A Geometric Proof of the Siebeck–Marden Theorem

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Abstract. The Siebeck–Marden theorem relates the roots of a third degree polynomial and the roots of its derivative in a geometrical way. A few geometric arguments imply that every inellipse for a triangle is uniquely related to a certain logarithmic potential via its focal points. This fact provides a new direct proof of a general form of the result of Siebeck and Marden.

Given three noncollinear points \(a, b, c \in \mathbb{C}\), we can consider the cubic polynomial \(P(z) = (z - a)(z - b)(z - c)\), whose derivative \(P'(z)\) has two roots \(f_1, f_2\). The Gauss–Lucas theorem is a well-known result which states that given a polynomial \(Q\) with roots \(z_1, \ldots, z_n\), the roots of its derivative \(Q'\) are in the convex hull of \(z_1, \ldots, z_n\).

In the simple case where we have only three roots, there is a more precise result. The roots \(f_1, f_2\) of the derivative polynomial are situated in the interior of the triangle \(\Delta abc\) and they have an interesting geometric property: \(f_1\) and \(f_2\) are the focal points of the unique ellipse that is tangent to the sides of the triangle \(\Delta abc\) at its midpoints. This ellipse is called the Steiner inellipse associated to the triangle \(\Delta abc\). In the rest of this note, we use the term inellipse to denote an ellipse situated in a triangle that is tangent to all three of its sides. This geometric connection between the roots of \(P\) and the roots of \(P'\) was first observed by Siebeck (1864) \([12]\) and was reproved by Marden (1945) \([8]\). There has been substantial interest in this result in the past decade: see \([3],[5, pp. 137–140],[7],[9],[10],[11]\). Kalman \([7]\) called this result Marden’s theorem, but in order to give credit to Siebeck, who gave the initial proof, we call this result the Siebeck–Marden theorem in the rest of this note. Apart from its purely mathematical interest, the Siebeck–Marden theorem has a few applications in engineering. In \([2]\) this result is used to locate the stagnation points of a system of three vortices and in \([6]\) this result is used to find the location of a noxious facility location in the three-city case.

The proofs of the Siebeck–Marden theorem found in the references presented above are either algebraic or geometric in nature. The initial motivation for writing this note was to find a more direct proof, based on geometric arguments. The solution was found by answering the following natural question: Can we find two different inellipses with the same center? Indeed, let’s note that \((a + b + c)/3 = (f_1 + f_2)/2\), which means that the centers of ellipses having focal points \(f_1, f_2\) coincide with the centroid of the triangle \(\Delta abc\). The geometric aspects of the problem can be summarized in the following questions.

1. Is an inellipse uniquely determined by its center?
2. Which points in the interior of the triangle \(\Delta abc\) can be centers of an inellipse?
3. What are the necessary and sufficient conditions required such that two points \(f_1\) and \(f_2\) are the focal points of an inellipse?
4. Is there an explicit connection between the center of the inellipse and its tangency points?

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We give precise answers to all these questions in the next section, dedicated to the geometric properties of inellipses. Once these properties are established, we are able to prove a more general version of the Siebeck-Marden theorem. The proof of the original Siebeck-Marden result will follow immediately from the two main geometric properties of the critical points $f_1$, $f_2$.

• The midpoint of $f_1 f_2$ is the centroid of $\Delta abc$.
• The points $f_1$, $f_2$ are isogonal conjugates relative to triangle $\Delta abc$.

We recall that two points $f_1$, $f_2$ are isogonal conjugates relative to triangle $\Delta abc$ if the pairs of lines $(af_1, af_2)$, $(bf_1, bf_2)$, $(cf_1, cf_2)$ are symmetric with respect to the bisectors of the angles $\alpha$, $\beta$, $\gamma$, respectively.

1. GEOMETRIC PROPERTIES OF INELLIPSES. We start by answering the third question raised above: Which pairs of points can be the foci of an inellipse? In order to have an idea of what the expected answer is, we can look at the following general configuration. Suppose we have an ellipse $E$ with foci $f_1$, $f_2$ and an exterior point $a$. Consider the two tangents at $t_1$, at $t_2$ to $E$ that go through $a$. Then the angles $\angle t_1 af_1$ and $\angle t_2 af_2$ are equal.

A simple proof of this fact goes as follows. Construct $g_1$, $g_2$, the reflections of $f_1$, $f_2$ with respect to lines $at_1$, $at_2$, respectively (see Figure 1). Then the triplets of points $(f_1, t_2, g_2)$, $(f_2, t_1, g_1)$ are collinear. To see this, recall the result, often called Heron’s problem, which says that the minimal path from a point $a$ to a point $b$ that touches a line $\ell$ not separating $a$ and $b$ must satisfy the reflection angle condition. Now, it is enough to note that $f_1 g_2 = f_1 t_2 + f_2 t_2 = f_1 t_1 + f_2 t_1 = g_1 f_2$. Thus, triangles $\Delta af_1 g_2$, $\Delta ag_1 f_2$ are congruent, which implies that the angles $\angle t_1 a f_1$ and $\angle t_2 a f_2$ are equal.

As a direct consequence, the foci of an inellipse for $\Delta abc$ are isogonal conjugates relative to $\Delta abc$. The converse is also true and this results dates back to the work of Steiner [13] (see [1]).

**Theorem 1.** (Steiner). Suppose that $\Delta abc$ is a triangle.

1. If $E$ is an inellipse for $\Delta abc$ with foci $f_1$ and $f_2$, then $f_1$ and $f_2$ are isogonal conjugates relative to $\Delta abc$.
2. If $f_1$ and $f_2$ are isogonal conjugates relative to $\Delta abc$, then there is a unique inellipse for $\Delta abc$ with foci $f_1$ and $f_2$. 

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**Figure 1.** Left: Basic property of the tangents to an ellipse. Right: Construction of an inellipse starting from two isogonal conjugate points.
Proof. The proof of 1. was discussed above, so it only remains to prove 2. Consider the points \( x_1, x_2 \), the reflections of \( f_1, f_2 \) with respect to the lines \( \overline{ab} \) and \( \overline{bc} \) (see Figure 1 right). The construction implies that \( bf_1 = bx_1, \ b f_2 = bx_2 \) and \( \angle x_1 b f_2 = \angle f_1 b x_2 \), which, in turn, implies that \( x_1 f_2 = f_1 x_2 \). We denote their common value with \( m \). We denote \( a_1 = f_1 x_2 \cap \overline{bc} \) and \( c_1 = x_1 f_2 \cap \overline{ab} \). The construction of \( x_1, x_2 \) implies that \( f_1 a_1 + f_2 a_1 = f_1 x_2 = f_2 x_1 = f_1 c_2 + f_2 c_1 = m \). Heron’s problem cited above implies that \( a_1 \) is the point that minimizes \( x \mapsto f_1 x + f_2 x \) with \( x \in \overline{bc} \) and \( c_1 \) is the point that minimizes \( x \mapsto f_1 x + f_2 x \) with \( x \in \overline{ab} \).

Thus, the ellipse characterized by \( f_1 x + f_2 x = m \) is tangent to \( \overline{bc} \) and \( \overline{ab} \) in \( a_1 \) and, respectively, \( c_1 \). A similar argument proves that this ellipse is, in fact, also tangent to \( \overline{ac} \). The unicity of this ellipse comes from the fact that \( m \) is defined as the minimum of \( f_1 x + f_2 x \) where \( x \) is on one of the sides of \( \triangle abc \), and this minimum is unique and independent of the chosen side.

We are left to answer questions 1, 2, and 4. The first two questions were answered by Chakerian in [4] using an argument based on orthogonal projection. We provide a slightly different argument, which, in addition, gives us information about the relation between the barycentric coordinates of the center of the inellipse and its tangency points. In the proof of the following results we use the properties of real affine transformations of the plane.

Theorem 2. 1. An inellipse for \( \triangle abc \) is uniquely determined by its center.

2. The locus of the set of centers of inellipses for \( \triangle abc \) is the interior of the medial\(^1\) triangle for \( \triangle abc \).

3. If the center of the inellipse \( E \) is \( \alpha a + \beta b + \gamma c \), where \( \alpha, \beta, \gamma > 0 \) and \( \alpha + \beta + \gamma = 1 \), then the points of tangency of the inellipse divide the sides of \( \triangle abc \) in the ratios \( (1 - 2\beta)/(1 - 2\gamma) \), \( (1 - 2\gamma)/(1 - 2\alpha) \), \( (1 - 2\alpha)/(1 - 2\beta) \).

Proof. 1. We begin with the particular case where the inellipse \( E \) is the incircle with center \( 0 \). Suppose \( E' \) is another inellipse, with center \( 0 \), and denote by \( f_1, f_2 \) its focal points. We know that \( f_1, f_2 \) are isogonal conjugates relative to \( \triangle abc \) and the midpoint of \( f_1 f_2 \) is \( 0 \), the center of the inellipse. Thus, if \( f_1 \neq f_2 \), then \( \overline{ao} \) is at the same time a median and a bisector in triangle \( \overline{af_1 f_2} \). This implies that \( \overline{ao} \perp f_1 f_2 \). A similar argument proves that \( \overline{bo} \perp f_1 f_2 \) and \( \overline{co} \perp f_1 f_2 \). Thus \( a, b, c \) all lie on a line perpendicular to \( f_1 f_2 \) in \( 0 \), which contradicts the fact that \( \triangle abc \) is nondegenerate. The assumption \( f_1 \neq f_2 \) leads to a contradiction, and therefore we must have \( f_1 = f_2 \), which means that \( E' \) is a circle and \( E' = E \).

Consider now the general case. Suppose that the inellipses \( E, E' \) for \( \triangle abc \) have the same center. Consider an affine mapping \( h \) that maps \( E \) to a circle. Since \( h \) maps ellipses to ellipses and preserves midpoints, the image of our configuration by \( h \) is a triangle where \( h(E) \) is the incircle and \( h(E') \) is an inscribed ellipse with the same center. This case was treated in the previous paragraph and we must have \( h(E') = h(E) \). Thus \( E = E' \).

2. To find the locus of the centers of inellipses for \( \triangle abc \), it is enough to see which barycentric coordinates are admissible for the incircle of a general triangle. We recall that the barycentric coordinates of a point \( p \) are proportional to the areas of the triangles \( \triangle pbc, \triangle pca, \triangle pab \), and their sum is chosen to be 1. Thus, barycentric coordinates are preserved under affine transformations. The barycentric coordinates of the center of an inellipse with respect to \( a, b, c \) are the same as the barycentric coordinates

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\(^1\)The medial triangle is the triangle formed by the midpoints of the edges of a triangle.
coordinates of the incenter of the triangle \( h(a), h(b), h(c) \). As before, \( h \) is the affine transformation which transforms the ellipse into a circle. Conversely, if the barycentric coordinates of the incenter 0 with respect to \( \Delta a'b'c' \) are \( x, y, z \), then we consider the affine transformation that maps the triangle \( \Delta a'b'c' \) onto the triangle \( \Delta abc \). The circle is transformed into an inellipse, with center having barycentric coordinates \( x, y, z \).

The barycentric coordinates of the incenter have the form

\[
\begin{align*}
x &= \frac{u}{u + v + w}, \\
y &= \frac{v}{u + v + w}, \\
z &= \frac{w}{u + v + w},
\end{align*}
\]

where \( u, v, w \) are the lengths of the sides of \( \Delta a'b'c' \). Thus, we can see that \( x + y + z = 1 \) and \( x < y + z, y < z + x, z < x + y \). One simple consequence of these relations is the fact that \( x, y, z < 1/2 \). Furthermore, since

\[
\begin{align*}
x &= \frac{\text{Area}(\Delta b'c')}{\text{Area}(\Delta a'b'c')}, \\
y &= \frac{\text{Area}(\Delta c'a')}{\text{Area}(\Delta a'b'c')}, \\
z &= \frac{\text{Area}(\Delta a'b')}{\text{Area}(\Delta a'b'c')},
\end{align*}
\]

we can see that the previous relations for \( x, y, z \) are satisfied if and only if 0 is in the interior of the medial triangle for \( \Delta a'b'c' \). Thus, the locus of the center of an inscribed ellipse is the interior of the medial triangle.

3. If the center of the inellipse \( E \) is \( \alpha a + \beta b + \gamma c \) with \( \alpha + \beta + \gamma = 1 \), then consider an affine map \( h \) that transforms \( E \) into a circle. Let \( \Delta a'b'c' \) be the image of \( \Delta abc \) by \( h \). It is known that \( \alpha, \beta, \gamma \) are proportional with the sidelengths of the triangle \( \Delta a'b'c' \). Thus, the tangency points of \( h(E) \) with respect to \( \Delta a'b'c' \) divide its sides into ratios

\[
\frac{\alpha + \gamma - \beta}{\alpha + \beta - \gamma}, \quad \frac{\alpha + \beta - \gamma}{\beta + \gamma - \alpha}, \quad \frac{\beta + \gamma - \alpha}{\alpha + \gamma - \beta}.
\]

The affine map \( h \) does not modify the ratios of collinear segments, thus, \( E \) divides the sides of \( \Delta abc \) into the same ratios.

2. INELLIPSES AND CRITICAL POINTS OF LOGARITHMIC POTENTIALS. The properties of inellipses described above allow us to state and prove a result which is a bit more general than Siebeck–Marden theorem. In fact, every inellipse relates to the critical points of a logarithmic potential of the form

\[
L(z) = \alpha \log(z - a) + \beta \log(z - b) + \gamma \log(z - c).
\]

The following result gives a precise description of this connection.

**Theorem 3.** Given \( \Delta abc \) and \( \alpha, \beta, \gamma > 0 \) with \( \alpha + \beta + \gamma = 1 \), the function \( L(z) = \alpha \log(z - a) + \beta \log(z - b) + \gamma \log(z - c) \) has two critical points \( f_1 \) and \( f_2 \). These critical points are the foci of an inellipse that divides the sides of \( \Delta abc \) into ratios \( \beta/\gamma, \gamma/\alpha, \alpha/\beta \).

Conversely, given an inellipse \( E \) for \( \Delta abc \), there exists a function of the form \( L(z) \) as above whose critical points \( f_1, f_2 \) are the foci of \( E \).

**Proof.** Denote by \( f_1, f_2 \) the roots of

\[
L'(z) = \frac{\alpha}{z - a} + \frac{\beta}{z - b} + \frac{\gamma}{z - c},
\]
which means that $f_1$, $f_2$ are roots of

\[ z^2 - (\alpha(b + c) + \beta(a + c) + \gamma(a + b))z + \alpha \beta c + \beta c \alpha + \gamma a b = 0. \]

Without loss of generality, we can suppose that $a = 0$ and that the imaginary axis is the bisector of the angle $\angle \overline{b}ac$ (equivalently $bc < 0$). In this case we have $f_1 f_2 = \alpha b c < 0$, and thus the imaginary axis is the bisector of the angle $\angle f_1 af_2$. Repeating the same argument for $b$ and $c$, we deduce that $f_1$, $f_2$ are isogonal conjugates relative to $\triangle_1 abc$. Steiner’s result (Theorem 1) implies that $f_1$, $f_2$ are the foci of an inellipse $\mathcal{E}$ for $\triangle_1 abc$. The center of this inellipse has barycentric coordinates

\[ 0 = \frac{\beta + \gamma}{2} a + \frac{\alpha + \gamma}{2} b + \frac{\alpha + \beta}{2} c, \]

which, according to Theorem 2, implies that $\mathcal{E}$ is the unique inellipse for $\triangle_1 abc$, which divides the sides of $\triangle_1 abc$ in ratios $\beta/\gamma, \gamma/\alpha, \alpha/\beta$.

Conversely, given an inellipse $\mathcal{E}$ for $\triangle_1 abc$, its tangency points must be of the form $\beta/\gamma, \gamma/\alpha, \alpha/\beta$ for some $\alpha, \beta, \gamma > 0$, $\alpha + \beta + \gamma = 1$. We choose $L(z) = \alpha \log(z - a) + \beta \log(z - b) + \gamma \log(z - c)$ and, according to the first part of the proof, the critical points $f_1, f_2$ of $L'(z)$ are the foci of an ellipse $\mathcal{E}'$ that divides the sides of $\triangle_1 abc$ into ratios $\beta/\gamma, \gamma/\alpha, \alpha/\beta$. This means that $\mathcal{E} = \mathcal{E}'$ and $L(z)$ is the associated logarithmic potential.

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