Spectral optimization on variable domains

Beniamin Bogosel

Laboratoire Jean Kuntzmann, Grenoble

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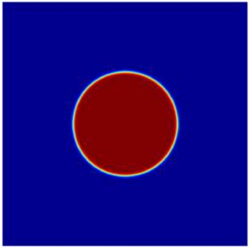
Shape optimization

$$\min_{\Omega \in \mathcal{A}} J(\Omega)$$

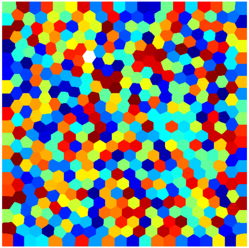
Questions?

- existence of optimal shapes
- regularity
- find the optimal shape explicitly
- qualitative results
- numerical computations

Examples

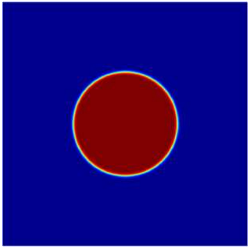


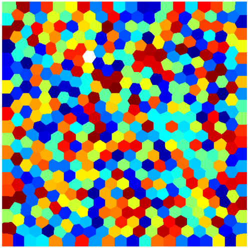
isoperimetric problem



optimal partitioning

Examples





isoperimetric problem   optimal partitioning

- optimal design of structures (bridges, cranes)
- modelization, population segregation

Spectral optimization

Lord Rayleigh - *The Theory of Sound* (1877)

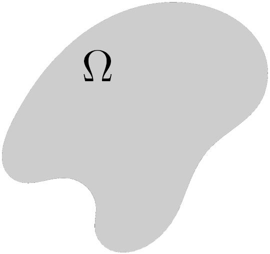
The drum

The disk minimises area at fixed frequency

Faber-Krahn (1920-1923)

The disk minimizes $\lambda_1(\Omega)$ at fixed area

Eigenvalues of the Laplacian



$$\begin{cases} -\Delta u & = \lambda u \\ u & \in H_0^1(\Omega) \end{cases}$$

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \to +\infty$$

Rayleigh quotients :

$$\lambda_k(\Omega) = \min_{S_k \subset H_0^1(\Omega)} \max_{\phi \in S_k \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 dx}{\int_\Omega \phi^2 dx}$$

Scaling : $\lambda_k(t\Omega) = \frac{1}{t^2}\lambda_k(\Omega)$.
Monotonicity : $\Omega_1 \subset \Omega_2 \Rightarrow \lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$

Optimization

- Constraints are necessary.

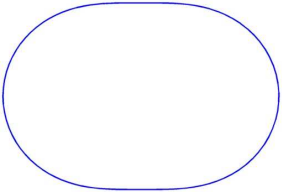
Volume constraint

- explicit optimal shapes:
  - $k = 1$ - the ball (Faber-Krahn)
  - $k = 2$ - two identical balls (Polya-Szego)
- existence of optimal shapes (quasi-open sets)
  (Bucur, Mazzolleni - Pratelli 2012)
- numerical computations
  (Oudet 2004 and Antunes, Freitas 2012)

Perimeter constraint

- explicit optimal shape for $k = 1$ - the ball
- qualitative results for $k = 2$ in 2D

  (Bucur, Buttazzo, Henrot 2009)



- The optimal shape Ω^* for λ_2 exists
- Ω^* is C^∞

- existence results (De Philippis, Velichkov 2012)
  - the optimal shape Ω^* exists
  - $\partial\Omega^*$ is of class $C^{1,\alpha}$, $\alpha \in (0,1)$
  - the weak curvature of Ω^* is positive

What happens for $k \geq 3$?

- no explicit optimal shapes are known
- do optimal shapes have multiple k-th eigenvalue?

Results

- numerical simulations
- optimality condition in the non-differentiable case of multiple eigenvalues
- the optimal shapes are C^∞

Constraint-free formulation

$$\min_{\text{Per}(\Omega)=c} \lambda_k(\Omega)$$

is equivalent up to an homothety to

$$\min_{\Omega\subset\mathbb{R}^d} \left(\lambda_k(\Omega) + \text{Per}(\Omega)\right).$$

- in 2D optimal shapes are convex
- the optimal shape for $k = 1$ is a ball

Numerical simulations

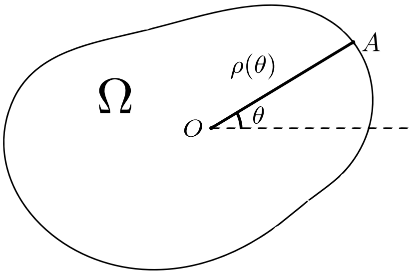
Challenges

- choose the right discretization
- compute the cost functional in an efficient way
- shape derivative formulas - gradient descent

Numerical simulations

radial parametrization

$$\Omega \longrightarrow \rho : [0, 2\pi) \to \mathbb{R}_+.$$



Fourier coefficients

$$\rho(\theta) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\theta) + b_k \sin(k\theta)).$$

- a_n, b_n decrease rapidly with n

$$\rho(\theta) \approx \rho_N(\theta) = a_0 + \sum_{k=1}^{N} (a_k \cos(k\theta) + b_k \sin(k\theta))$$

Gradient computation

$$\lambda_k(\Omega) \approx \lambda_k(a_0, a_1, ..., a_n, b_1, ..., b_n).$$

For $-\Delta u_n = \lambda_n(\Omega) u_n,\ \ u_n \in H_0^1(\Omega)$

$$\frac{\partial \lambda_n}{\partial a_k} = -\int_0^{2\pi} \rho(\theta) \cos(k\theta) \left(\frac{\partial u_n}{\partial n}(\rho(\theta), \theta) \right)^2 d\theta$$

$$\frac{\partial \lambda_n}{\partial b_k} = -\int_0^{2\pi} \rho(\theta) \sin(k\theta) \left(\frac{\partial u_n}{\partial n}(\rho(\theta), \theta) \right)^2 d\theta$$

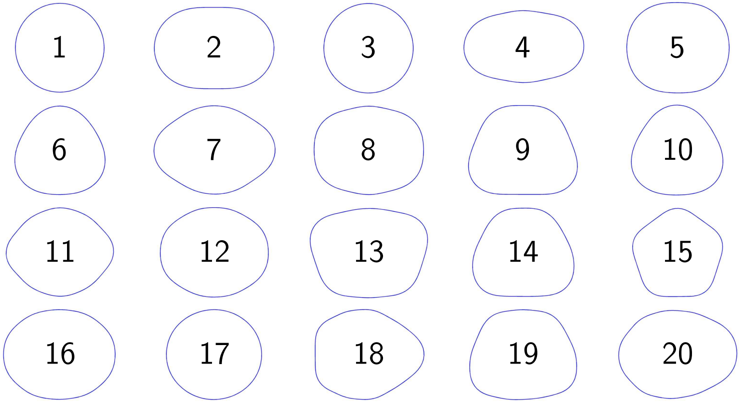
Optimization

$\longrightarrow$ quasi-Newton algorithm (LBFGS)

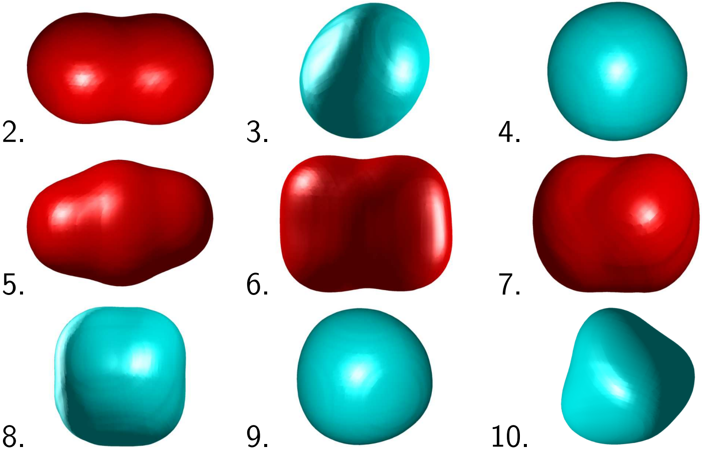
$$\mathbf{x} \to \mathbf{x} - \alpha \nabla J(\mathbf{x})$$

- random initial condition - avoid local minima
- computation of eigenvalues/eigenfunctions with MpsPack

Results



Some 3D computations



Observation

- if Ω^* is optimal for λ_k then it is often multiple, i.e.

$$\lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*) = ...$$

$\Rightarrow$ eventual loss of differentiability

Classical optimality conditions

Main idea: make small perturbations of optimal domains

- if λ_k is simple: $\Omega_\varepsilon = (\mathrm{Id} + \varepsilon V)(\Omega)$, V arbitrary

$$\frac{d\lambda_k(\Omega)}{dV} = -\int_{\partial\Omega} \left(\frac{\partial u_k}{\partial n}\right)^2 V.nd\sigma$$

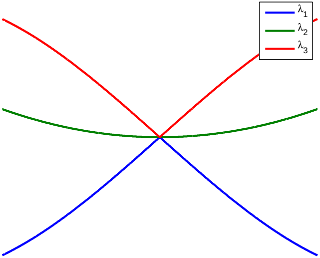
- the derivative of the perimeter is

$$\frac{d\,\mathrm{Per}(\Omega)}{dV} = \int_{\partial\Omega} \mathcal{H} V.nd\sigma$$

$\longrightarrow$ at the optimum we have $\left(\frac{\partial u_k}{\partial n}\right)^2 = \mathcal{H}$

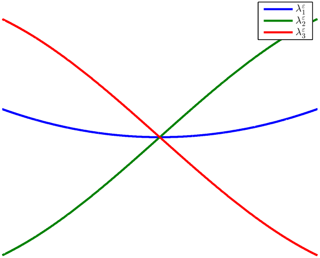
Optimality condition: multiplicity > 1

We may lose differentiability!



Optimality condition: multiplicity > 1

... but recover left and right derivatives



Optimality condition: multiplicity > 1

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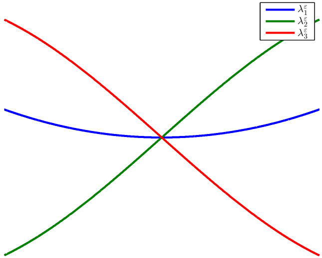


Rellich

If $A(\varepsilon)$ is a sequence of self adjoint operators with compact resolvents, having the same definition domain depending analytically on ε then it is possible to **parametrize analytically the eigenvalues and eigenvectors**.

What can we do?

- $\Omega_\varepsilon = (\mathrm{Id} + \varepsilon V)(\Omega)$.
- $F(\varepsilon) = \lambda_k(\Omega_\varepsilon) + \mathrm{Per}(\Omega_\varepsilon)$
- F may not be differentiable when λ_k is multiple
- we have left and right derivatives in 0
- optimality $\longrightarrow \dfrac{dF}{d\varepsilon+}(0) \cdot \dfrac{dF}{d\varepsilon-}(0) \leq 0$

Main idea

If $(\mathfrak{u}_i)$ is an orthonormal basis for the associated eigenspace, then for any V

$$\exists i \text{ such that } -\int_{\partial\Omega}\left(\frac{\partial \mathfrak{u}_i}{\partial n}\right)^2 V.n + \int_{\partial\Omega} \mathcal{H} V.n \leq 0$$

$$\exists j \text{ such that } -\int_{\partial\Omega}\left(\frac{\partial \mathfrak{u}_j}{\partial n}\right)^2 V.n + \int_{\partial\Omega} \mathcal{H} V.n \geq 0$$

Hahn-Banach theorem $\Rightarrow$ we can find $\mu_i \in [0,1]$ with $\mu_1 + ... + \mu_m = 1$ such that for any perturbation V

$$-\sum_{i=1}^m \mu_i \int_{\partial\Omega}\left(\frac{\partial \mathfrak{u}_i}{\partial n}\right)^2 V.n + \int_{\partial\Omega} \mathcal{H} V.n = 0$$

General optimality conditions

- if $(u_i)_{i=1}^m$ is an orthonormal basis for the eigenspace associated to λ_k then

$$\mathcal{H} \textbf{ is a convex combination of } (\partial_n u_i)^2.$$

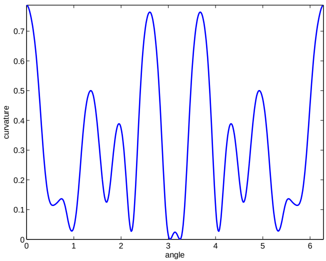
Consequences

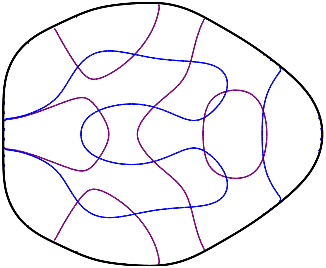
1. The optimal shape Ω^* is C^∞.
   - we know Ω^* is $C^{1,\alpha}$ so $u_i \in C^{1,\alpha}$, $\partial_n u_i \in C^{0,\alpha}$
   - then we use the optimality condition to see that $\mathcal{H} \in C^{0,\alpha}$
   - thus Ω^* is $C^{2,\alpha}$ (not trivial...)
   - inductively we deduce that Ω^* is C^∞.
2. Validate the numerical simulations

(the shapes obtained verify the optimality condition)

Consequences

3. $\mathcal{H}(x) = 0 \Rightarrow \frac{\partial u_i}{\partial n}(x) = 0$





4. The boundary of a local minimizer does not contain segments (flat parts in higher dimensions)
5. The multiplicity cluster (if it exists) ends at λ_k

Open questions?

- $\lambda_3(\Omega)$ is minimized by the disk (area and perimeter constraints)
- are optimal shapes analytic? (perimeter constraint)
- regularity issue for the volume constraint? (no bootstrap argument possible)

Thank you!