Sendov's conjecture and the geometry of cubic polynomials

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Every polynomial is characterized by its complex roots, up to the leading coefficient. Moreover, since the complex numbers have a well established geometric structure, it is natural to investigate geometric aspects related to polynomial roots. In the following we will often identify a point in the plane with the associated complex number. Given a non-constant polynomial P of degree at least equal to two, consider the derivative P' and its roots, called critical points pf P. The well known Gauss-Lucas theorem says that the critical points lie in the convex hull of the roots of P. Various works in the literature search for relations between roots and critical points. Among these, there is the following famous conjecture by Sendov [11], solved for $\deg P \leq 8$ in [4] and for all sufficiently large degrees in [16].

Conjecture 1. *Suppose the roots of P lie in the unit disk. Then if $\mathbf{a}$ is one of these roots, there is a critical point at distace at most 1 from $\mathbf{a}$.*

There is one particular case where the connection between the roots of P and its critical points is made explicit geometrically. Given three noncolinear points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{C}$, consider the cubic poynomial $P(z) = (z - \mathbf{a})(z - \mathbf{b})(z - \mathbf{c})$, whose derivative $P'(z)$ has two roots $\mathbf{f}_1, \mathbf{f}_2$. It was first observed by Siebeck [15] and later on by Marden [10] that $\mathbf{f}_1, \mathbf{f}_2$ are the focal points of the Steiner inellipse associated to the triangle $\Delta\mathbf{abc}$, the unique ellipse tangent to the sides of $\Delta\mathbf{abc}$ at its midpoints. This result generated a lot of interest in the past years. Various elementary proofs exploiting aspects related to complex numbers were given in [2], [6], [9], [12], [13], [14]. A proof based solely on geometric arguments was given in [3].

When presenting Sendov's conjecture in [10], Marden already gave the geometric interpretation, that if $\Delta\mathbf{abc}$ is contained in the unit disk, then each one of the vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is at a distance at most one from the focal points $\mathbf{f}_1$ or $\mathbf{f}_2$ of the Steiner inellipse. A direct proof, using complex numbers may be found in [8, p. 22]. The goal of this note is to give a purely geometrical proof of Sendov's conjecture for cubic polynomials. Moreover, the sharpness of this result can be explored geometrically, investigating polynomials of high degree having only three distinct roots.

1. A SURPRISING PROPERTY RELATED TO THE STEINER INELLIPSE

In [1] the following identity is proved for any inellipse tangent to the sides of the triangle $\Delta\mathbf{abc}$ and having focal points $\mathbf{f}_1, \mathbf{f}_2$:

$$\frac{\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}} + \frac{\mathbf{bf}_1 \cdot \mathbf{bf}_2}{\mathbf{ba} \cdot \mathbf{bc}} + \frac{\mathbf{cf}_1 \cdot \mathbf{cf}_2}{\mathbf{ca} \cdot \mathbf{cb}} = 1. \tag{1}$$

The proof given in [1] is elegant and uses synthetic geometry arguments, by symmetrizing one of the focal points $\mathbf{f}_i$ about the sides of the triangle. For the Steiner inellipse, one has the stronger property that all three terms in (1) are equal

$$\frac{\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}} = \frac{\mathbf{bf}_1 \cdot \mathbf{bf}_2}{\mathbf{ba} \cdot \mathbf{bc}} = \frac{\mathbf{cf}_1 \cdot \mathbf{cf}_2}{\mathbf{ca} \cdot \mathbf{cb}} = \frac{1}{3}. \tag{2}$$

Proofs of (2), based on the Siebeck-Marden theorem, using relations between polynomial roots and critical points are rather straightforward and well known. Nevertheless, it is possible to prove (2) with purely geometric arguments, using only the basic properties of the Steiner inellipse, which we recall below.

Theorem 2. *1.* **(Reflection property)** *If the inellipse is tangent to the side* $\mathbf{ab}$ *at the interior point* $\mathbf{d}$ *then the angle bisector of* $\angle \mathbf{f}_1\mathbf{d}\mathbf{f}_2$ *is orthogonal to* $\mathbf{ab}$.

2. The focal ponts $\mathbf{f}_1, \mathbf{f}_2$ *of any inellipse are* isogonal conjugates *in* $\Delta\mathbf{abc}$.

3. An inellipse is uniquely determined by its center. In particular, the Steiner inellipse is the unique inellipse whose center coincides with the centroid of $\Delta\mathbf{abc}$.

Proofs of these facts can be found in many classical references. The proof of 1. is a simple consequence of the minimality of $\mathbf{xf}_1 + \mathbf{xf}_2$ for $\mathbf{x} \in \mathbf{ab}$, also known as Heron's problem. A geometric proof of 2. is recalled in [3]. The proof of 3. may be found in [5] or [3, Theorem 2].

In order to prove the sequence of equalities shown in (2) consider the reflection $\mathbf{f}_1'$ of $\mathbf{f}_1$ with respect to $\mathbf{ab}$ and denote by $\mathbf{d}$ the tangency point of the Steiner inellipse with $\mathbf{ab}$, as shown in Figure 1. Of course, $\mathbf{d}$ is the midpoint of $\mathbf{ab}$ and $\mathbf{f}_1', \mathbf{d}, \mathbf{f}_2$ are colinear, in view of the reflection property recalled in Theorem 2. Then one can write the following equalities regarding triangle areas:

$$S_{\Delta\mathbf{af}_1'\mathbf{f}_2} = S_{\Delta\mathbf{af}_1'\mathbf{d}} + S_{\Delta\mathbf{adf}_2} = S_{\Delta\mathbf{adf}_1} + S_{\Delta\mathbf{adf}_2} = 2S_{\Delta\mathbf{adg}},$$

where $\mathbf{g}$ is the midpoint of $\mathbf{f}_1, \mathbf{f}_2$, i.e. the center of the Steiner inellipse and the centroid of $\Delta\mathbf{abc}$. The last of the above area equalities comes from the fact that the corresponding triangles have a common basis $\mathbf{ad}$ and the average of the distances from $\mathbf{f}_1$ and $\mathbf{f}_2$ to $\mathbf{ad}$ is equal to the distance from $\mathbf{g}$ to $\mathbf{ad}$ (see Figure 1).

Since $\mathbf{d}$ is the midpoint of $\mathbf{ab}$ and $\mathbf{g}$ is the centroid, we conclude by observing that

$$S_{\Delta\mathbf{af}_1'\mathbf{f}_2} = 2S_{\Delta\mathbf{adg}} = S_{\Delta\mathbf{abg}} = \frac{1}{3}S_{\Delta\mathbf{abc}}.$$

Triangles $\Delta\mathbf{af}_1'\mathbf{f}_2$ and $\Delta\mathbf{abc}$ have equal angles in the vertex $\mathbf{a}$, since $\mathbf{f}_1, \mathbf{f}_2$ are isogonal conjugates. Therefore we have

$$\frac{1}{3} = \frac{S_{\Delta\mathbf{af}_1'\mathbf{f}_2}}{S_{\Delta\mathbf{abc}}} = \frac{\mathbf{af}_1' \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}} = \frac{\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}},$$

hence (2) holds.

Remark 3. It should be noted that (2) provides *yet another geometric proof* of the Siebeck-Marden theorem. Indeed, since $\mathbf{f}_1, \mathbf{f}_2$ are isogonal conjugates and (2) implies the equality $|\mathbf{a}-\mathbf{b}||\mathbf{a}-\mathbf{c}| = 3|\mathbf{a}-\mathbf{f}_1||\mathbf{a}-\mathbf{f}_2|$, we also have $(\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{c}) = 3(\mathbf{a}-\mathbf{f}_1)(\mathbf{a}-\mathbf{f}_2)$. Analogue identies are obtained for vertices $\mathbf{b}$ and $\mathbf{c}$. This it implies that the second degree polynomials

$$P'(z) = (z-\mathbf{a})(z-\mathbf{b}) + (z-\mathbf{b})(z-\mathbf{c}) + (z-\mathbf{c})(z-\mathbf{a})$$

and

$$Q(z) = 3(z-\mathbf{f}_1)(z-\mathbf{f}_2)$$

are equal for three distinct points $z \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and have the same leading coefficient. Therefore, $P'(z) = Q(z)$.

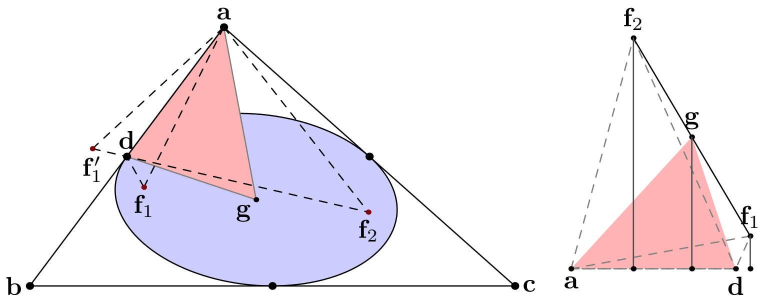

Figure 1. (left) The Steiner inellipse: symmetrize the focal point $\mathbf{f}_1$ with respect to $\mathbf{ab}$. (right) Proving that $2S_{\Delta\mathbf{agd}} = S_{\Delta\mathbf{af}_1\mathbf{d}} + S_{\Delta\mathbf{af}_2\mathbf{d}}$: observe that $2d(\mathbf{g}, \mathbf{ad}) = d(\mathbf{f}_1, \mathbf{ad}) + d(\mathbf{f}_2, \mathbf{ad})$.

2. GEOMETRIC PROOF OF SENDOV'S CONJECTURE FOR CUBIC POLYNOMIALS The geometric interpretation of Sendov's conjecture is the following: if $\mathbf{f}_1, \mathbf{f}_2$ are the focal points for the Steiner inellipse then at least one of the lengths $\mathbf{af}_1, \mathbf{af}_2$ is smaller than R, the circumradius of $\Delta\mathbf{abc}$. Observing that $\mathbf{f}_1, \mathbf{f}_2$ can get arbitrarily close and they coincide for an equilateral triangle, it is reasonable to attempt proving that a certain *mean* of $\mathbf{af}_1, \mathbf{af}_2$ is smaller than R.

Since we have precise information regarding the product of $\mathbf{af}_1$ and $\mathbf{af}_2$, let us first compare the geometric mean of $\mathbf{af}_1, \mathbf{af}_2$ with R. In view of (2) and the law of sines we have

$$\sqrt{\mathbf{af}_1 \cdot \mathbf{af}_2} = \sqrt{\frac{\mathbf{ab} \cdot \mathbf{ac}}{3}} = \sqrt{\frac{4 \sin \widehat{\mathbf{b}} \sin \widehat{\mathbf{c}}}{3}} R.$$

Since there exist triangles with angles $\widehat{\mathbf{b}} = \widehat{\mathbf{c}} = \pi/2 - \varepsilon$, the geometric mean can get arbitrarily close to $\frac{2}{\sqrt{3}}R$. Therefore, R cannot be an upper bound for this mean.

The next classical mean, smaller than the geometric one is the harmonic mean. This mean contains $\mathbf{af}_1 + \mathbf{af}_2$ at the denominator, therefore a lower bound is needed for this quantity. It is classical, and immediate to prove, that the median is at most equal to the average of the neighboring sides, implying that $\mathbf{af}_1 + \mathbf{af}_2 \geq 2\mathbf{ag}$. A classical proof of this fact constructs the parallelogram $\mathbf{af}_1\mathbf{a}'\mathbf{f}_2$ and uses the triangle inequality in $\Delta\mathbf{af}_1\mathbf{a}'$, showing moreover that equality can hold if and only if $\mathbf{a}, \mathbf{f}_1, \mathbf{f}_2$ are colinear. Denoting by $\mathbf{m}$ the midpoint of $\mathbf{bc}$ we have $\mathbf{ag} = \frac{2}{3}\mathbf{am}$ which, using again the law of sines $\mathbf{a} = 2R \sin \widehat{\mathbf{a}}$, gives

$$\min\{\mathbf{af}_1, \mathbf{af}_2\} \leq \frac{2\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{af}_1 + \mathbf{af}_2} \leq \frac{\mathbf{ab} \cdot \mathbf{ac}}{2\mathbf{am}} = \frac{2S_{\Delta\mathbf{abc}}}{2\mathbf{am} \cdot \sin \widehat{\mathbf{a}}} = \frac{h_{\mathbf{a}}}{\mathbf{am}} R, \tag{3}$$

where $h_{\mathbf{a}}$ is the length of the height of $\Delta\mathbf{abc}$ from vertex $\mathbf{a}$. Since the height always has a smaller length than the median, we are done. We have, therefore proved the following result.

Theorem 4. *The harmonic mean of* $\mathbf{af}_1$ *and* $\mathbf{af}_2$ *is at most equal to* R*. As a consequence, Sendov's conjecture holds for cubic polynomials.*

When presenting Conjecture 1 in [11], Marden talks about *extremal polynomials*, i.e. polynomials for which equality is attained in Sendov's estimate. Assuming that $\min\{\mathbf{af}_1, \mathbf{af}_2\} = R$, the sequence of inequalities in (3) becomes a sequence of equalities. The equality of the minimum and the harmonic mean implies $\mathbf{af}_1 = \mathbf{af}_2$.

The equality $\mathbf{af}_1 + \mathbf{af}_2 = \mathbf{ag}$ can hold only if $\mathbf{a}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{g}$ are colinear. Moreover, $h_{\mathbf{a}} = \mathbf{am}$, implying that $\Delta\mathbf{abc}$ is isosceles. Since $\mathbf{a}, \mathbf{f}_1, \mathbf{f}_2$ are colinear and $\mathbf{af}_1 = \mathbf{af}_2$ it follows that $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{g}$. This implies that the Steiner inellipse is a circle, therefore $\Delta\mathbf{abc}$ is equilateral. Thus, we arrive at a geometric proof of [11, Conjecture II] for cubic polynomials.

Theorem 5. *If $\min\{\mathbf{af}_1, \mathbf{af}_2\} = R$ then $\Delta\mathbf{abc}$ is equilateral. Polynomials of degree 3 for which equality is attained in Sendov's estimate have three equidistant roots on the unit disk.*

3. SHARPNESS OF SENDOV'S CONJECTURE It is well known that Sendov's result is sharp as the following well known examples illustrate:

- $P(z) = z^n - z$ has a root at the origin, while $P'(z)$ has n roots with modulus $n^{1/n} \to 1$ as $n \to \infty$.
- $P(z) = z^n - 1$ has n roots on the unit circle, while $P'(z)$ has all roots equal to 0.

However, it turns out that considering polynomials of the form $P(z) = (z-\mathbf{a})^m(z-\mathbf{b})^n(z-\mathbf{c})^p$, which in view of [3, 10] are also related to inscribed ellipses, one can find examples where the roots of $P'(z)$ *different from* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are at distance larger than 1 from at least one of the vertices of the triangle.

As already observed in [10], a polynomial of the form

$$P(z) = (z-\mathbf{a})^m(z-\mathbf{b})^n(z-\mathbf{c})^p \tag{4}$$

has only two critical points lying strictly inside $\Delta\mathbf{abc}$ which are the focal points of an inellipse. More generally, in [3] it was observed that for $\alpha, \beta, \gamma > 0$ the critical points of the logarithmic potential $L(z) = \alpha\log(z-\mathbf{a}) + \beta\log(z-\mathbf{b}) + \gamma\log(z-\mathbf{c})$ are the focal points of an inellipse dividing the sides of $\Delta\mathbf{abc}$ into ratios $\beta/\gamma, \gamma/\alpha, \alpha/\beta$. Conversely, given any inellipse $\mathcal{E}$, there exists a logarithmic potential $L(z)$ of the same form whose critical points are the focal points of $\mathcal{E}$.

Counterexample 1. Let $\Delta\mathbf{abc}$ be a non-equilateral triangle having two angles $\widehat{\mathbf{b}}, \widehat{\mathbf{c}}$ greater than $\pi/3$. The distance from the incenter to $\mathbf{a}$ is given by $4R\sin(\widehat{\mathbf{b}}/2)\sin(\widehat{\mathbf{c}}/2)$ and is greater than R in this case. Then there exist positive integers m, n, p such that the critical points $\mathbf{f}_1, \mathbf{f}_2$ of (4) different from $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in an ε neighborhood of the incenter, not containing the circumcenter. It is enough to consider m, n, p positive integers such that $\frac{m}{m+n+p}, \frac{n}{m+n+p}, \frac{p}{m+n+p}$ are approximations of the coefficients of the logarithmic potential $L(z)$ whose associated inellipse is the incircle. Therefore, for the vertex $\mathbf{a}$ and the considered inellipse we have $\min\{\mathbf{af}_1, \mathbf{af}_2\} > R$. It may be observed that if m, n, p give such an example, choosing exponents km, kn, kp, for any integer $k \geq 1$ in (4) produces the same critical points.

Counterexample 2. Furthermore, consider the case of only one multiple root, given by $P(z) = (z-\mathbf{a})^m(z-\mathbf{b})(z-\mathbf{c})$ for $m \geq 2$. The critical points of P are the focal points $\mathbf{f}_1^m, \mathbf{f}_2^m$ of an inellipse $\mathcal{E}_m$ tangent to the sides at points dividing the sides into ratios $m/1, 1/1, 1/m$. Let us observe the behavior of $\mathbf{f}_1^m, \mathbf{f}_2^m$ as $m \to \infty$. See Figure 2 for a graphical representation. The inellipse $\mathcal{E}_m$ is tangent to $\mathbf{bc}$ at its midpoint $\mathbf{m}$ and at $\mathbf{ab}, \mathbf{ac}$ at $\mathbf{p}_m, \mathbf{n}_m$, respectively. The points $\mathbf{n}_m, \mathbf{p}_m$ divide $\mathbf{ac}, \mathbf{ab}$ into segments having ratios $m/1$. It is classical that the line joining $\mathbf{b}$ to the midpoint $\mathbf{q}_m$ of $\mathbf{mp}_m$ passes through the center of $\mathcal{E}_m$. For a proof, it is enough to transform $\mathcal{E}_m$ into a circle via an affine transformation. In the same way the line going through $\mathbf{c}$ and the midpoint $\mathbf{r}_m$ of $\mathbf{mn}_m$ passes through the center of $\mathcal{E}_m$. Thus, the center $\mathbf{c}_m$ of $\mathcal{E}_m$ is given by $\mathbf{bq}_m \cap \mathbf{cr}_m$.

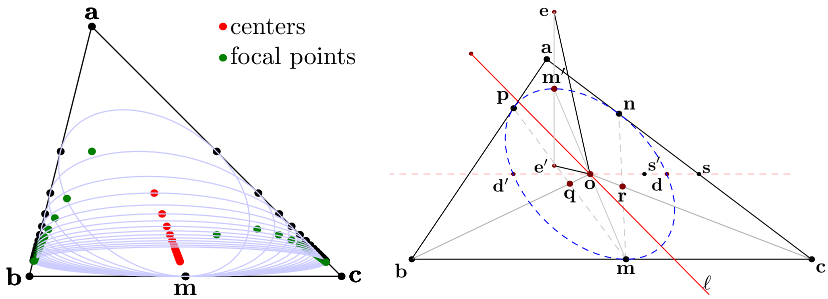

Figure 2. (left) Construction of $\mathcal{E}_m$ for $m = 1, ..., 15$. The centers $\mathbf{c}_m$ and focal points are also represented. The focal points converge to $\mathbf{b}$ and $\mathbf{c}$ as $m \to \infty$. (right) Constructing an inellipse starting from tangency points.

It is straightforward to observe that $\mathbf{c}_m$ converges to $\mathbf{m}$ and $\mathbf{f}_1^m, \mathbf{f}_2^m$ converge to $\mathbf{b}, \mathbf{c}$ as $m \to \infty$. When $\min\{\mathbf{ab}, \mathbf{ac}\} > R$, or equivalently, $\min\{\widehat{\mathbf{b}}, \widehat{\mathbf{c}}\} > \pi/6$, this produces a class of polynomials of arbitrarily large degree for which the distance from the only multiple root $\mathbf{a}$ to the critical points different from $\mathbf{a}$ is larger than R.

Therefore, there exist polynomials P of arbitrarily large degree with roots in the unit disk such that the distance from one zero of P to all critical points which are not roots is greater than 1.

Remark 6. For more geometric constructions related to ellipses [7, Chapter IV] is a great reference. All figures involving inellipses in this paper are constructed using the software Metapost and constructive ideas from this reference. For the sake of completeness, let us describe the steps for constructing an inellipse $\mathcal{E}$ starting from the tangency points $\mathbf{m} \in \mathbf{bc}, \mathbf{n} \in \mathbf{ac}, \mathbf{p} \in \mathbf{ab}$. It is classical that a necessary and sufficient condition for $\mathcal{E}$ to exist is that $\mathbf{am}, \mathbf{bn}, \mathbf{cp}$ are concurrent.

1. Let $\mathbf{q}$ be the midpoint of $\mathbf{mp}$ and $\mathbf{r}$ be the midpoint of $\mathbf{mn}$. Then the center of the inellipse is $\mathbf{o} \in \mathbf{bq} \cap \mathbf{cr}$.
2. Construct $\mathbf{m}'$ the symmetric of $\mathbf{m}$ through $\mathbf{o}$. Thus $\mathbf{mm}'$ is a *diameter* of $\mathcal{E}$.
3. Draw the line d through $\mathbf{o}$ parallel to $\mathbf{bc}$. Define $\mathbf{s} \in d \cap \mathbf{ac}$ and let $\mathbf{s}'$ be the intersection of d with the parallel to $\mathbf{mm}'$ through $\mathbf{n}$. Construct $\mathbf{d} \in d$ such that $\mathbf{od}^2 = \mathbf{os} \cdot \mathbf{os}'$. Then $\mathbf{d} \in \mathcal{E}$ [7, p. 107]. Construct $\mathbf{d}'$, the symmetric of $\mathbf{d}$ through $\mathbf{o}$. In this way we constructed another diameter $\mathbf{dd}'$ *conjugate* to $\mathbf{mm}'$.
4. Construct the segment $\mathbf{ee}'$, orthogonal to $\mathbf{dd}'$, having midpoint at $\mathbf{m}'$ such that $\mathbf{ee}' = \mathbf{dd}'$. The angle bisector of $\angle \mathbf{eoe}'$ is the principal axis of $\mathcal{E}$. [7, p. 111]
5. The lengths of the axes of the ellipse are given by $\mathbf{oe} + \mathbf{oe}'$ and $|\mathbf{oe} - \mathbf{oe}'|$.

The construction is depicted in Figure 2.

REFERENCES

1. P. R. Allaire, J. Zhou, and H. Yao, Proving a nineteenth century ellipse identity, *The Mathematical Gazette* **96** (2012) 161–165.
2. E. Badertscher, A simple direct proof of Marden's theorem, *Amer. Math. Monthly* **121** (2014) 547–548.
3. B. Bogosel, A geometric proof of the Siebeck-Marden theorem, *Amer. Math. Monthly* **124** (2017) 459–463.
4. J. E. Brown and G. Xiang, Proof of the Sendov conjecture for polynomials of degree at most eight, *J. Math. Anal. Appl.* **232** (1999) 272–292.
5. G. Chakerian, A distorted view of geometry, in *Mathematical Plums*, R. Honsberger, ed., The Dolciani Mathematical Expositions, vol. 4, Mathematical Association of America, Washington, D.C., 1979.

6. V. Dragović and M. Radnović, *Poncelet porisms and beyond*, Frontiers in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2011, Integrable billiards, hyperelliptic Jacobians and pencils of quadrics.
7. T. H. Eagles, *Constructive geometry of plane curves, with numerous examples*, Macmillan and Co, 1885.
8. Z. Jin and L. Zeng, Study on inscribed ellipse of triangle.
9. D. Kalman, An elementary proof of Marden's theorem, *Amer. Math. Monthly* **115** (2008) 330–338.
10. M. Marden, A note on the zeros of the sections of a partial fraction, *Bull. Amer. Math. Soc.* **51** (1945) 935–940.
11. ———, Conjectures on the critical points of a polynomial, *Amer. Math. Monthly* **90** (1983) 267–276.
12. D. Minda and S. Phelps, Triangles, ellipses, and cubic polynomials, *Amer. Math. Monthly* **115** (2008) 679–689.
13. S. Northshield, Geometry of cubic polynomials, *Math. Mag.* **86** (2013) 136–143.
14. J. L. Parish, On the derivative of a vertex polynomial, *Forum Geom.* **6** (2006) 285–288.
15. J. Siebeck, Ueber eine neue analytische behandlungweise der brennpunkte, *J. Reine Angew. Math.* **64** (1864) 175–182.
16. T. Tao, Sendov's conjecture for sufficiently high degree polynomials, 2020.

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