PHASE FIELD APPROACH TO OPTIMAL PACKING PROBLEMS AND RELATED CHEEGER CLUSTERS

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Abstract. This paper stems from the idea of adopting a new appraoch to solve some classical optimal packing problems for balls. In fact, we attack this kind of problems (which are of discrete nature) by means of shape optimization techniques, applied to suitable Γ-converging sequences of energies associated to Cheeger type problems. More precisely, in a first step we prove that different optimal packing problems are limits of sequences of optimal clusters associated to the minimization of energies involving suitable (generalized) Cheeger constants. In a second step, we propose an efficient phase field approach based on a multiphase Γ-convergence result of Modica-Mortola type, in order to compute those generalized Cheeger constants, their optimal clusters and, as a consequence of the asymptotic result, optimal packings. Numerical experiments are carried over in two and three space dimensions. Our continuous shape optimization approach to solve discrete packing problems circumvents the NP-hard character of these ones, and efficiently leads to configurations close to the global minima.

1. Introduction

In recent years, some discrete shape optimization problems which are NP-hard, like the computation of a Steiner tree or some minimal partition problems, have been approached by continuous models involving tools of geometric measure theory, as well as shape optimization techniques associated with PDEs. The main point is that such discrete problems have, in general, a huge number of local minima, and their resolution through combinatorial algorithms is quite involved. For these reasons, several authors proposed a continuous approach which can be roughly described as follows. The basic idea consists in finding a sequence of continuous models, depending on some parameter, possibly involving PDEs, whose energy functionals converge in a variational sense (*e.g.*, Γ-convergence) to the discrete problem energy. The advantage of using those continuous models depending on some parameter is often related to the decomposition of the energy functional of the approximated problem into the sum of a convex and a non-convex part. More precisely, choosing the parameter sufficiently large may enhance the convex term, thus leading close to a global minimum of the approximating problem, whereas, decreasing the value of the parameter, one expects to get close to a global minimum of the discrete model.

The reader can find successful applications of this variational approach, for instance, in [27, 5, 15, 24] regarding the computation of Steiner trees in two and three dimensions and in [8, 31], concerning the optimization in two dimensions of free interfaces of circular type and their junctions (emerging, for instance, between different fluids).

This paper was originally motivated by the classical optimal packing problem which consists in finding a family of equal balls of maximal volume fitting a given two or three-dimensional container. Precisely, given a box of arbitrary shape and a fixed positive integer k, one has to find the maximal size of k equal balls fitting in the container, as well as their position. This problem, which is NP-hard, is related to the Kepler conjecture solved by Hales (see [20]).

We approach this optimal packing problem from the variational point of view recalled above. More precisely, we find a suitable family of continuous shape optimization problems which approximate the discrete one. In fact, we prove that a solution to the optimal packing problem is the limit, when some parameter converges to a critical value, of optimal clusters involving

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generalized Cheeger constants (according to the definition given in the next section). We emphasize that when we are given a value of the parameter, the associated Cheeger clusters do not consist of balls. However when the parameter becomes closer and closer to the critical value these clusters do converge to an optimal packing of balls.

As stated above, the motivation for developing this variational formulation is to be able to build a numerical framework for computing optimal Cheeger clusters and optimal packings of circles and balls. One of the key ingredients in the variational and numerical approximation is the Γ-convergence result of Modica-Mortola type. For the perimeter alone, such numerical approaches were already described in [31] for the approximation of minimal partitions of honeycomb type. Additional difficulties arise in our case, as the Cheeger constant is related to the ratio perimeter/volume, and thus we need to treat carefully the volume term in the denominator.

We point out that, to the best of our knowledge, no general numerical method seems to be available in the literature to compute Cheeger clusters, even for planar domains. On the other hand, the literature related to circle packings is quite vast, and different algorithms are known to work for thousands of cells. However, they rely on heuristic combinatorial and geometric ideas in order to search for the optimal configuration (see for instance [18, 26]). Based on our Γ-convergence results, our approach is a global one: we initialize each cell with a random density function and we perform a direct gradient based optimization to reach the final configuration. This approach works rather well, as our examples show in the last section. We underline the fact that the search of optimal circle packings is a non-smooth problem. The desired configuration corresponds to maximizing the minimal distance between centers satisfying the inclusion constraints in the container (see (6) for a precise mathematical formulation). Any standard gradient descent algorithm will get stuck as soon as the pairwise distances between centers contain repeated minimal values. The method we present can consistently get near an optimal candidate, or produce a good starting point for other algorithms. Nevertheless, we point out that our method does not provide an exact mathematical solution to the optimal packing problem.

The paper is organized as follows: theoretical results are stated and discussed in Section 2; proofs and numerical results are then given respectively in Sections 3 and 4.

2. Statement of the results

In order to fix ideas and introduce our results, let $N \geq 2$ be the space dimension and $\alpha > \frac{N-1}{N}$ be a fixed constant. For every bounded measurable subset E of $\mathbb{R}^N$, we introduce the following generalized Cheeger constant. Precisely, we define the α-Cheeger constant of E by

$$h_\alpha(E) := \min\{\frac{\mathcal{H}^{N-1}(\partial^*\Omega)}{|\Omega|^\alpha} \ : \ \Omega \subseteq E, \ \Omega \text{ measurable}\}. \tag{1}$$

Above, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and, if Ω has finite perimeter, $\partial^*\Omega$ is its reduced boundary, in the measure theoretical sense. If Ω is negligible or it has positive measure but does not have finite perimeter, the ratio $\frac{\mathcal{H}^{N-1}(\partial^*\Omega)}{|\Omega|^\alpha}$ is assumed by convention to be equal to $+\infty$.

For $\alpha = 1$, definition (1) corresponds to the classical Cheeger constant, which was thoroughly studied in the last years, see for instance the review papers [25, 32]; for $\alpha \neq 1$, and strictly larger than the scale invariance exponent $\frac{N-1}{N}$, the notion of α-Cheeger constant is a variant which has appeared in the literature more recently, we refer in particular to [33] and references therein.

Object of this paper are the following optimal partition problems

$$\min\left\{\max_{i=1,\dots,k} h_\alpha(E_i) : \ (E_1,\dots,E_k) \in \mathcal{P}_k(D)\right\} \tag{2}$$

$$\min\left\{\sum_{i=1}^{k} h_\alpha(E_i) : \ (E_1,\dots,E_k) \in \mathcal{P}_k(D)\right\}, \tag{3}$$

where D is a given open bounded subset of $\mathbb{R}^N$, and

$$\mathcal{P}_k(D) = \Big\{(E_1, \ldots, E_k) \ : \ \forall i, j = 1, \ldots, k, \ E_i \subseteq D, \ E_i \cap E_j = \emptyset, \ E_i \text{ measurable}\Big\}.$$

We point out that the hypothesis that D has Lipschitz boundary is not necessary for most of our results. Indeed, the main role of the "box" D in our problem is merely to contain the sets E_i, of which we compute the full perimeter, not only in D but in the whole $\mathbb{R}^N$ (*i.e.*, the BV-norm of the characteristic function 1_{E_i} in $\mathbb{R}^N$); in particular, the perimeters of the common boundaries of E_i and D are counted in our problem. Consequently, for most of our results we do not need the compactness of the embedding of the space $BV(D)$ into $L^1(D)$ (which is a consequence of the Lipschitz regularity of D), but just the compactness of the embedding

$$\{u \in BV(B^*) : u = 0 \text{ a.e. on } B^* \setminus D\} \hookrightarrow L^1(B^*), \tag{4}$$

where B^* denotes an open ball containing $\overline{D}$ (which does not require any regularity on D). On the other hand, the Lipschitz regularity of D will be required in order to validate the phase field approach via the Modica-Mortola theorem, so it must be assumed essentially for numerical purposes. In each statement we give, we shall specify if the Lipschitz regularity of D is required.

Let us also emphasize that the condition that the union of the sets E_i covers the given box D (up to a negligible set) is not required in the definition $\mathcal{P}_k(D)$, so that there is some abuse of notation in adopting the usual epithet of optimal "partitions" as done above, and in the sequel we prefer to speak rather about "clusters". Thus, solutions of (2)-(3) will be generically called *α-Cheeger clusters*.

For $\alpha = 1$, problem (3) has been firstly studied by Caroccia in the paper [13], where the existence of solutions and some regularity results for the free boundaries are obtained. In fact, for arbitrary $\alpha > \frac{N-1}{N}$, the existence of solutions $(E_1, \ldots, E_k)$ for both problems (2)-(3) is quite immediate, and for convenience of the reader it will be briefly discussed in Section 3 below. On the other hand, the analysis of their qualitative properties may require some more attention (in particular for the *maximum* problem (2)), but it is not our purpose to discuss here regularity issues.

Let us also mention that the asymptotics as $k \to +\infty$ of the energies in (2)-(3) has received a lot of attention in some recent works focused on the honeycomb conjecture. This celebrated conjecture, which was proved by Hales in [19], states loosely speaking that the hexagonal honeycomb solves the optimal partition problem in which the criterion is minimizing the total perimeter of k mutually disjoint cells having equal area and covering a given planar domain. A similar conjecture was formulated in [13] for the problem of minimizing the sum of the Cheeger constants of k mutually disjoint cells contained into a planar box (in fact, this was inspired by a ten years old conjecture by Caffarelli and Lin involving optimal partitions of spectral type [11]). A proof of the honeycomb conjecture for the α-Cheeger constant when $\alpha = 1$ and $\alpha = 2$ has been obtained very recently, respectively in [10] and in [9], under the restriction that the admissible clusters are made by convex cells.

We are now going to focus on the study of α-Cheeger clusters, with the following twofold aim:

I. To get qualitative results describing the behaviour of α-Cheeger clusters, in the limit when $\alpha \to \left(\frac{N-1}{N}\right)_+$ or $\alpha \to +\infty$.

II. To give an efficient phase field numerical approach for the computation of optimal α-Cheeger clusters and, as a consequence of their asymptotic behaviour, of optimal packings of balls in arbitrary boxes D.

The theoretical results are presented and discussed in the two subsections hereafter.

2.1. Limiting behaviour of α-Cheeger clusters.

Our main asymptotical results show that, as $\alpha \to \left(\frac{N-1}{N}\right)_+$, solutions to problems (2) and (3) converge to solutions to two different optimal packing problems for balls. More precisely: solutions of problem (2) converge to a solution of the classical packing problem, which consists in finding k mutually disjoint equal balls with maximal radius in D; solutions of problem (3) converge to a solution of a more peculiar packing problem, which consists in finding k mutually disjoint balls in D maximizing the product of their volumes. The statements read as follows, where the $L^1(D,\mathbb{R}^k)$-convergence has to be meant as the convergence of the characteristic functions $\{1_{\Omega_1^\alpha},\dots,1_{\Omega_k^\alpha}\}$ of solutions of (5) or (7) to the characteristic functions of balls $\{1_{B(x_1,r_1)},\dots,1_{B(x_k,r_k)}\}$ solving (6) or (8).

Theorem 1. *Let D be an open bounded subset of $\mathbb{R}^N$. Then a solution to problem*

$$\min\Big\{\max_{i=1,\dots,k}\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha}\ :\ \{\Omega_i\}\in\mathcal{P}_k(D)\Big\}\tag{5}$$

exists and, as $\alpha\to\left(\frac{N-1}{N}\right)_+$, it converges in $L^1(D,\mathbb{R}^k)$ (up to subsequences) to a family of balls solving the following optimal packing problem

$$\max\Big\{r\ :\ \exists\{x_i,r_i\geq r\}_{i=1,\dots,k}\,,\ B(x_i,r_i)\subset D\,,\ B(x_i,r_i)\cap B(x_j,r_j)=\emptyset\Big\}.\tag{6}$$

Theorem 2. *Let D be an open bounded subset of $\mathbb{R}^N$. Then a solution to problem*

$$\min\Big\{\sum_{i=1}^k\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha}\ :\ \{\Omega_i\}\in\mathcal{P}_k(D)\Big\}\tag{7}$$

exists and, as $\alpha\to\left(\frac{N-1}{N}\right)_+$, it converges in $L^1(D,\mathbb{R}^k)$ (up to subsequences) to a family of balls solving the following optimal packing problem

$$\max\Big\{\prod_{i=1}^k r_i\ :\ \exists\{x_i,r_i\}_{i=1,\dots,k}\,,\ B(x_i,r_i)\subset D\,,\ B(x_i,r_i)\cap B(x_j,r_j)=\emptyset\Big\}.\tag{8}$$

Remark 3. We point out that the above statements are not phrased in terms of Γ-convergence results for the functionals

$$\Phi_\alpha(\{\Omega_i\}):=\max_{i=1,\dots,k}\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha}\,,\qquad \Psi_\alpha(\{\Omega_i\}):=\sum_{i=1}^k\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha}\,.$$

Actually, the scale-invariant behaviour of their limit functionals as $\alpha\to\left(\frac{N-1}{N}\right)_+$ makes *any* family of balls optimal in the limit, so that studying the Γ-convergence of Φ_α and Ψ_α is not meaningful to the purpose of determining optimal packings.

Remark 4. To get an intuitive glance about the proof of Theorems 1 and 2, let us say in advance that we will repeatedly use the equality

$$\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha}=\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^{\frac{N-1}{N}}}\frac{1}{|\Omega_i|^{\alpha-\frac{N-1}{N}}}\,,$$

which allows to decompose the α-Cheeger quotient as the product between a scaling invariant quantity (which by the isoperimetric inequality favours balls-shaped sets), and the volume to a negative power (which favours large volumes).

When $\alpha\to+\infty$, we are not able to give a complete description of the asymptotic behaviour of α-Cheeger clusters. Nevertheless we observe that, in case of problem (5), such behaviour seems to be related to an optimal partition problem into cells of equal volume which minimize the product of their perimeters. This question is naturally linked to the possible validity of a stronger version of the classical honeycomb conjecture solved by Hales. The picture is detailed in the statement given hereafter, along with a conjecture. For every $k\in\mathbb{N}$, we call k-cell a connected region obtained as the union of k unit area regular hexagons taken from the hexagonal tiling of $\mathbb{R}^2$.

Proposition 5. *As $\alpha\to+\infty$, up to subsequences a solution to problem* (5) *converges in $L^1(D,\mathbb{R}^k)$ to a partition of D into k mutually disjoint subsets of equal measure. Moreover, if $N=2$, D is a k-cell, and Conjecture 6 below holds true, this partition is the hexagonal one.*

Conjecture 6. Let D be a k-cell. Then we expect that Hales' celebrated result [19] can be strengthened into the following product-version of the honeycomb conjecture

$$\min\Big\{\prod_{i=1}^{k}\mathcal{H}^1(\partial^*\Omega_i)\ :\ \{\Omega_i\}\in\mathcal{P}_k(D)\,,\ |\Omega_i|=1\Big\}=\big(\mathcal{H}^1(\partial H)\big)^k.$$

Here H denotes the unit area regular hexagon, and the hexagonal configuration is expected to be the unique minimizer up to negligible sets (*cf.* [19, Theorem 3], [30, Theorem 4.1]).

The proof of this new conjecture seems to be non-trivial, as it would require a product form of the hexagonal isoperimetric inequality in the spirit of Theorem 4 in [19]. It is not a purpose of this paper to discuss such issue in detail. We limit ourselves to support the validity of the conjecture by performing a few simulations using algorithms similar to those introduced in [31]. In order to avoid numerical instabilities, we minimize the logarithm of the product of perimeters, which leads to the following equivalent problem

$$\min\Big\{\sum_{i=1}^{k}\log\big(\mathcal{H}^1(\partial^*\Omega_i)\big)\ :\ \{\Omega_i\}\in\mathcal{P}_k(D)\,,\ |\Omega_i|=1\Big\}.$$

The numerical results presented in Figure 1 are obtained using the method presented in [31]. In particular, the perimeter is computed using the Modica-Mortola approximation and the partition is represented as a family of density functions with sum identically equal to one and with prescribed integrals, in order to have equal area cells. In Figure 1 we present results for $8, 16$ and 32 cells in the flat torus. The choice of the torus instead of a k-cell is based on the observation that, if Conjecture 6 would fail on some k-cell which tiles the plane, then it will also fail on the torus.

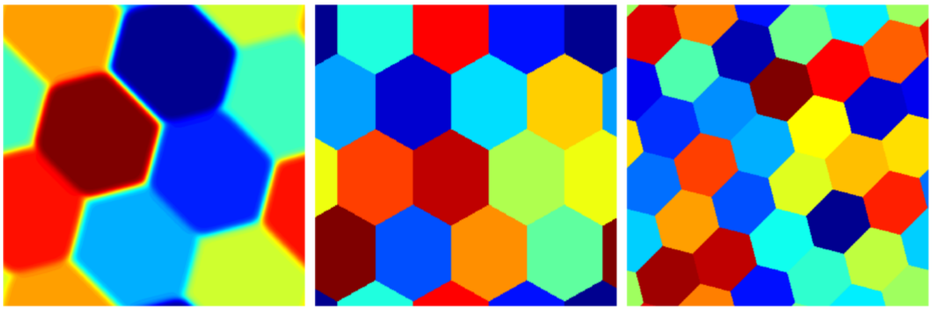


FIGURE 1. Illustration of Conjecture 6 for $8, 16, 32$ cells in the torus

Concerning the asymptotic behaviour of α-Cheeger clusters for problem (7) when $\alpha\to+\infty$, the same arguments used to obtain Proposition 5 can be used to show that, up to subsequences, a solution to such problem converges as well in L^1 to a partition of D into k mutually disjoint subsets of equal measure. Anyway in this case we have no conjecture about the optimization problem solved by this limit configuration.

2.2. **Phase field approach for computing α-Cheeger clusters and optimal packings.** There exist various works in the literature dealing with the computation of the Cheeger sets of a given domain D (which corresponds to take $\alpha=1$ and $k=1$), using techniques related to convex optimization as, for instance, [14]. In [12] the authors exploit a projection algorithm in order to solve a more general weighted Cheeger problem, while in [23] a convex hull method is used for the optimization in the class of convex bodies. In [21] a characterization of the Cheeger set for convex domains is given, which can easily provide an algorithm for the computation of the Cheeger set for convex polygons. To our knowledge, no algorithm was implemented for the numerical computation of Cheeger clusters. An algorithm for computing numerically partitions of minimal length was introduced in [31], based on the approximation of perimeter by Γ-convergence using the classical Modica-Mortola theorem [29].

Our approach is as well of Γ-convergence type and relies on the Modica-Mortola theorem for the approximation of the perimeter term, at the numerator in the α-Cheeger ratio. Since in problem (5) we have to minimize a maximum, in order to launch a phase field approach it is convenient to introduce a "p-approximation", namely to consider the functional $F(\Omega_1, \dots, \Omega_k)$ defined as in (9) below, in the perspective of passing then to the limit as $p \to +\infty$. Nevertheless, in studying the approximation by Γ-convergenge of the functional $F(\Omega_1, \dots, \Omega_k)$ for a fixed exponent p, two additional difficulties arise: on one hand, even for the computation of the α-Cheeger constant of a set (corresponding to the case $k = 1$), one has to handle the approximation of the measure term in the denominator. This is done by considering a L^q-norm of the phase field function for q sufficiently high, as the more natural L^1 or L^2 norms are not suitable. Precisely, the measure of the set is efficiently approximated by the $L^{\frac{2N}{N-1}}$-norm, this choice being available for every α. On the other hand, one has to cope with the collective behaviour of the different phases and the empty regions, this being handled in the spirit of the partitioning algorithm introduced in [6], with the novelty of the presence of the empty regions. The outcoming Γ-convergence result reads as follows. For the definition and main properties of Γ-convergence, we refer the reader to [7, 16].

Theorem 7. *Let D be a bounded, open and Lipschitz domain in $\mathbb{R}^N$. For any fixed $\alpha > \frac{N-1}{N}$ and $p > 1$, consider the sequence of functionals defined on $L^1(D, \mathbb{R}^k)$ by*

$$F_{p,\varepsilon}(u_1, \dots, u_k) := \sum_{i=1}^{k} \Big(\frac{\varepsilon \int_D |\nabla u_i|^2 dx + \frac{9}{\varepsilon} \int_D u_i^2 (1-u_i)^2}{\Big(\int_D |u_i|^{\frac{2N}{N-1}} dx \Big)^{\alpha}} \Big)^p$$

if $u_i \in H_0^1(D)$, $u_i \geq 0$, $\sum_{i=1}^k u_i \leq 1$, and $+\infty$ if not. Then, for every sequence $u_i^\varepsilon \in H_0^1(D) \setminus \{0\}$ such that $\limsup_{\varepsilon \to 0} F_\varepsilon(u_1^\varepsilon, \dots, u_k^\varepsilon) < +\infty$, $(u_1^\varepsilon, \dots, u_k^\varepsilon)$ converges up to subsequences to some limit $(u_1, \dots, u_k)$ in $L^1(D, \mathbb{R}^k)$. Moreover, the sequence $F_{p,\varepsilon}$ Γ-converges as $\varepsilon \to 0$ in $L^1(D, \mathbb{R}^k)$ to the functional

$$F_p(\Omega_1, \dots, \Omega_k) := \sum_{i=1}^{k} \Big(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} \Big)^p . \tag{9}$$

In the above statement, the separation of phases is related to the constraints $u_i \geq 0$, $\sum_{i=1}^k u_i \leq 1$, and to the fact that each phase will naturally converge to a bang-bang configuration (namely to a function which takes only the values 0 and 1). In order to avoid handling the constraint $\sum_{i=1}^k u_i \leq 1$, we use a penalization approach inspired by the paper of Caffarelli and Lin [11]. Namely, we slightly change our functional by introducing the penalty term $\frac{1}{\varepsilon} \sum_{1 \leq i < j \leq k} \int_D u_i^2 u_j^2 dx$. This term enhances the separation of phases during the computational process and avoids (at least the) local minima consisting on flat functions at an intermediate level between 0 and 1. At the same time, the presence of this term, together with the bounds $0 \leq u_i \leq 1$, still ensure that the Γ-convergence holds. Thus we state the following proposition, which is the main practical tool for the numerical implementation our phase field approach:

Proposition 8. *Let D be a bounded, open and Lipschitz domain in $\mathbb{R}^N$. For any fixed $\alpha > \frac{N-1}{N}$ and $p > 1$, the sequence of functionals defined on $L^1(D, \mathbb{R}^k)$ by*

$$\widetilde{F}_{p,\varepsilon}(u_1, \dots, u_k) := \sum_{i=1}^{k} \Big(\frac{\varepsilon \int_D |\nabla u_i|^2 dx + \frac{9}{\varepsilon} \int_D u_i^2 (1-u_i)^2}{\Big(\int_D |u_i|^{\frac{2N}{N-1}} dx \Big)^{\alpha}} \Big)^p + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq k} \int_D u_i^2 u_j^2 dx \tag{10}$$

if $u_i \in H_0^1(D)$, $0 \leq u_i \leq 1$ and $+\infty$ if not, Γ*-converges as $\varepsilon \to 0$ in $L^1(D, \mathbb{R}^N)$ to the functional*

$$F_p(\Omega_1, \dots, \Omega_k) := \sum_{i=1}^{k} \Big(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} \Big)^p$$

Moreover, if for any $p > 1$ we denote $(\Omega_1^p, \ldots, \Omega_k^p)$ a minimizer in $\mathcal{P}_k(D)$ of F_p then, for $p \to +\infty$, possibly extracting a subsequence, we have

$$(\Omega_1^p, \ldots, \Omega_k^p) \overset{L^1(D,\mathbb{R}^k)}{\longrightarrow} (\Omega_1, \ldots, \Omega_k),$$

being $(\Omega_1, \ldots, \Omega_k)$ a solution to (5).

Our algorithm is very efficient in computing clusters or just Cheeger sets in arbitrary (Lipschitz) boxes D, in two or three space dimensions. In Section 4, we give several numerical experiments with different values of α and k, in $\mathbb{R}^2$ and in $\mathbb{R}^3$. In particular, for α close to the critical value $\left(\frac{N-1}{N}\right)_+$, we recover some classical results for optimal circle/sphere packings, underlining the interest of Theorems 1 and 2.

3. Proofs

Throughout this section, we set $\omega_N := \mathcal{H}^{N-1}(\partial B)/|B|^{\frac{N-1}{N}}$, being B a ball in $\mathbb{R}^N$. Moreover, as done in (4), we denote by B^* a fixed open ball containing the closure of the bounded box D.

Existence of α-Cheeger clusters. Before giving the proofs of our results, for the benefit of the reader we briefly discuss the existence of solutions to problems (2) and (3) for a fixed $\alpha > \frac{N-1}{N}$. This question relies on classical compactness results in BV-spaces and, for $\alpha = 1$ and problem (3), has been extensively discussed in [13]. For shortness, we only deal with problem (2) (the arguments being the same for problem (3)). Assume that $\{\Omega_1^n, \ldots, \Omega_k^n\}$ is an element of $\mathcal{P}_k(D)$ such that

$$\max_{i=1,\ldots,k} \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^n)}{|\Omega_i^n|^\alpha} \longrightarrow \inf\Big\{ \max_{i=1,\ldots,k} \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^\alpha} \ : \ \{\Omega_i\} \in \mathcal{P}_k(D)\Big\} \quad \text{as} \quad n \to +\infty.$$

One has first to observe that

$$\liminf_{n\to+\infty} |\Omega_i^n| > 0 \qquad \forall i = 1, \ldots, k\,. \tag{11}$$

Indeed, we can assume that there exists some $M > 0$ such that $\max_{i=1,\ldots,k} \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^n)}{|\Omega_i^n|^\alpha} \le M$. Then, using the isoperimetric inequality on each cell Ω_i^n we get

$$\omega_N \frac{1}{|\Omega_i^n|^{\alpha - \frac{N-1}{N}}} \le \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^n)}{|\Omega_i^n|^\alpha} \le M,$$

hence, (11) is true. Exploiting also the fact that

$$\mathcal{H}^{N-1}(\partial^*\Omega_i^n) \le M|D|^k,$$

we get that the sequence $(1_{\Omega_i^n})_n$ is bounded in $BV(B^*)$. By the compactness of the injection (4), we can assume that, up to a subsequence, $(1_{\Omega_i^n})_n$ converges strongly in $L^1(D)$ to some limit which can be written as 1_{Ω_i}. Since $|\Omega_i^n| \to |\Omega_i|$ as $n \to +\infty$, the sets Ω_i have strictly positive measure and, for $i \neq j$, they have a negligible intersection. Moreover, by lower semicontinuity of the perimeter, we have

$$\liminf_{n\to+\infty} \mathcal{H}^{N-1}(\partial^*\Omega_i^n) \ge \mathcal{H}^{N-1}(\partial^*\Omega_i),$$

hence $\{\Omega_1, \ldots, \Omega_k\}$ is a solution to problem (2).

Proof of Theorem 1. We write $\alpha = \frac{N-1}{N} + \delta$ for some $\delta > 0$, and we consider the auxiliary problems

$$\begin{aligned} M_{k,\delta}(D) \quad &:= \min\Big\{ \max_{i=1,\ldots,k} \left(\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{\omega_N|\Omega_i|^{\frac{N-1}{N}+\delta}}\right)^{\frac{1}{\delta}} \ : \ \{\Omega_i\} \in \mathcal{P}_k(D)\Big\} \\ &= \min\Big\{ \max_{i=1,\ldots,k} \left(\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{\omega_N|\Omega_i|^{\frac{N-1}{N}}}\right)^{\frac{1}{\delta}} \frac{1}{|\Omega_i|} \ : \ \{\Omega_i\} \in \mathcal{P}_k(D)\Big\}, \end{aligned}$$

which have the same optimal clusters as our initial problems (5).

If D contains k mutually disjoint balls of radius r_D, we have

$$M_{k,\delta}(D) \le \frac{1}{|B_{r_D}|}\,. \tag{12}$$

Then, if $\{\Omega^1_\delta, \dots \Omega^k_\delta\}$ is a competitor for an optimal cluster for $M_{k,\delta}(D)$, we have

$$\left(\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\delta)}{\omega_N|\Omega_i^\delta|^{\frac{N-1}{N}}}\right)^{\frac{1}{\delta}} \frac{1}{|\Omega_i^\delta|} \le \frac{1}{|B_{r_D}|} \qquad \forall i = 1, \dots, k\,. \tag{13}$$

Using (13) and the isoperimetric inequality, we get

$$|\Omega_i^\delta| \ge |B_{r_D}| \qquad \forall i = 1, \dots, k\,. \tag{14}$$

Moreover, using (13) and the upper bound $|\Omega_i^\delta| \le |D|$, we get

$$\mathcal{H}^{N-1}(\partial^*\Omega_i^\delta) \le \omega_N |D|^{\frac{N-1}{N}} \Big(\frac{|D|}{|B_{r_D}|}\Big)^\delta \qquad \forall i = 1, \dots, k\,. \tag{15}$$

The previous observations insure that good competitors for optimal clusters for $M_{k,\delta}(D)$ have measures uniformly bounded from below and above, and uniformly bounded perimeters. In other words, the class of candidates is compact in $L^1(D)$, so that we get immediately the existence of an optimal cluster $\{\Omega^1_\delta, \dots \Omega^k_\delta\}$ for $M_{k,\delta}(D)$.

Moreover, we deduce that a (not relabeled) subsequence of $\{\Omega^\delta_1, \dots \Omega^\delta_k\}$ converges in $L^1(D, \mathbb{R}^k)$ to a limit cluster, that we denote by $\{\Omega_1, \dots, \Omega_k\}$, satisfying $|\Omega_i| \ge |B_{r_D}|$. Let us show first that all the sets Ω_i's are balls, and then that they solve the optimal packing problem (6).

By using the definitions of $M_{k,\delta}(D)$ and r_D, we have the following estimate:

$$\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\delta)}{|\Omega_i^\delta|^{\frac{N-1}{N}+\delta}} \le \omega_N (M_{k,\delta}(D))^\delta \le \frac{\omega_N}{|B_{r_D}|^\delta} \qquad \forall i = 1, \dots, k\,.$$

Then, passing to the liminf as $\delta \to 0$, we obtain

$$\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{|\Omega_i|^{\frac{N-1}{N}}} \le \omega_N \qquad \forall i = 1, \dots, k\,,$$

which implies that the sets Ω_i's are balls, as a consequence of the uniqueness part in the isoperimetric inequality.

Let us show that they solve the optimal packing problem (6). Note that this problem is finite dimensional and the existence of a solution follows directly by a compactness/continuity argument. Denote by $r_1, \dots r_k$ the radii of the sets $\Omega_1, \dots \Omega_k$ found above, set $r_{\min} := \min\{r_1, \dots, r_k\}$ and let r_* denote the maximal radius in the optimal packing problem (6). Clearly, by the definition of r_*, it holds $r_* \ge r_{\min}$. On the other hand, since D contains k mutually disjoint balls of radius r_*, by definition of $M_{k,\delta}(D)$ we have

$$M_{k,\delta}(D) \le \frac{1}{|B_{r_*}|}\,.$$

Hence the inequalities (13) and (14) are in force with r_* in place of r_D and therefore, passing to the limit as $\delta \to 0$, we obtain

$$|\Omega_i| \ge |B_{r_*}| \qquad \forall i = 1, \dots, k\,.$$

Then $r_{\min} \ge r_*$ and we conclude that $r_{\min} = r_*$. $\square$

Proof of Theorem 2. For fixed $\alpha > \frac{N-1}{N}$ the existence of a solution for problem (7) is obtained as in Theorem 1. We write $\alpha = \frac{N-1}{N} + \delta$ for some $\delta > 0$, and we set

$$m_{k,\delta}(D) := \min\Big\{ \sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i)}{\omega_N|\Omega_i|^{\frac{N-1}{N}+\delta}} \;:\; \{\Omega_i\} \in \mathcal{P}_k(D)\Big\}\,,$$

(where we have introduced just for convenience the constant ω_N with respect to our initial problems (7)).

Let $(\Omega_1^\delta, \dots, \Omega_k^\delta)$ be an optimal cluster for $m_{k,\delta}(D)$. From the inclusion $\Omega_i^\delta \subset D$, we know that

$$\limsup_{\delta\to 0} |\Omega_i^\delta|^\delta \le \limsup_{\delta\to 0} |D|^\delta = 1 \qquad \forall i = 1, \dots, k\,. \tag{16}$$

We claim that

$$\lim_{\delta\to 0} |\Omega_i^\delta|^\delta = 1 \qquad \forall i = 1, \dots, k\,. \tag{17}$$

Indeed, assume by contradiction that (17) is not satisfied. Then for some index i we have $\liminf_{\delta\to 0} |\Omega_i^\delta|^\delta < 1$, so that for a subsequence $\delta_n \to 0$ we have $\lim_{n\to+\infty} |\Omega_i^{\delta_n}|^{\delta_n} < 1$.

On the other hand notice that, if D contains k mutually disjoint balls of radius r_D, it holds

$$\sum_{i=1}^k \frac{1}{|\Omega_i^\delta|^\delta} \le \sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_i^\delta|^\delta} = m_{k,\delta}(D) \le \frac{k}{|B_{r_D}|^\delta}\,. \tag{18}$$

By passing to the limit as $\delta \to 0$ in (18), we get

$$\sum_{i=1}^k \frac{1}{\limsup_{\delta\to 0} |\Omega_i^\delta|^\delta} \le k\,.$$

which in view of (16) implies $\limsup_{\delta\to 0} |\Omega_i^\delta|^\delta = 1$ for every $i = 1, \dots, k$. Then, for the sequence δ_n, the passage to the limit in (18) would give

$$\frac{1}{\limsup_{n\to+\infty} |\Omega_i^{\delta_n}|^{\delta_n}} < 1\,,$$

contradiction.

We write

$$\sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_i^\delta|^\delta} = m_{k,\delta}(D) \le \frac{k}{|B_{r_D}|^\delta}\,.$$

In the limit as $\delta \to 0_+$, thanks to (17) and to the isoperimetric inequality, we deduce that for every index i

$$\limsup_{\delta\to 0} \frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \le 1\,, \tag{19}$$

which implies in particular

$$\limsup_{\delta\to 0} \mathcal{H}^{N-1}(\partial^*\Omega_i^\delta) \le \omega_N |D|^{\frac{N-1}{N}}\,. \tag{20}$$

Moreover, we get that $m_{k,\delta}(D) \to k$ as $\delta \to 0$.

Next, let us show that the measures of Ω_i^δ are bounded from below for every $i = 1, \dots, k$. We use the Jensen inequality, taking advantage from the concavity of the logarithm function. We have

$$\frac{1}{k}\sum_{i=1}^k \ln\Big(\frac{1}{|\Omega_i^\delta|^\delta}\Big) \le \ln\Big(\frac{1}{k}\sum_{i=1}^k \frac{1}{|\Omega_i^\delta|^\delta}\Big) \le \ln\Big(\frac{1}{|B_{r_D}|^\delta}\Big),$$

the last inequality being a consequence of (18). We conclude that $|\Omega_i^\delta|$ is bounded below for every $i = 1, \dots, k$.

So far, we have obtained that, for every $i = 1, \dots, k$, the sets Ω_i^δ have perimeters bounded from above and measures bounded from below *uniformly* as $\delta \to 0_+$; hence, up to a (not relabeled) subsequence, they admit a k-uple of (non trivial) limit sets, and from inequalities (19) we see that these limit sets are necessarily balls. Let us denote them by $(B_1^0, \dots, B_k^0)$, and let us show that they solve the optimal packing problem (8).

To that aim, we are going to estimate from above and from below the quotient
$$\frac{m_{k,\delta}(D)-k}{\delta}\,.$$
We already know that $m_{k,\delta}(D)\to k$ as $\delta\to 0$. In the sequel we estimate the first order term in the development of $m_{k,\delta}(D)$ as $\delta\to 0$. On one hand, if $(B_1^*,\dots,B_k^*)$ are balls which solve the optimal packing problem (8), taking them as a test cluster in the definition of $m_{k,\delta}(D)$, we get the upper bound
$$\frac{m_{k,\delta}(D)-k}{\delta}\le\frac{1}{\delta}\sum_{i=1}^k\Big(\frac{1}{|B_i^*|^\delta}-1\Big)=\sum_{i=1}^k\frac{1}{\delta}\Big(\exp\Big(\delta\log\frac{1}{|B_i^*|}\Big)-1\Big)\,.$$
Hence,
$$\limsup_{\delta\to 0}\frac{m_{k,\delta}(D)-k}{\delta}\le\log\Big(\prod_{i=1}^k\frac{1}{|B_i^*|}\Big)\,. \tag{21}$$
On the other hand, by applying as usual the isoperimetric inequality, we have
$$\frac{m_{k,\delta}(D)-k}{\delta}\ge\frac{1}{\delta}\sum_{i=1}^k\Big(\frac{1}{|\Omega_i^\delta|^\delta}-1\Big)\,.$$
Now we exploit the fact that $|\Omega_i^\delta|\to|B_i^0|$ as $\delta\to 0_+$, so that, for every fixed $\eta>0$, we can find $\overline{\delta}=\overline{\delta}(\eta)$ such that
$$|\Omega_i^\delta|\le|(1+\eta)B_i^0|\qquad\forall\delta\le\overline{\delta}(\eta)\,,\ \forall i=1,\dots,k\,.$$
Therefore,
$$\frac{m_{k,\delta}(D)-k}{\delta}\ge\frac{1}{\delta}\sum_{i=1}^k\Big(\frac{1}{|(1+\eta)B_i^0|^\delta}-1\Big)\,.$$
Then, by arguing as above and using the arbitrariness of η, we arrive at
$$\liminf_{\delta\to 0}\frac{m_{k,\delta}(D)-k}{\delta}\ge\log\Big(\prod_{i=1}^k\frac{1}{|B_i^0|}\Big)\,. \tag{22}$$
Eventually, combining (21) and (22), we obtain
$$\prod_{i=1}^k\frac{1}{|B_i^0|}\le\prod_{i=1}^k\frac{1}{|B_i^*|}\,.$$
By the definition of B_i^*, the above inequality holds necessarily with equality sign, which amounts to say that the limit balls B_i^0 maximize the product of volumes of a family of k mutually disjoint balls contained into D, namely they solve problem (8). □

Proof of Proposition 5. Let $(\Omega_1^\alpha,\dots,\Omega_k^\alpha)$ by a solution to problem (5). Assuming without loss of generality that $|D|=k$, we consider a k-uple $(\widehat{\Omega}_1,\dots,\widehat{\Omega}_k)\in\mathcal{P}_k(D)$ of sets with finite perimeter, such that $|\widehat{\Omega}_i|=1$ for every $i=1,\dots,k$. Then we have
$$\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\alpha)}{|\Omega_i^\alpha|^\alpha}\le C:=\max_{i=1,\dots,k}\mathcal{H}^{N-1}(\partial^*\widehat{\Omega}_i)\qquad\forall i=1,\dots,k\,. \tag{23}$$
By applying the isoperimetric inequality, we obtain
$$\frac{\omega_N|\Omega_i^\alpha|^{\frac{N-1}{N}}}{|\Omega_i^\alpha|^\alpha}\le C\qquad\forall i=1,\dots,k\,,$$
or equivalently
$$|\Omega_i^\alpha|\ge\Big(\frac{\omega_N}{C}\Big)^{\frac{1}{\alpha-\frac{N-1}{N}}}\qquad\forall i=1,\dots,k\,.$$

Hence,

$$\liminf_{\alpha\to+\infty} |\Omega_i^\alpha| \geq 1 \qquad \forall i = 1,\dots,k\,. \tag{24}$$

Moreover, from inequality (23), we know that

$$\sum_{i=1}^k \Big(\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\alpha)}{C}\Big)^{\frac{1}{\alpha}} \leq \sum_{i=1}^k |\Omega_i^\alpha| \leq k\,.$$

Using again the concavity of the logarithm function, we get directly

$$\frac{1}{\alpha}\sum_{i=1}^k \log\Big(\frac{\mathcal{H}^{N-1}(\partial^*\Omega_i^\alpha)}{C}\Big) \leq 0\,,$$

which means that

$$\prod_{i=1}^k \mathcal{H}^{N-1}(\partial^*\Omega_i^\alpha) \leq C^k\,. \tag{25}$$

We observe that, by (24), $\mathcal{H}^{N-1}(\partial^*\Omega_i^\alpha)$ is bounded from below for every $i = 1,\dots,k$; then inequality (25) ensures that $\mathcal{H}^{N-1}(\partial^*\Omega_i^\alpha)$ is also bounded from above for every $i = 1,\dots,k$. We conclude that $(\Omega_1^\alpha,\dots,\Omega_k^\alpha)$ admits a limit in L^1, hereafter denoted by $(\Omega_1^\infty,\dots,\Omega_k^\infty)$, and in view of (24) this limit turns out to be a partition of D.

Finally, if $N = 2$ and D is a k-cell, by taking the sets $\widehat{\Omega}_i$'s in (23) equal to the k-copies of H which compose the k-cell, inequality (25) implies

$$\prod_{i=1}^k \mathcal{H}^1(\partial^*\Omega_i^\infty) \leq \big(\mathcal{H}^1(\partial H)\big)^k\,. \tag{26}$$

Therefore, if Conjecture 6 is satisfied, we conclude that

$$\prod_{i=1}^k \mathcal{H}^1(\partial^*\Omega_i^\infty) \leq \min\Big\{\prod_{i=1}^k \mathcal{H}^1(\partial^*\Omega_i) \;:\; \{\Omega_i\}\in\mathcal{P}_k(D)\,,\ |\Omega_i| = 1\Big\}\,.$$

Hence the above inequality holds with equality sign, and the partition $\{\Omega_i^\infty\}$ is optimal for the minimization problem at the right hand side. $\square$

Proof of Theorem 7. We prove separately the so-called Γ-liminf and Γ-limsup inequalities (see the monograph [16] for an introduction to Γ-convergence).

Γ-*liminf inequality.* Let $u_i^\varepsilon \in H_0^1(D)\setminus\{0\}$, such that $u_i^\varepsilon \geq 0, \sum_{i=1}^k u_i^\varepsilon \leq 1$ and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_1^\varepsilon,\dots,u_k^\varepsilon) < +\infty.$$

In a first step, we notice the existence of a constant M, such that for small ε and for every i we have

$$\varepsilon\int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon}\int_D (u_i^\varepsilon)^2(1-u_i^\varepsilon)^2 dx \leq M|D|^\alpha.$$

This inequality, together with the L^∞ bound of u_i^ε, ensures the boundedness of the sequence $(u_i^\varepsilon)_\varepsilon$ in $BV(B^*)$, hence compactness in $L^1(D)$. Up to passing to subsequences, we can assume that $(u_1^\varepsilon,\dots,u_k^\varepsilon) \stackrel{L^1(D,\mathbb{R}^k)}{\longrightarrow} (u_1,\dots,u_k)$. The limit functions u_i belong to $BV(B^*,\{0,1\})$ (we refer the reader to the Modica-Mortola theorem, see for instance [28]). Thus we can write $u_i = 1_{\Omega_i}$, where Ω_i are pairwise disjoint and satisfy

$$\mathcal{H}^{N-1}(\partial^*\Omega_i) \leq \liminf_{\varepsilon\to 0}\varepsilon\int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon}\int_D (u_i^\varepsilon)^2(1-u_i^\varepsilon)^2 dx.$$

We notice that $1 = 2\int_0^1 3t(1-t)dt$, which ensures that the constant multiplying the perimeter on the left hand side above, equals to 1. The fact that Ω_i are pairwise disjoint is a consequence of the passage to the limit as $\varepsilon \to 0$ in the inequality $\sum_{i=1}^k u_i^\varepsilon \le 1$ a.e., leading to $\sum_{i=1}^k 1_{\Omega_i} \le 1$ a.e.

Let us prove that $\Omega_i \neq \emptyset$. We know that

$$\varepsilon \int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon}\int_D (u_i^\varepsilon)^2(1-u_i^\varepsilon)^2 dx \le M\Big(\int_D (u_i^\varepsilon)^{\frac{2N}{N-1}} dx\Big)^\alpha.$$

Setting $\delta := \frac{2N}{N-1}\alpha - 2 > 0$ and using the Cauchy-Schwartz inequality on the left hand side, we get

$$6\int_D |\nabla u_i^\varepsilon| u_i^\varepsilon (1-u_i^\varepsilon) dx \le M \|u_i^\varepsilon\|^{2+\delta}_{L^{\frac{2N}{N-1}}(D)},$$

or, by the chain rule,

$$6\int_D \Big|\nabla\Big(\frac{(u_i^\varepsilon)^2}{2} - \frac{(u_i^\varepsilon)^3}{3}\Big)\Big| dx \le M \|u_i^\varepsilon\|^{2+\delta}_{L^{\frac{2N}{N-1}}(D)}.$$

Using the Sobolev inequality with a dimensional constant S_N, we get

$$6S_N \|\frac{(u_i^\varepsilon)^2}{2} - \frac{(u_i^\varepsilon)^3}{3}\|_{L^{\frac{N}{N-1}}(D)} \le M \|u_i^\varepsilon\|^{2+\delta}_{L^{\frac{2N}{N-1}}(D)}.$$

Since $0 \le u_i^\varepsilon \le 1$ we get $\frac{(u_i^\varepsilon)^2}{2} - \frac{(u_i^\varepsilon)^3}{3} \ge \frac{(u_i^\varepsilon)^2}{6}$ so

$$S_N \|u_i^\varepsilon\|^2_{L^{\frac{2N}{N-1}}(D)} \le M \|u_i^\varepsilon\|^{2+\delta}_{L^{\frac{2N}{N-1}}(D)},$$

or

$$\frac{S_N}{M} \le \|u_i^\varepsilon\|^{\delta}_{L^{\frac{2N}{N-1}}(D)}.$$

Passing to the limit, we get $|\Omega_i| \ge \frac{S_N}{M}$. Since $\int_D (u_i^\varepsilon)^{\frac{2N}{N-1}} dx \to |\Omega_i| \neq 0$, the Γ-liminf property occurs.

Γ-*limsup inequality.* For one single set, the Modica-Mortola theorem gives the procedure of constructing the recovering sequence. For partition problems we refer to the paper by Baldo [1], where the recovering sequence requires more attention because of the exact partition requirement which is obtained via the constraint $\forall x \in D$, $\sum_{i=1}^k u_i(x) = 1$. Since in our problem we do not have a complete partition of the set D, we can give a direct proof as follows.

Let $(\Omega_1, \dots, \Omega_k)$ be pairwise disjoint measurable subsets of D with finite perimeter. Relying on the regularity of D, we can assume that the distances from $\partial^*\Omega_i$ to ∂D are strictly positive, otherwise we make an approximation of each Ω_i by inner sets having such properties, and then take a diagonal sequence. Let us fix a positive constant $\delta > 0$. For a standard sequence of mollifiers $(\rho_\eta)_\eta$ satisfying $\int \rho_\eta dx = 1$, we have

$$(\sum_{i=1}^k 1_{\Omega_i}) * \rho_\eta \le 1, \tag{27}$$

and we claim that we can choose η small enough such that for every $i = 1, \dots, k$ we can find $t_i \in (\frac{1}{2}, 1)$ such that the set $A_i^\delta := \{1_{\Omega_i} * \rho_\eta > t_i\}$ is smooth and

$$\mathcal{H}^{N-1}(\partial A_i^\delta) \le \mathcal{H}^{N-1}(\partial^*\Omega_i) + \delta,$$
$$|A_i^\delta \Delta \Omega_i| \le \delta.$$

Indeed, from the Sard theorem, we know that almost all levels sets are smooth. From the strict convergence $\rho_\eta * 1_{\Omega_i} \xrightarrow{BV} 1_{\Omega_i}$, we get

$$\int_{\mathbb{R}^N} |\nabla \rho_\eta * 1_{\Omega_i}| dx \to \mathcal{H}^{N-1}(\partial^*\Omega_i),$$

and the co-area formula gives the existence of a level $t > \frac{1}{2}$ such that

$$\mathcal{H}^{N-1}(\partial^*\Omega_i) \ge \limsup_{\eta \to 0} \mathcal{H}^{N-1}(\partial\{\rho_\eta * 1_{\Omega_i} > t\}).$$

On the other hand, from (27) and the choice of $t_i > \frac{1}{2}$, the sets A_i^δ are disjoint, at positive distance. Using now the Modica-Mortola theorem, we can find recovering sequences $u_i^\varepsilon \stackrel{L^1(D)}{\to} 1_{A_i^\delta}$ with

$$\limsup_{\varepsilon \to 0} \varepsilon \int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon} \int_D (u_i^\varepsilon)^2 (1 - u_i^\varepsilon)^2 dx \leq \mathcal{H}^{N-1}(\partial A_i^\delta).$$

Choosing, a sequence $\delta_n \to 0$, by a diagonal procedure we can find ε_n small enough such that the Γ-limsup property holds, and moreover $u_i^{\varepsilon_n} u_j^{\varepsilon_n} = 0$. □

Proof of Proposition 8. The proof of the first part of Proposition 8 is implicitly contained into the one of Theorem 7. In particular, we point out that the choice of the recovering sequence for the Γ-limsup property is also suitable for the penalized functionals.

The passage to the limit $p \to +\infty$ is standard, being a consequence of the approximation of the $\| \cdot \|_\infty$- norm in $\mathbb{R}^k$ by the $\| \cdot \|_p$- norm. □

4. Numerical results

In order to discretize the functional (10), we consider a rectangular box D in $\mathbb{R}^2$ or $\mathbb{R}^3$ endowed with a finite differences uniform grid with M discretization points along each axis direction. A function u will be numerically represented by its values at the grid points. We use basic order 1 centered finite differences in order to compute the gradient terms $|\nabla u_i|$, and basic quadrature formulas to compute all integrals. Similar approaches were already used in [31] and [4]. Moreover, in [4, Section 4] detailed expressions of the gradient with respect to each of the grid point variables of (10) are given for the Modica-Mortola term. The other integrals, like the denominator of F_ε and the penalization terms, are approximated by their arithmetic mean along the grid. The optimization is done using a LBFGS quasi-Newton method implemented in Matlab [34]. The algorithm uses information on a number of previous gradients, 5 in our computations, in order to build an approximation of the Hessian. In addition to being more efficient in avoiding local minima than a simple gradient descent algorithm, the LBFGS algorithm recalled above allows us to impose pointwise bounds for every variable, fact which is important in our approach. This also motivated us to use a penalized approach, rather than a projected gradient approach. The optimization procedure is presented in Algorithm 1.

Since we perform an optimization of a non-convex functional, there is no guarantee of convergence to a global minimum. In order to avoid local minima, we choose as initialization some random densities for each of the functions u_i. Moreover, in order to avoid the rapid convergence to a characteristic function and to diminish the effects of the non-convex potential, we choose the parameter ε equal to $1/M$, where M is the number of discretization points along each axis direction. The parameter ε dictates the width of the interface where the functions u_i go from 0 to 1. Sometimes it is useful to consider larger values of ε, $2/M$ or $4/M$, to allow the cells to move more freely.

In order to reduce the number of iterations needed to reach an optimum, we propose a grid refinement procedure as already noted in [6] and [4]. We perform an initial optimization on a grid of rather low size with $M \in [20, 50]$. Then we interpolate the result on a finer grid, usually doubling the discretization parameter, and we continue the optimization on this refined grid. We continue until we reach the desired level of accuracy. In 2D we can easily perform computations on grids of size up to 400×400 while in 3D we go up to $100 \times 100 \times 100$. Our current algorithm works well if the number of cells is not too large. When considering many phases, computations become more costly and memory costs become larger. It is possible to use techniques from [2] in order to address these issues. The idea is that, when dealing with partitioning or multiphase problems in the phase-field context, one could restrict the computation in a neighborhood of the significant part of each cell, for example $\{u_i > \delta\}$, for a given threshold $\delta > 0$. This could significantly reduce the computational and memory cost of the computations and could allow the use of the algorithm for many cells. However, this goes beyond the scope of this article.

Algorithm 1 Optimization procedure

Require: · M: initial discretization parameter ($M = 20$ in 2D, $M = 10$ in 3D)
· k: number of cells
· d: dimension
· $maxit$: maximal number of iterations
· $\varepsilon \in [1/M, 4/M]$: Modica-Mortola parameter
· tol: stopping criterion
· n: number of refinements chosen so that the final resolution is as good as we want (in our case $2^{n-1}M > 300$ in 2D and $2^{n-1}M > 100$ in 3D)

Initialize densities $u_1, .., u_k$ as k random vectors of M^d elements
step = 1
repeat
  if step>1 **then**
    $M \leftarrow 2M$
    interpolate linearly the previous densities $(u_i)_{i=1}^k$ on the new grid
  end if
  Run the LBFGS optimization procedure [34] with the following parameters
  · starting point $(u_i)_{i=1}^k$ (random for first step, interpolated from the previous optimization result for the next steps)
  · tolerances and maximum number of iterations: $tol = 10^{-8}$, $maxit = 10000$
  · function to optimize: (10). Value and gradient are computed at each iteration
  · pointwise upper and lower bounds $0 \leq u_i \leq 1$.
  The previous algorithm returns the optimized densities $(u_i)_{i=1}^n$
  step = step+1 (go to next step)
until step$> n$
  return the k density functions

Even if we choose to work on finite differences grid we may still compute α-Cheeger sets and α-Cheeger clusters corresponding to non-rectangular domains. For a general domain D we consider a rectangular box $D' \supset D$ on which we construct the finite differences grid. We set all functions involved in the computations to be equal to zero on grid points outside D and set the gradient to be equal to zero on the same points lying outside D. In this way the optimization is made only on points inside the desired domain D.

It is also possible to use a finite element framework in order to minimize (10) on general domains. In [4, Section 3] one can find a detailed presentation of such a finite element framework in the context of Modica-Mortola functionals. Once the mass and rigity matrices for the Lagrange P1 finite elements are obtained, all functionals needed in our computations can be expressed as vector matrix products.

Now we exemplify the use of the proposed algorithm for computing α-Cheeger sets, α-Cheeger clusters and optimal packings. We make available an implementation of the algorithm described above which can be found online at the following link: `https://github.com/bbogo/Cheeger_patch`. This implementation uses the finite element framework for the optimization of (10). As detailed below it is also possible, for convex domains, to compare the Cheeger sets found by minimizing (10) with the exact Cheeger sets obtained by using the representation formula provided by Kawohl and Lachand-Robert in [21].

• **Computation of α-Cheeger sets.** In this case, corresponding to $k = 1$, there is no need to use the penalization term. We optimize directly the non-penalized ratio between the Modica-Mortola ratio and the volume term with constraints $0 \leq u \leq 1$. Examples can be seen in Figure 2 for a domain in $\mathbb{R}^2$ and in Figure 3 for a domain in $\mathbb{R}^3$.

In order to test the accuracy of our method we compare our algorithm with an implementation of the Kawohl & Lachand-Robert explicit formula for finding Cheeger sets ($\alpha = 1$) associated to convex sets in 2D [21]. As can be seen in Figure 4 the relaxation algorithm we propose is quite precise. We represent with red the ε-level set of the result obtained when minimizing

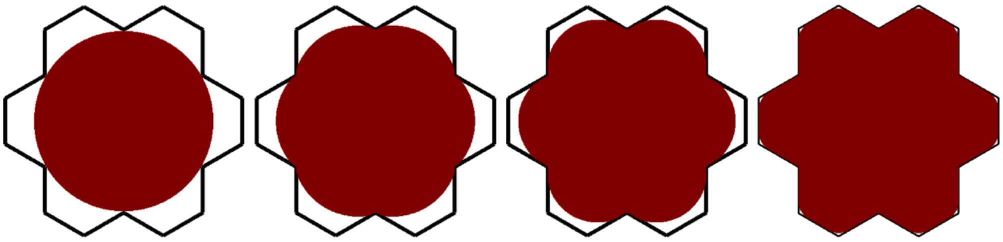


FIGURE 2. The α-Cheeger set for a non-convex set in $2D$, for $\alpha \in \{0.5001, 0.75, 1, 2\}$.

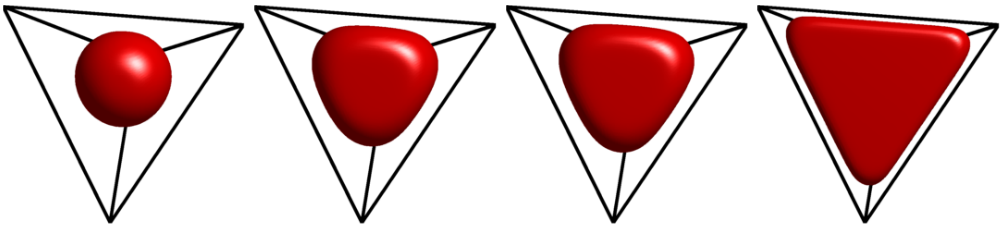
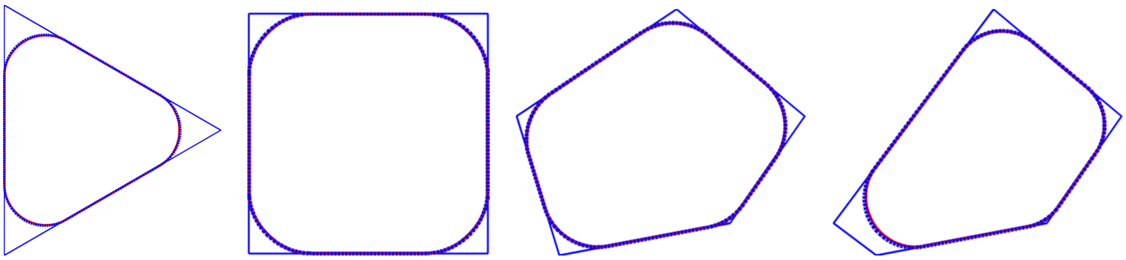


FIGURE 3. The α-Cheeger set for a regular tetrahedron in $3D$, for $\alpha \in \{0.667, 0.9, 1, 2\}$.



0.0040 0.0030 0.0030 0.0037

FIGURE 4. Comparison between results obtained when minimizing (10) (red) and the Kawohl & Lachand-Robert formula (dotted-blue). Relative errors for the Cheeger constants are also given.

(10) and with dotted blue the result obtained using the algorithm described in [21]. The value of ε used here is the same as the one used in the relaxed formulation (Algorithm 1). Choosing a level set corresponding to a larger value would correspond to a contour which does not touch the boundary of D, contrary to the known behavior of Cheeger sets. In the test cases presented below the results given by the two algorithms are almost indistinguishable. We denote by ω, ω_{ap} the analytic Cheeger set and the approximate one obtained when minimizing (10). In order to quantify the precision of our algorithm, we show the the relative errors for the corresponding Cheeger constants. These errors are computed using the following formula: $\frac{|\mathcal{H}^1(\omega_{ap})/|\omega_{ap}|-\mathcal{H}^1(\omega)/|\omega||}{\mathcal{H}^1(\omega)/|\omega|}$. The errors obtained are small and, as expected, working on finer meshes leads to better approximations both of the Cheeger sets and of the Cheeger constants.

• **Computation of α-Cheeger clusters.** Some examples of Cheeger clusters can be seen in Figure 5. One can notice immediately that the cells are not necessarily convex, for instance when D is a square and $n = 5$. The results in the periodic case are in accordance with results in [10, 9].

• **Computation of optimal packings.** To this aim, we exploit Theorem 1 combined with Proposition 8 with a suitable choice for p (see as well the beginning of Section 2.2). Precisely, we compute α-Cheeger clusters for α very close to $\frac{N-1}{N}$ and p very "large", which in our computations means at most 100. Choosing the parameter α close to $\frac{N-1}{N}$ forces the cells in the optimal configurations to be close to disks. We choose to use a p-norm approach since this regularizes the non-smooth problem of minimizing the maximal radius of a family of disks. The minimization

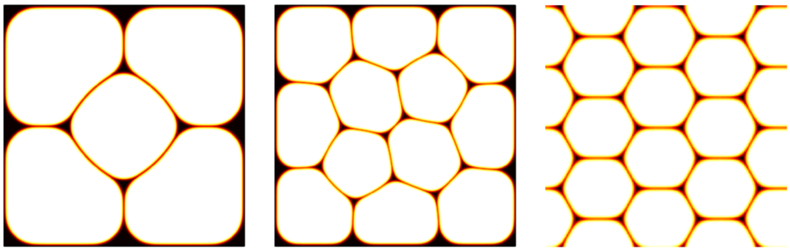


Figure 5. Cheeger clusters for problem (3) in a square: 5 cells, 12 cells, 16 cells (periodical).

of a p-norm instead of the ∞-norm is a natural idea, already used in [3] for the study of partitions of a domain which minimize the largest fundamental eigenvalue of the Dirichlet-Laplace operator.

We want to be able to quantify our results, so we use a local refinement procedure as a post-treatment. At the end of the optimization process we have access to the density functions associated to the α-Cheeger cells, which are approximately smoothed characteristic functions of disks. Using these density functions we approximate the precise location of the centers of the disks by computing the barycenter of the 0.5 level-set of each density function. Then we use a very basic local-optimization routine in order to get a precise description of the circle packing that can be compared with existing results in the literature. The algorithm just computes the pairwise distances between centers of the disks and between the centers and the boundary and uses Matlab's algorithm `fmincon` with the option `active-set` to perform a gradient-free local optimization of the current configuration. We note that the refinement algorithm is not at all adapted for solving *alone* the problem, given random starting points for the centers. It is only useful for locally optimizing the circle packing configuration once localization is obtained by our approximation procedure.

It is possible to apply this algorithm for computing both optimal circle packings for domains in $\mathbb{R}^2$ or optimal sphere packings for domains in $\mathbb{R}^3$. In the planar case, we present some computational results in Figure 6. In our test cases the numerical algorithm based on the Γ-convergence result combined with the post-treatment algorithm generally produce configurations which are comparable to the best known results in the literature. We recall that one of the first papers regarding the circle packings in a circle was authored by Kravitz in 1967 [22]. In this paper we can find a conjecture regarding the 19-circle packing in a disk. The optimality of this 19-packing, presented in Figure 6, was proved by Fodor in [17]. Extensive numerical results up to thousands of circles were performed and collected on the website `http://www.packomania.com/`, maintained by Eckard Specht. In all cases, we compared our results with best ones available, listed on the above cited website. The numerical algorithm manages to capture the right results in cases where the optimal circle packing configuration is unique and rigid, like the case of 19 disks in a circle or 28 disks in an equilateral triangle. Moreover, we are able to capture the best known results even in cases where the solution is not unique, like in the case when we have 18 disks in a circle. One may notice that the best known configuration for 18 disks inside a circle contains disks of the same radius as the best known configuration for 19 disks. Therefore, when dealing with the circle, removing a disk from the 19-disk optimal packing gives a solution for the 18 disks case. This shows that the optimal configuration is not unique in this case. Our relaxed algorithm finds a configuration which is equivalent to the best known configurations given by other algorithms. We may observe slight differences between configurations in the relaxed setting and the refined results, which are due to the fact that when working with densities cells are not constrained to be disks.

Some examples of computations of optimal spherical packings for domains in $\mathbb{R}^3$ are presented in Figure 7. In this case, we observe again a good convergence to the best known configurations.

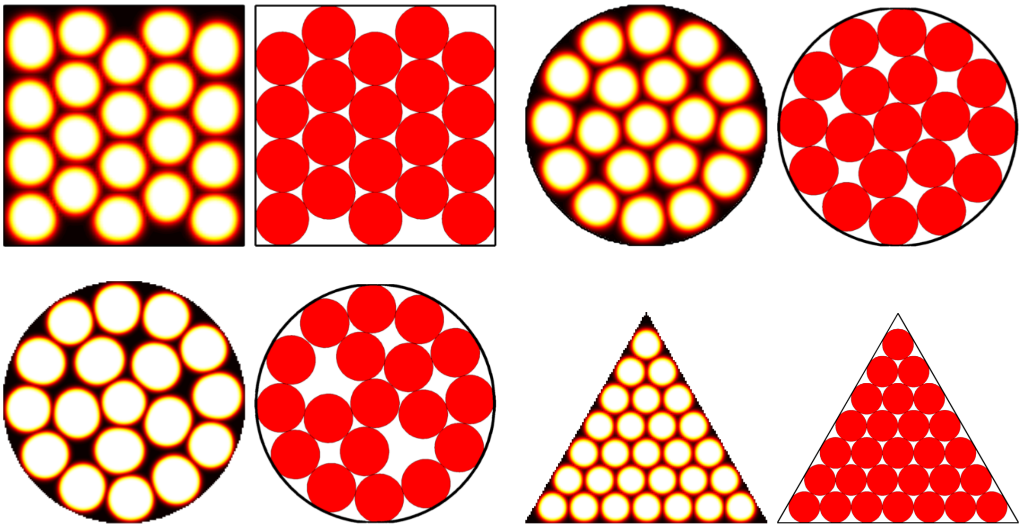


FIGURE 6. Circle packing examples in 2D for problem (6): density representation and local optimization.

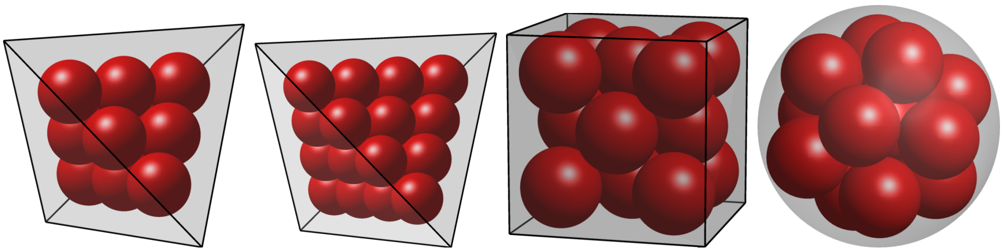


FIGURE 7. Sphere packing examples in 3D for problem (6).

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