omega_1^p, ..., \Omega_k^p)$ a minimizer of F_p in $\mathcal{P}_k(D)$ then for $p \to +\infty$, up to a subsequence we have

$$(\Omega_1^p, ..., \Omega_k^p) \overset{L^1(D,\mathbb{R}^k)}{\longrightarrow} (\Omega_1, ..., \Omega_k),$$

where $(\Omega_1, ..., \Omega_k)$ is a solution of (M).

3. Numerical Results

The Γ-convergence result shown above is used in order to obtain numerical approximation of α-Cheeger sets and α-Cheeger clusters. The functions u_i are discretized using a finite difference grid defined on the unit square. The gradients are approximated by finite differences and the integrals are computed using a basic quadrature method. The parameter ε is chosen in relation to the step size h of the grid. In general the choice $\varepsilon \in [h, 4h]$ gives good results.

Random densities are chosen as initialization so that local minima are successfully avoided. In order to accelerate the computations, multiple grid resolutions are considered: an initial solution is computed on a coarse grid, which is then interpolated on a finer grid and the process is continued. When domains D with different shapes are considered, the finite difference grid points in the exterior of D are simply discarded in the computations.

We start by computing some α-Cheeger sets for various domains D by minimizing (GC) for $k = 1$, $p = 1$.

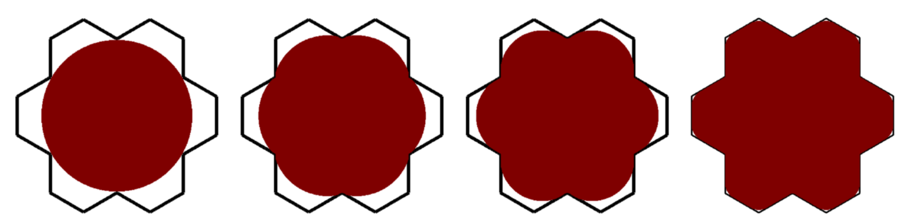

Figure 1: *The α-Cheeger set for a non-convex set in $2D$, for $\alpha \in \{0.5001, 0.75, 1, 2\}$.*

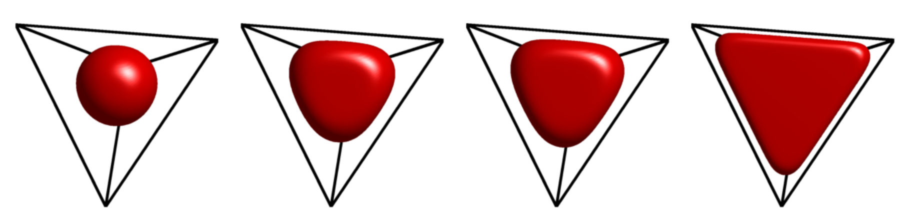

Figure 2: *The α-Cheeger set for a regular tetrahedron in $3D$, for $\alpha \in \{0.667, 0.9, 1, 2\}$.*

In the case of 2D polygons Kawohl and Lachand-Robert [2] gave an explicit characterization for the Cheeger sets. Below you can see a comparison of the Cheeger sets computed by minimizing (GC) directly and the exact solutions obtained using [2]. Relative errors for the Cheeger constants are also given.

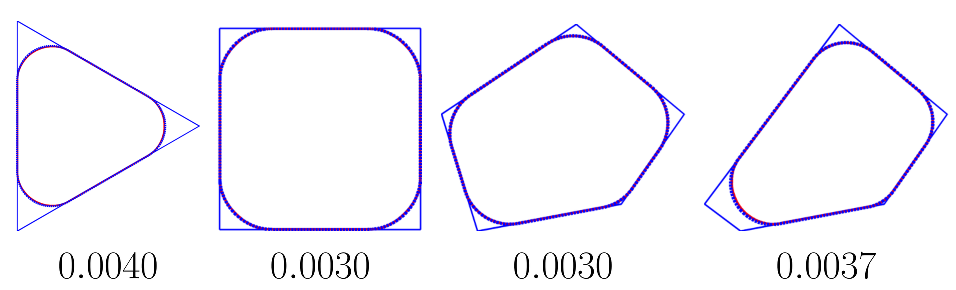

Figure 3: *Comparison between results obtained when minimizing* (GC) *(red) and the Kawohl & Lachand-Robert formula (dotted-blue).*

Next we compute some numerical approximations of Cheeger clusters, corresponding to the classical case $\alpha = 1$, $p = 1$ in the case of the square. It can be observed that the optimal partition does not seem to be made of convex sets.

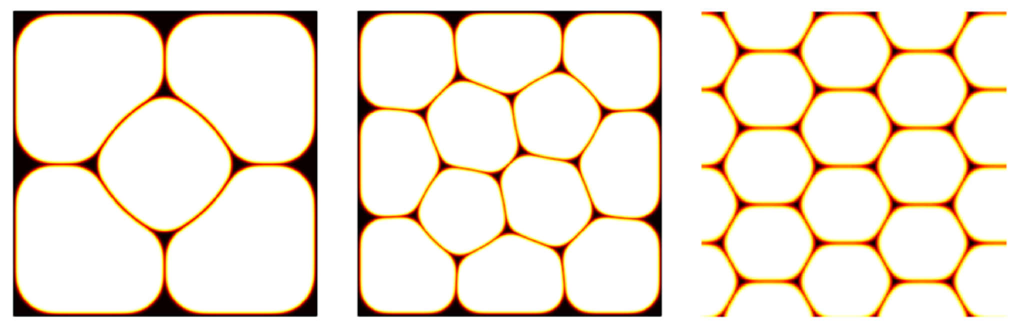

Figure 4: *Cheeger clusters in a square: 5 cells, 12 cells, 16 cells (periodical).*

Now we arrive to the main objective of our approach: the computation of optimal circle packings. In order to approximate solutions to problem (M) we use again (GC) with $\alpha = (N-1)/N + \delta$ (with $\delta > 0$) and p "large" ($p \le 100$ in our computations). Various two dimensional results are shown below in the case of the square, the disk and the equilateral triangle. A very basic refinement procedure is used starting from the densities obtained when minimizing (GC), which allows us to compare the results with the best known packings (see `http://www.packomania.com/`). The Γ-convergence approximation behaves well even in cases where the optimal packing is not unique.

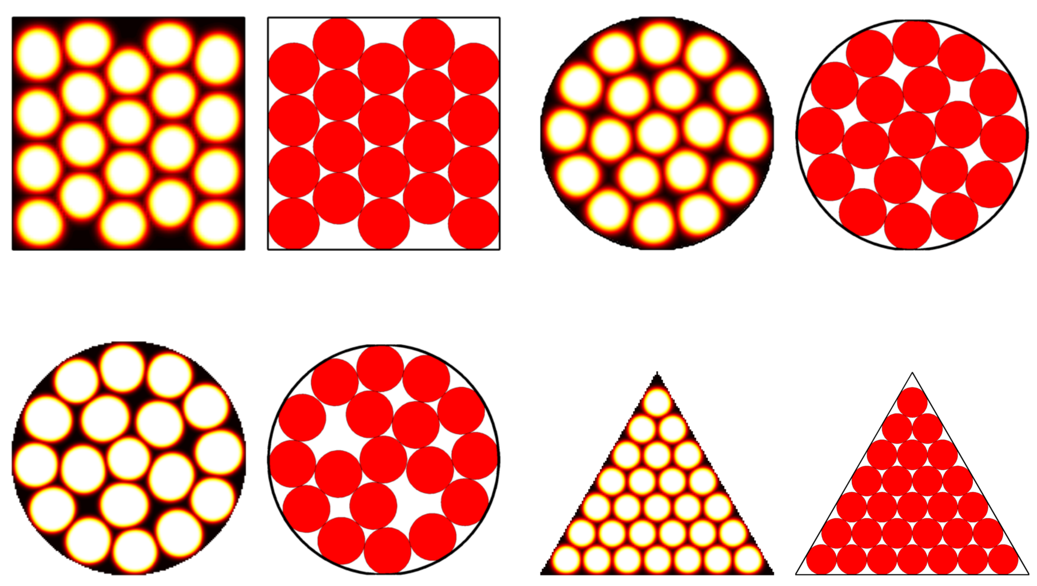

Figure 5: *Circle packing examples in 2D: density representation and local optimization.*

Some examples of computations of optimal spherical packings for domains in $\mathbb{R}^3$ are presented below. In this case, we observe again a good convergence to the best known configurations.

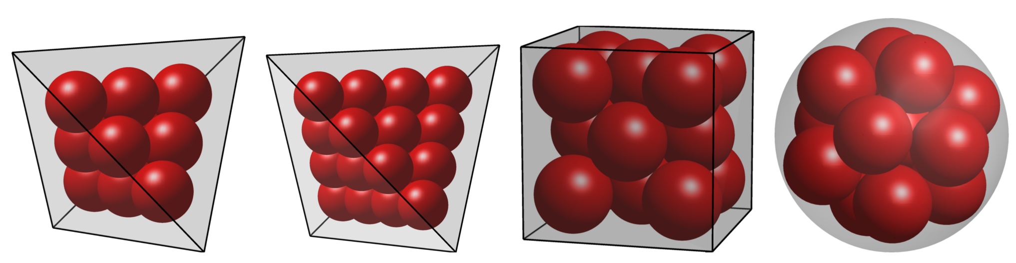

Figure 6: *Sphere packing examples in 3D.*

4. Conclusion

The Γ-convergence approach presented here allows us to approximate solutions to a complex combinatorial problem using a direct optimization algorithm. The optimal circle and sphere packings obtained are in accordance with best known configurations.

References

[1] Oudet, Édouard, Approximation of partitions of least perimeter by Γ-convergence: around Kelvin's conjecture, Exp. Math 2011.

[2] B. Kawohl and T. Lachand-Robert, *Characterization of Cheeger sets for convex subsets of the plane*, Pacific J. Math. **225** (2006)

[3] B. Bogosel, D. Bucur, I. Fragalà Phase field approach to optimal packing problems and related Cheeger clusters, Appl. Math. Optim, 2018.