A Geometric proof for the Polygonal Isoperimetric Inequality

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Abstract

Gradients of the perimeter and area of a polygon have straightforward geometric interpretations. The use of optimality conditions for constrained problems and basic ideas in triangle geometry show that polygons with prescribed area minimizing the perimeter must be regular.

Given a closed curve γ in the plane bounding a region ω having a given area A_0, what is the shortest possible value for the length L of γ? This problem is called the *isoperimetric inequality* and has been widely studied since the antiquity. The solution to the isoperimetric problem is, without surprise, the circle. Historical details and various proof ideas are shown in [1]. For a positive integer $n \geq 3$, the same question can be formulated, restricting to the family of n-gons. Given $A_0 > 0$, the polygonal isoperimetric inequality can be formulated as follows:

$$\text{Which } n\text{-gon } P \text{ having prescribed area } A(P) = A_0 \text{ has the smallest perimeter } L(P)? \tag{1}$$

Drawing a parallel between the continuous and the discrete case, it is natural to expect that the solution to problem (1) is the regular n-gon.

Since any n-gon depends on $2n$ real variables, the coordinates of the vertices, (1) is in fact a constrained minimization problem of the form

$$\min_{g(x)=0} f(x), \tag{2}$$

where $f, g : \mathbb{R}^{2n} \to \mathbb{R}$. Classical theory of optimality conditions, sometimes called the Lagrange multipliers method, for this kind of problems states the following:

$$\text{If } x^* \text{ solves (2) and } \nabla g(x^*) \neq 0, \text{ then there exists } \lambda \in \mathbb{R} \text{ such that } \nabla f(x^*) = \lambda \nabla g(x^*). \tag{3}$$

See [2, Section 5.5.3] for more details. In [1] an algebraic proof based on (3) shows that the regular n-gon is the only possible critical point for (1).

The purpose of this note is to show the geometric interpretation of (3) when applied to the solution of the polygonal isoperimetric problem. A geometric description of the gradients of the area and perimeter of an n-gon is provided. Then, the optimality condition (3) combined with well known facts from triangle geometry give a straightforward argument showing that the regular n-gon is the only solution to problem (1).

1 Geometric description of the gradients

Denote by $\mathbf{a}_0, ..., \mathbf{a}_{n-1}$ the vertices of a simple polygon P oriented in a counter-clockwise direction. For simplicity we assume that the indices are taken modulo n and that the edges may intersect

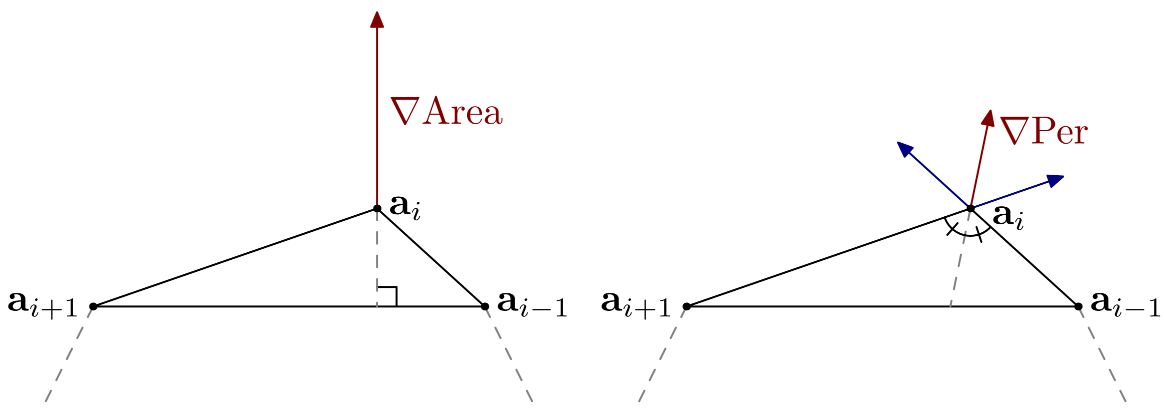

Figure 1: Geometric interpretation for the gradient of the area and perimeter of a polygon.

only at their end points. The perimeter and the area of an n-gon vary smoothly, as long as two vertices do not coincide. This can readily be seen by writing the analytical expressions of the area and perimeter in terms of the vertex coordinates $\mathbf{a}_i(x_i, y_i)$, $i = 0, ..., n-1$:

$$A(P) = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1} - x_{i+1} y_i), \quad L(P) = \sum_{i=0}^{n-1}\sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}.$$

Instead of computing the gradients directly, like in [1], let us give their geometric interpretation.

Start by observing that when moving the vertex $\mathbf{a}_i$ the variation of the perimeter and area are the same for the polygon P and for the triangle $\Delta\mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$. Therefore, to compute the gradients of the perimeter and area with respect to coordinates of $\mathbf{a}_i$ we can work on $\Delta\mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$.

Gradient of the Area. Choose a coordinate system such that $\mathbf{a}_{i-1}, \mathbf{a}_{i+1}$ are on the x-axis and $\mathbf{a}_i$ has positive y-coordinate. Then $A(\Delta\mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}) = \frac{1}{2}|\mathbf{a}_{i-1}\mathbf{a}_{i+1}|y_i$. Therefore, the gradient of the area of P with respect to the coordinates of $\mathbf{a}_i$ is the vector $\vec{v}_i = (0, \frac{1}{2}|\mathbf{a}_{i-1}\mathbf{a}_{i+1}|)$ and it has the following geometric properties:

(a) $\vec{v}_i$ is orthogonal to the segment $\mathbf{a}_{i-1}\mathbf{a}_{i+1}$ and points towards the exterior of P.

(b) $\vec{v}_i$ has length equal to $\frac{1}{2}|\mathbf{a}_{i-1}\mathbf{a}_{i+1}|$.

Thus, we have the more precise description given by:

$$\vec{v}_i \text{ equals } \frac{1}{2}\overrightarrow{\mathbf{a}_{i-1}\mathbf{a}_{i+1}} \text{ rotated in the clockwise direction with angle } \pi/2. \tag{4}$$

An illustration is given in Figure 1.

Gradient of the Perimeter. The gradient of the distance to a point is simply the unit vector directed from this point. Therefore the gradient of the perimeter of P with respect to $\mathbf{a}_i$ is the sum of the gradients of $|\mathbf{a}_{i-1}\mathbf{a}_i|, |\mathbf{a}_{i+1}\mathbf{a}_i|$ with respect to $\mathbf{a}_i$ given by

$$\vec{w}_i = \frac{1}{|\mathbf{a}_{i-1}\mathbf{a}_i|}\overrightarrow{\mathbf{a}_{i-1}\mathbf{a}_i} + \frac{1}{|\mathbf{a}_{i+1}\mathbf{a}_i|}\overrightarrow{\mathbf{a}_{i+1}\mathbf{a}_i}. \tag{5}$$

Since the sum of two vectors having the same magnitude is aligned with the bisector of the angle made by the two vectors we have the following:

(a) $\vec{w}_i$ is aligned with the bisector of the angle $\angle\mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$.

(b) $\vec{w}_i$ points towards the exterior when $\theta_i = \angle\mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1} < \pi$ and towards the interior if $\theta_i > \pi$. Moreover, $\vec{w}_i = \vec{0}$ if and only if $\mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}$ are aligned in this order.

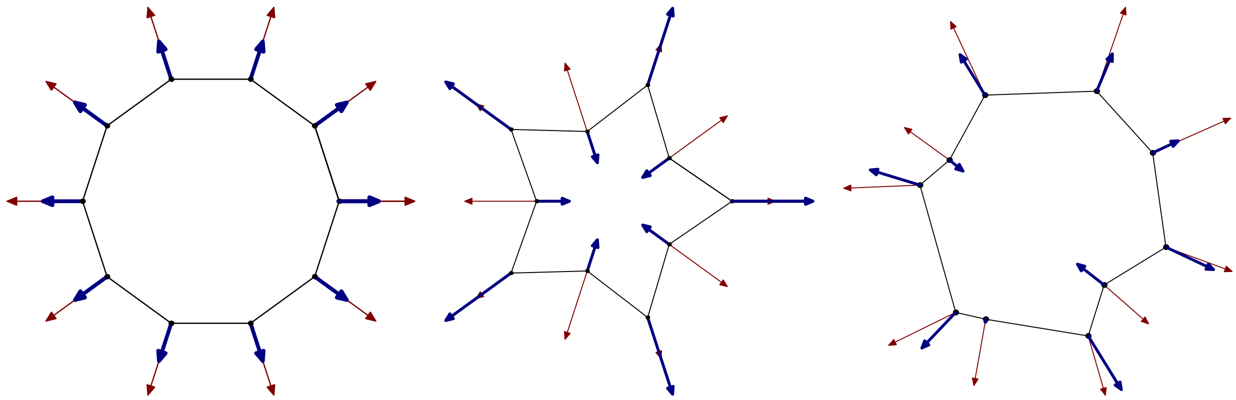

Figure 2: Gradients of the perimeter and area for regular, star-shaped and arbitrary n-gons. Bold arrows represent the gradient of the perimeter.

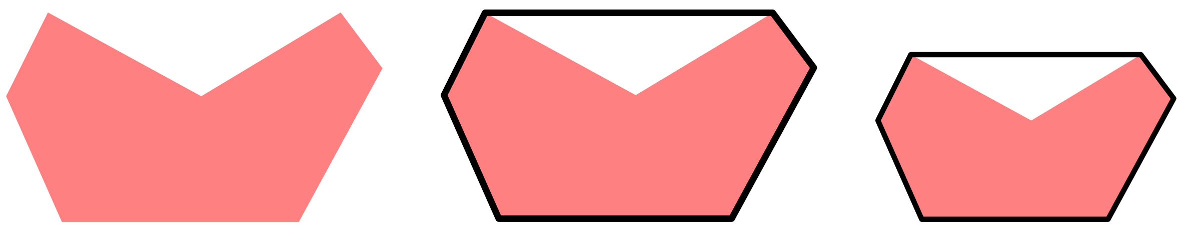

Figure 3: Replacing a non-convex polygon (left) with its convex hull (center) decreases the perimeter and increases the area. Rescaling to have the original area decreases the perimeter further (right).

(c) $\vec{w}_i$ has length equal to $2\cos\frac{\theta_i}{2}$.

An illustrative example is shown in Figure 1.

To further illustrate the geometry of the gradients, in Figure 2 three situations are plotted, an arbitrary n-gon, a star-shaped n-gon and the regular one. For each vertex, the corresponding components of the gradient are plotted. Bold arrows represent the gradient of the perimeter. It can be observed that in the regular case, gradients are colinear with the same factor of proportionality for each vertex. In the next section we show that this only occurs when the n-gon is regular. For a star-shaped polygon, the gradients have the same direction at a fixed vertex, but the vectors do not scale with the same factor. For arbitrary n-gons, the gradients may not even have the same directions for a given vertex.

2 Any optimal polygon must be regular

We are now ready to characterize solutions of (1) via optimality conditions. The proof of existence of an optimal n-gon is straightforward and recalled in [1]. Furthermore, taking the convex hull of a non-convex polygon decreases its perimeter and increases its area. Shrinking it to preserve the area constraint further decreases the perimeter (see Figure 3). Thus a non-convex n-gon is is never optimal for (1).

Consider P^* a convex n-gon which is optimal in (1). Without loss of generality, suppose the vertices are pairwise distinct. If two vertices would coincide, simply replace one of them with the midpoint of one of the edges of P^*. Thus, for each i we have $|\mathbf{a}_{i-1}\mathbf{a}_{i+1}| \neq 0$, the gradients of the area given in (4) verify $\vec{v}_i \neq 0$ for every $i = 0, ..., n-1$, which implies $\nabla A(P^*) \neq 0$. The characterization of the optimality conditions given in (3) applies: there exists $\lambda \in \mathbb{R}$ such that

$$\nabla L(P^*) = \lambda \nabla A(P^*).$$

In other words, for $\vec{v}_i$ and $\vec{w}_i$ defined by (4) and (5), respectively, we have $\vec{w}_i = \lambda \vec{v}_i$.

There are at least three consecutive vertices $\mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}$, which are not colinear. Otherwise, all vertices of P^* would lie on the same line and its area would be zero. Therefore $\vec{w}_i \neq 0$, which implies $\lambda \neq 0$. Furthermore, since $\vec{v}_i \neq 0$ for all $i = 0, ..., n-1$ and $\lambda \neq 0$ we also have $\vec{w}_i \neq 0$ for all $i = 0, ..., n-1$.

Let us now use the three implications of a the vectorial equalities $\vec{w}_i = \lambda \vec{v}_i$.

(a) **Two equal vectors have the same direction.** For each $i = 0, ..., n-1$ vectors $\vec{w}_i$ and $\vec{v}_i$ are non-zero and collinear. Since the first one is aligned with the bisector of $\angle \mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$ and the second is aligned with the height from $\mathbf{a}_i$ in $\Delta \mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$, it follows that this triangle is isosceles with respect to vertex $\mathbf{a}_i$. Therefore for every $i = 0, ..., n-1$ we have $|\mathbf{a}_{i-1}\mathbf{a}_i| = |\mathbf{a}_i\mathbf{a}_{i-1}|$, which means that all edges of P^* have the same length ℓ.

(b) **Two equal vectors have the same orientation.** First, let us notice that $\lambda > 0$. Indeed, the sum of the interior angles of an n-gon is $(n-2)\pi$, which shows that there is at least one vertex corresponding to an inner angle smaller than π. For this vertex, both the gradient for the area $\vec{v}_i$ and the gradient of the perimeter $\vec{w}_i$ point towards the exterior of the polygon. Therefore $\lambda > 0$.

Since $\vec{v}_i$ points always towards the exterior of the polygon it follows that $\vec{w}_i$ will always point towards the exterior. Therefore an optimal polygon P^* does not have angles greater than π. This fact was already established, since P^* is convex.

(c) **Two equal vectors have the same length.** In the isosceles triangle $\Delta \mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$ we have $|\mathbf{a}_{i-1}\mathbf{a}_{i+1}| = 2\ell \sin \frac{\theta_i}{2}$ where $\theta_i = \angle \mathbf{a}_{i-1}\mathbf{a}_i\mathbf{a}_{i+1}$ and ℓ is the length of the edges of the polygon. Recalling that the length of $\vec{w}_i$ is $2\cos\frac{\theta_i}{2}$ we find that

$$\frac{1}{2}\ell \sin \frac{\theta_i}{2} = 2\lambda \cos \frac{\theta_i}{2},$$

for every $i = 0, ..., n-1$. Therefore $\tan \frac{\theta_i}{2} = \frac{4\lambda}{\ell}$ showing that all angles of P^* are equal.

In conclusion P^* has equal edge lengths and equal angles, therefore P^* is the regular n-gon. Viewing gradients geometrically, rather than algebraically, turns the optimality condition (3) verified by an optimal polygon in (1) into a series of elementary observations showing that the optimal n-gon must be the regular one.

References

[1] V. Blasjö. The isoperimetric problem. *Amer. Math. Monthly*, 112(6):526–566, 2005.

[2] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, 2004.

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