Wishart Stochastic Volatility: Asymptotic Smile and Numerical Framework

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08/03/2010
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Conceptual stakes

1. Requirement of an efficient model which fits market data (short and long maturities)
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2. Study of the influence of parameters
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Let $X_1, \ldots, X_n$ be $n$ independent Gaussian vectors in $\mathbb{R}^p$ such as $X_i \sim \mathcal{N}(0, \Sigma)$. The law of the $p \times p$ random matrix $S = \sum_{i=1}^n X_i X_i^T$ is called *Wishart distribution*.

For $X_i \sim \mathcal{N}(\mu, \Sigma)$, the law of $S = \sum_{i=1}^n X_i X_i^T$ is called *non central Wishart distribution*. 
Wishart process (M.F. Bru)

Let \( \{ W_t, t \geq 0 \} \) denote a \( n \times n \) matrix-valued Brownian motion under the probability measure \( Q \). The matrix-valued process is said to be a Wishart process if it satisfies the following diffusion equation

\[
\begin{align*}
dV_t &= (\beta Q^T Q + MV_t + V_t M^T) dt + \sqrt{V_t} dW_t Q + Q^T dW_t^T \sqrt{V_t}, \\
V_0 &= v_0.
\end{align*}
\]

where \( Q \in GL_n(\mathbb{R}) \) is a \( n \times n \) invertible matrix, \( M \) is a \( n \times n \) nonpositive matrix, \( v_0 \in \tilde{S}_n^+ \) is a strictly nonnegative symmetric matrix and \( \beta \) a real such as \( \beta > (n - 1) \).
The infinitesimal generator of a Wishart process is given by

\[ \mathcal{L}^V = \text{Tr}[(\beta Q^T Q + MV + VM^T)D + 2VDQ^T QD] \]

where \( D \) denotes the operator \( \left( \frac{\partial}{\partial V_{ij}} \right)_{1 \leq i,j \leq n} \).
Laplace transform (C. Gourieroux and al.)

Let \( \Psi_t(u, \Theta) = \mathbb{E}[\exp(-Tr(\Theta V_{t+h}))|V_t] \) define the Laplace transform of the Wishart process. For \( \Theta \in S_n^+ \) and \( t, h \geq 0 \),

\[
\Psi_t(u, \Theta) = \exp \left[ -\operatorname{Tr} \left( \Delta(h)^T \Theta (I_n + 2\Sigma(h)\Theta)^{-1} \Delta(h) V_t \right) \right] \frac{\exp(hM) \beta^2}{(\det[I_n + 2\Sigma(h)\Theta])^{\frac{\beta}{2}}},
\]

with

\[
\Delta(h) = \exp(hM), \quad \Sigma(h) = \int_0^h \Delta(s)Q^TQ\Delta(s)^T ds.
\]
Determinant dynamic (M.F Bru)

Let \( \{\zeta_t = (\zeta_{ij}^t)_{1 \leq i, j \leq n}\} \) be a matrix process taking values in \( S_n(\mathbb{R}) \) whose components are continuous semi-martingales. \( U_t \in O_n(\mathbb{R}) \) is defined such as \( U^T_t \zeta_t U_t = \text{diag}(\lambda_1^t, \ldots, \lambda_n^t) \) where \( \{\lambda_i^t, 1 \leq i \leq n\} \) are the eigenvalues of \( \zeta_t \).

Let \( \{A_t = (A_{ij}^t)_{1 \leq i, j \leq n}, t \geq 0\} \) be the process defined as

\[
A_t = \int_0^t U_s^T d\zeta_s U_s, \forall t \geq 0 \quad \text{such as} \quad d < A_{ij}, A_{ji} >= \Gamma_{ij}^t dt.
\]

Then, the dynamic of the eigenvalues can be written as

\[
d\lambda_i^t = dM_i^t + dJ_i^t, \quad dJ_i^t = \sum_{j \neq i} \frac{1}{\lambda_i^t - \lambda_j^t} \Gamma_{ij}^t dt + d\Theta_i^t.
\]

where \( dM_i^t \) and \( d\Theta_i^t \) are respectively the martingale part and the finite variation part of \( dA_{ii}^t \).
Determinant dynamic of the Wishart process

Let us denote $\tilde{Q}_t = U_t^T Q U_t$ and $\tilde{M}_t = U_t^T M U_t$ and $Z_t = \int_0^t U_s^T dW_s U_s$.

$$d\lambda_t^i = \left[ \beta (\tilde{Q}_t^T \tilde{Q}_t)^{ii} + 2\lambda_t^i \tilde{M}_t^{ii} + \sum_{j \neq i} \frac{(\lambda_t^i \tilde{Q}_t^T \tilde{Q}_t)^{ij} + (\lambda_t^j \tilde{Q}_t^T \tilde{Q}_t)^{ii}}{\lambda_t^i - \lambda_t^j} \right] dt + 2 \sqrt{\lambda_t^i} \sum_{k=1}^n \tilde{Q}_t^{ki} dZ_t^{ik}.$$  

$$d(\text{det}(V_t)) = \text{det}(V_t) \sum_{i=1}^n \frac{d\lambda_t^i}{\lambda_t^i}$$  

By using the trace invariance by a base change, the dynamic of the determinant is obtained by

$$\frac{d(\text{det}(V_t))}{\text{det}(V_t)} = [(\beta - n+1) \text{Tr}(V_t^{-1} Q^T Q) + 2 \text{Tr}(M)] dt + 2 \text{Tr}(\sqrt{V_t^{-1}} dW_t Q).$$
Change of probability measure

Taking the floor allows to write $\beta = K + 2\nu$ with $K = \lfloor \beta \rfloor \geq n + 1$ and $\nu$ a real number such as $0 \leq \nu \leq \frac{1}{2}$.

Let's give a change of probability measure $\mathbb{Q}|_{\mathcal{F}_T}^*$ in order to change the generalized Wishart diffusion into the simple one where $K$ is an integer.

If $f_T(\mathbb{Q}, \mathbb{Q}^*) = \frac{d\mathbb{Q}|_{\mathcal{F}_T}}{d\mathbb{Q}^*|_{\mathcal{F}_T}}$ defines the Radon-Nikodym derivative of $\mathbb{Q}|_{\mathcal{F}_T}$ with respect to $\mathbb{Q}^*|_{\mathcal{F}_T}$, then

$$f_T(\mathbb{Q}, \mathbb{Q}^*) = \frac{\text{det}(V_T)}{\text{det}(V_0)} \nu^{\frac{1}{2}} \exp[-\nu T \text{Tr}(M)] \exp \left[ -\frac{\nu}{2} (K + \nu - n - 1) \int_0^T \text{Tr}(V_s^{-1}Q^TQ)ds \right].$$
Proof

A probability measure $\mathbb{Q}^*$ can be specified with an exponential martingale (Yor and al.)

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*} \bigg|_{\mathcal{F}_T} = \exp \left[ \nu \int_0^T \text{Tr}(\sqrt{V_s^{-1}} dW_s^* Q) - \frac{\nu^2}{2} \int_0^T \text{Tr}(V_s^{-1} Q^T Q) ds \right].$$

Therefore, this expression suggests to define a new process $W^*$ by

$$W_t^* = W_t + \nu \int_0^t \sqrt{V_s^{-1}} Q^T ds.$$

Girsanov theorem $\implies W^*$ is a matrix-valued Brownian motion under the probability measure $\mathbb{Q}^*$.

$$dV_t = (KQ^T Q + MV_t + V_t M^T) dt + \sqrt{V_t} dW_t^* Q + Q^T (dW_t^*)^T \sqrt{V_t}.$$
Proof 2

The Radon-Nikodym derivative can be simplified using the determinant dynamic. Indeed, we have

$$\log \left[ \frac{\det(V_t)}{\det(V_0)} \right] = 2T \operatorname{Tr}(M) + (K-n-1) \int_0^T \operatorname{Tr}(V_s^{-1}Q^TQ) \, ds + 2 \int_0^T \operatorname{Tr}(\sqrt{V_s^{-1}}dW_s^*Q).$$

Finally, the change of the probability measure can be obtained by

$$dQ|\mathcal{F}_T = \frac{\det(V_T)}{\det(V_0)} \frac{\nu}{2} \exp[-\nu T \operatorname{Tr}(M)] \exp[-\frac{\nu}{2} (K + \nu - n - 1) \int_0^T \operatorname{Tr}(V_s^{-1}Q^TQ) \, ds] \cdot dQ^*|\mathcal{F}_T,$$

which completes the proof.
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Study of the Wishart model
Presentation of the Wishart model

Wishart model (J. Da Fonseca and al.)

The underlying volatility is given by the trace of a Wishart process. Under the risk neutral probability measure, the asset dynamic is given by the following expression

\[
\frac{dS_t}{S_t} = r dt + \text{Tr}[\sqrt{V_t}(dW_t R + dZ_t \sqrt{I_n - RR^T})]
\]

\[
dV_t = (\beta Q^T Q + MV_t + V_t M^T)dt + \sqrt{V_t}dW_t Q + Q^T dW_t^T \sqrt{V_t}
\]

\[V_0 = v_0.\]

where \(\{V_t, t \geq 0\}\) is a Wishart matrix-valued process as introduced in the previous section, \(r\) is the interest rate considered constant, \(S_t\) the price of the asset at date \(t\), \(R\) a matrix such as \(\rho(R) \leq 1\), \(W, Z\) are independent matrix-valued Brownian motions.
Extension of multidimensional Heston model

\[ dV_t^{ii} = \left( \beta (Q^{ii})^2 + 2M^{ii} V_t^{ii} \right) dt + 2Q^{ii} \sum_{k=1}^{n} \sqrt{V_t^{ki}} \, dW_t^{ki}. \]

\[ d < V^{ii}, V^{ii} >_t = 4(Q^{ii})^2 V_t^{ii}. \]

Let \( B = (B^1, \ldots, B^n) \) a vector of \( n \) independent Brownian motions. Then,

\[ dV_t^{ii} = \left( \beta (Q^{ii})^2 + 2M^{ii} V_t^{ii} \right) dt + 2Q^{ii} \sqrt{V_t^{ii}} dB_t^{i}. \]

This particular case underlines the fact that the Wishart model extends the multifactor Heston model.
Stochastic correlation

\[
\frac{dS_t}{S_t} = rdt + \sqrt{\text{Tr}(V_t)}dB_t
\]

\[
d\text{Tr}(V_t) = (\beta \text{Tr}(Q^TQ) + 2\text{Tr}(MV_t))dt + 2\sqrt{\text{Tr}(Q^TQV_t)}(\rho_t dB_t + \sqrt{1 - \rho_t^2}d\tilde{B}_t)
\]

\[
\rho_t = \frac{\text{Tr}(R^TQV_t)}{\sqrt{\text{Tr}(V_t)}\sqrt{\text{Tr}(Q^TQV_t)}}
\]

with \(B, \tilde{B}\) independent Brownians.
Skew, convexity and slope at the money (V. Durrlemann)

Let us suppose that we are given adapted processes $a, \tilde{a}, b, c$ such that there exists a strictly positive solution to the following stochastic differential equation

$$d\sigma_t^2 = (b_t + a_t^2 + \sigma_t^2(a_t + c_t))dt - 2\sigma_t(a_t dW_t + \tilde{a}_t d\tilde{W}_t).$$

where a given initial condition $\sigma_0$ and such that

$$d < a, \mathcal{W} > t = -\sigma_t \left( \frac{3c_t}{2} + \frac{3a_t^2}{4\sigma_t^2} - \frac{\tilde{a}_t^2}{\sigma_t^2} \right) dt$$

Further let

$$S_t = S_0 \exp \left( \int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right)$$

be the stock process and $\Sigma_t(T, K)$ be the implied volatility.
Then

\[ \Sigma_t(t, S_t) = \sigma(t) \]

\[ \frac{\partial \Sigma_t}{\partial K}(t, S_t) = -\frac{a_t}{2S_t \Sigma_t(t, S_t)} \]

\[ \frac{\partial \Sigma_t}{\partial T}(t, S_t) = \frac{b_t}{4 \Sigma_t(t, S_t)} \]

\[ \frac{\partial^2 \Sigma_t}{\partial K^2}(t, S_t) = \frac{1}{2S_t^2 \Sigma_t(t, S_t)} \left( a_t + c_t - \frac{a_t^2}{2 \Sigma_t(t, S_t)^2} \right) \]
For the expression of the skew at the money ($K = S_t$) in the Wishart model, the application of this theorem with

$$a_t = -\frac{Tr(R^T Q V_t)}{Tr(V_t)}$$

provides the following expression

$$\frac{\partial \Sigma_t}{\partial K}(t, S_t) = \frac{Tr(R^T Q V_t)}{2 S_t (Tr(V_t))^\frac{3}{2}}.$$

Asymptotic skew ATM
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Simulation of the Wishart process

\[ dX_{k,t} = MX_{k,t}dt + QTdW_{k,t}. \]

The process defined by \( V_t = \sum_{k=1}^{\beta} X_{k,t}X_{k,t}^T \) follows

\[ dV_t = (\beta Q^TQ + MV_t + V_tM^T)dt + \sqrt{V_t}dW_tQ + QTdW_t^T \sqrt{V_t}, \]

with \( W \) a matrix-valued Brownian motion. \( W \) is determined by

\[ \sqrt{V_t}dW_t = \sum_{k=1}^{\beta} X_{k,t}dW_{k,t}^T. \]

Finally, for \( \beta \) an integer, the Wishart process can be discretized

\[ \bar{V}_{t_i} = \sum_{k=1}^{\beta} \bar{X}_{k,t_i} \bar{X}_{k,t_i}^T. \]
Rewriting the Wishart model under the probability $Q^*$

For the simulation, if $beta$ is not an integer, we use the change of probability.

\[
\frac{dS_t}{S_t} = [r - \nu \text{Tr}(Q^T R)]dt + \text{Tr}[\sqrt{V_t}(dW_t^* R + dZ_t \sqrt{I_n - RR^T})].
\]

\[
dV_t = (KQ^T Q + MV_t + V_t M^T)dt + \sqrt{V_t}dW_t^* Q + Q^T (dW_t^*)^T \sqrt{V_t}.
\]
Simulation of the asset

\[
\log(S_{t_i}) = \log(S_{t_{i-1}}) + \left( r\Delta t - \frac{1}{2} \int_{t_{i-1}}^{t_i} \text{Tr}(V_s)ds \right) \\
+ \text{Tr} \left[ \int_{t_{i-1}}^{t_i} \sqrt{V_s}dW_s R + \int_{t_{i-1}}^{t_i} \sqrt{V_s}dZ_s \sqrt{I_n - RR^T} \right].
\]

with

\[
\int_{t_{i-1}}^{t_i} \sqrt{V_s}dW_s = \sum_{k=1}^{\beta} \int_{t_{i-1}}^{t_i} X_{k,s} dW_{k,s} = \sqrt{\Delta t} \sum_{k=1}^{\beta} X_{k,t_i \epsilon_k,i}.
\]

\[
\int_{t_{i-1}}^{t_i} V_s ds = \bar{V}_{t_{i-1}} \Delta t.
\]

\[
\text{Tr} \left[ \int_{t_{i-1}}^{t_i} \sqrt{V_s}dZ_s \sqrt{I_n - RR^T} \right] = \sqrt{\Delta t} \sqrt{\text{Tr}(V_{t_{i-1}}(I_n - RR^T))} Z_i.
\]
Therefore, the price of an european option at the maturity $T$ with a payoff $f$ is evaluated by

$$
\pi_0 = \mathbb{E}_Q[\exp(-rT)f(S_T)]
$$

$$
= \exp[-(r + \nu T \text{Tr}(M))]\mathbb{E}_{Q^*}\left\{ \left( \frac{\text{det}(V_T)}{\text{det}(V_0)} \right)^{\frac{\nu}{2}} \exp\left[ -\frac{\nu}{2} (K + \nu + n - 1) \int_0^T \text{Tr}(V_s^{-1}Q^TQ)ds \right] f(S_T) \right\}.
$$
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Whishart model belongs to the Affine Class Models where we can find a closed formula for Fourier transform. This transform is used to find options prices by calculating inverse Fourier transform by FFT.

The price at the date $t$ of a Call with Strike $K = \log(k)$ and maturity $T$ is given by

$$c_t(T, K) = \exp[-r(T - t)] \mathbb{E}[(\exp(Y_T) - \exp(k))_+ | \mathcal{F}_t]$$
Call pricing by FFT 2

By applying a Fubini integration theorem, the Fourier transform of the modified price is given by

\[
\Psi_t(T, \nu) = \int_{-\infty}^{\infty} \exp(i\nu k) c_t(T, k) dk
\]

\[
= \exp[-r(T - t)] E \left[ \int_{-\infty}^{Y_T} \exp[i\nu k](\exp(Y_T) - \exp(k)) dk \right] |F_t|
\]

\[
= \exp[-r(T - t)] \frac{\Phi_t(T, \nu - i)}{(1 + i\nu)(i\nu)}
\]

So we can obtain the price of the call by inverse Fourier transform

\[
c_t(T, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\nu k) \Psi_t(T, \nu) d\nu
\]
Expression of the Fourier transform

The expression of the Fourier transform of \((V, Y, \int V)\) defined by

\[
\Psi_t(T, \theta, \gamma, \delta) = \mathbb{E}[e^{Tr(\theta V_t)+\gamma Y_T+\delta U_T}|\mathcal{F}_t]
\]

can be explicitly calculated.

\[
\Psi_t(T, \theta, \gamma, \delta) = \exp[Tr[A(T-t)V_t]+B(T-t)Y_t+C(T-t)U_t+D(T-t)],
\]

where \(A, B, C\) and \(D\) are solutions of the following Riccati equations

\[
\begin{align*}
A'(\tau) &= \left(\frac{\gamma(\gamma-1)}{2} - \delta\right) I_n + A(\tau)M + (M^T + 2\gamma R^T Q)A(\tau) + 2A(\tau)Q^T QA(\tau) \\
B(\tau) &= \gamma, \quad C(\tau) = \delta. \\
D'(\tau) &= r\gamma + \beta Tr[Q^T QA(\tau)].
\end{align*}
\]
Expression of the Fourier transform (J. Da Fonseca and al.)

The expression of the Fourier transform of log-price defined by
\( \Phi_t(T, \gamma) = \mathbb{E}[\exp(i\gamma Y_T)|\mathcal{F}_t] \) can be explicitly calculated.

\[
\Phi_t(T, \gamma) = \exp[Tr[A(T - t)V_t] + B(T - t)Y_t + C(T - t)],
\]

where \( A \) and \( C \) are solutions of the following Riccati equations

\[
\begin{align*}
B(\tau) &= i\gamma, \\
A'(\tau) &= \frac{i\gamma(i\gamma - 1)}{2} I_n + A(\tau)M + (M^T + 2i\gamma R^T Q)A(\tau) + 2A(\tau)Q^T QA(\tau), \\
C'(\tau) &= ir\gamma + \beta Tr[Q^T QA(\tau)].
\end{align*}
\]
Expression of the Fourier transform 2

By using the linearization method, the solutions are given by

\[ A(\tau) = F(\tau)^{-1} G(\tau). \]
\[ C(\tau) = i \gamma \tau [r - \beta \text{Tr}(R^T Q)] - \frac{\beta}{2} [\text{Tr}(M) \tau + \log(\text{det}F(\tau))]. \]

with

\[
\begin{bmatrix}
G(\tau) & F(\tau)
\end{bmatrix} = \begin{bmatrix} 0 & I_n \end{bmatrix} \exp(\tau Z(\gamma)),
\]
\[
Z(\gamma) = \begin{pmatrix}
\frac{M}{i \gamma(i \gamma - 1)} I_n & -2 Q^T Q \\
-(M^T + 2i \gamma R^T Q)
\end{pmatrix}.
\]
Numerical upgrade 1

We take a constant $\alpha$ such as $x \to \exp(\alpha x)(\zeta - \exp(x))_+ \in L^1(\mathbb{R})$ and in practice, Carr & Madan consider $\alpha = 1.1$ is an empirical good value for the Heston model. We define a modified price

$$c_t^{\alpha}(T, k) = \exp(\alpha k)c_t(T, k)$$

$$c_t(T, k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} \exp(-i\nu k)\Phi_t^\alpha(T, \nu)d\nu$$

$$= \exp[-r(T - t)]\frac{\exp(-\alpha k)}{\pi} \Re \int_0^{\infty} \exp(-i\nu k) \frac{\Phi_t(T, \nu - (1 + \alpha)i)}{(1 + \alpha + i\nu)(\alpha + i\nu)} d\nu$$
The idea is to use this formula for a simple model (like the Black & Scholes model with a constant volatility $\bar{\sigma}$) and to apply this expression for numerical applications. This improves the effectiveness of numerical integration. It is done by analogy with the variance reduction method for Monte Carlo options pricing by control variates. So we can write

$$c_t(T, k) = \exp[-r(T - t)] \frac{\exp(-\alpha k)}{\pi} \text{Re}[\int_0^\infty \exp(-i\nu k) \frac{(\Phi_t(T, \nu - (1 + \alpha)i) - \Phi^{BS}_t(T, \nu - (1 + \alpha)i))}{(1 + \alpha + i\nu)(\alpha + i\nu)} d\nu]$$

$$+ c_t^{BS}(T, k)$$
Numerical upgrade 2 bis

We consider that a good choice for the volatility $\bar{\sigma}$ is the Black and Scholes volatility as the realised mean variance in the Wishart model

$$\bar{\sigma} = \sqrt{\frac{1}{T} \int_0^T \mathbb{E}[\text{Tr}(V_t)] dt}$$

Calculations in the Wishart model gives us

$$\bar{\sigma}^2 = \frac{1}{T} \text{Tr}[(V_0 + \beta Q^T Q(2M)^{-1})(2M)^{-1}(\exp(2TM) - I_n)]$$

$$- \beta \text{Tr}[Q^T Q(2M)^{-1}]$$
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Smile dynamic in the Wishart model

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Set of parameters

\[ M = \begin{pmatrix} -m_1 & 0 \\ 0 & -m_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad R = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \]

Denote \( \nu_i = \frac{q_i}{\sqrt{m_i}} \) linked to the asymptotic variance covariance matrix of the Ornstein Uhlembeck processes.
We can write

\[
\mathbb{E}(V_{TT}^{11}\mid\mathcal{F}_t) = \frac{\beta\nu_1^2}{2} + (V_t^{11} - \frac{\beta\nu_1^2}{2})\exp[-2m_1(T-t)]
\]

\[
\mathbb{E}(V_{TT}^{22}\mid\mathcal{F}_t) = \frac{\beta\nu_2^2}{2} + (V_t^{22} - \frac{\beta\nu_2^2}{2})\exp[-2m_2(T-t)]
\]

\[
\mathbb{E}(V_{TT}^{12}\mid\mathcal{F}_t) = V_t^{12}\exp[-(m_1 + m_2)(T-t)]
\]

Consequently, \( \tau_i = \frac{1}{2m_i} \) for \( i \in \{1, 2\} \) are mean-reverting times of \( V_T \) and \( \frac{\beta\nu_i^2}{2} \) for \( i \in \{1, 2\} \) correspond to the asymptotic mean of the diagonal components of \( V_T \) when \( T \) goes to infinity.
Perturbation Method of the Riccati equations

Remember the homogeneous Riccati system associated to the Fourier transform

\[
A'(\tau) = \frac{i\gamma(i\gamma - 1)}{2}I_n + A(\tau)M + (M^T + 2i\gamma R^T Q)A(\tau) + 2A(\tau)Q^T QA(\tau).
\]

\[
C'(\tau) = ir\gamma + \beta \text{Tr}[Q^T QA(\tau)].
\]

Consequently, a development of the function \(A\) furnishes instantly a development for \(C\).
Consequently, let us focus on a development of the solution \(A(\tau)\) of the form

\[
A(\tau) = \sum_{i,j} \varepsilon^{i/2} \delta^{j/2} A^{i,j}(\tau).
\]
From a development of the Fourier transform to a price approximation

The main question of this procedure is how the price approximation can be inferred from the Fourier transform development.
In a general way, assume that there is a development for the Fourier transform under the form

$$
\Phi_t(T, \gamma) = \Phi_t^{BS}(\bar{\sigma})(T, \gamma) \sum_{i,j} \varepsilon_i^2 \delta_j^2 P_{i,j}(i\gamma).
$$

where $P_{i,j} \in \mathbb{R}_p[X]$. Noticing that

$$P\left[ \frac{\partial}{\partial y} \right] \Phi_t^{BS(\bar{\sigma})}(T, \gamma) = P(i\gamma) \Phi_t^{BS(\bar{\sigma})}(T, \gamma),$$

the price of a European option is given by

$$C_t(T, k) = \sum_{i,j} \varepsilon_i^2 \delta_j^2 P_{i,j} \left( \frac{\partial}{\partial y} \right) C_t^{BS(\bar{\sigma})}(T, k).$$
We assume the same kind of reduction for the implied volatility

$$\Sigma_t(T, K) = \sum_{i,j} \epsilon^{i/2} \delta^{j/2} \Sigma_{t}^{i,j}(T, K)$$

Using Taylor formula, we know that

$$C_t(T, k) = C_t^{BS}(T, K, \Sigma_t^{0,0}(T, K)) + [\sqrt{\epsilon} \Sigma_t^{1,0} + \sqrt{\delta} \Sigma_t^{0,1}] \frac{\partial C_t^{BS}}{\partial \sigma}(T, K, \Sigma_t^{0,0})$$

Finally, by identifying the developments of price, we obtain the implied volatility development.
The short maturity case is $(T - t) \ll \tau_1, \tau_2$ i.e $m_1, m_2$ small

\[ M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = -\varepsilon M_1 - \delta M_2. \]

\[ Q = \sqrt{\varepsilon \nu_1} M_1 + \sqrt{\delta \nu_2} M_2, \quad Q^2 = \varepsilon \nu_1^2 M_1 + \delta \nu_2^2 M_2. \]

\[ V_0 = \begin{pmatrix} u & v \\ v & w \end{pmatrix}, \quad \theta = i \gamma. \]
Development of the Riccati functions order 1

\[ A = A^{0,0} + \sqrt{\varepsilon}A^{1,0} + \sqrt{\delta}A^{0,1} + o \left( \max \left( \sqrt{\varepsilon}, \sqrt{\delta} \right) \right). \]

with

\[ A^{0,0} = \frac{\theta (\theta - 1)}{2} \tau \ast I_2. \]

\[ A^{1,0} = \frac{\theta^2 (\theta - 1)}{2} \nu_1 \tau^2 \left( R^T M_1 \right). \]

\[ A^{0,1} = \frac{\theta^2 (\theta - 1)}{2} \nu_2 \tau^2 \left( R^T M_2 \right). \]

\[ C = rT \theta + o \left( \max \left( \sqrt{\varepsilon}, \sqrt{\delta} \right) \right) \]
The Fourier transform can be developed as follows

\[ \Phi(\tau, \theta) = \exp \left[ \frac{\theta(\theta-1)}{2} \tau (u + w) + \theta \log(S_0) + \tau \theta \right] \ast \left[ 1 + \sqrt{\varepsilon} \text{Tr} (A_{1,0} V_0) + \sqrt{\delta} \text{Tr} (A_{0,1} V_0) \right] + o \left( \max \left( \sqrt{\varepsilon}, \sqrt{\delta} \right) \right) \]
Price approximation order 1

Using classic formulae in the Black-Scholes model (see Appendix B), the development for the call price can be determined

\[ P = P^{0,0} + \sqrt{\varepsilon} P^{1,0} + \sqrt{\delta} P^{0,1} + o\left( \max\left( \sqrt{\varepsilon}, \sqrt{\delta} \right) \right). \]

with

\[ P^{0,0} = C_{BS}(\sigma), \quad \sigma = \sqrt{u + w}. \]

\[ P^{1,0} = \frac{\nu_1 \tau^2}{2} \left( \rho_{11} u + \rho_{12} v \right) \left( \frac{\partial^3 C_{BS}(\sigma)}{\partial \log (S_0)^3} - \frac{\partial^2 C_{BS}(\sigma)}{\partial \log (S_0)^2} \right). \]

\[ P^{0,1} = \frac{\nu_2 \tau^2}{2} \left( \rho_{22} w + \rho_{21} v \right) \left( \frac{\partial^3 C_{BS}(\sigma)}{\partial \log (S_0)^3} - \frac{\partial^2 C_{BS}(\sigma)}{\partial \log (S_0)^2} \right). \]
Wishart Stochastic Volatility: Asymptotic Smile and Numerical Framework

Smile dynamic in the Wishart model
Smile dynamic for short maturities

**Implied volatility order 1**

\[
\Sigma = \Sigma^{0,0} + \sqrt{\varepsilon} \Sigma^{1,0} + \sqrt{\delta} \Sigma^{0,1} + o \left( \max \left( \sqrt{\varepsilon}, \sqrt{\delta} \right) \right).
\]

with

\[
\Sigma^{0,0} = \sqrt{u + w},
\]

\[
\Sigma^{1,0} = \frac{P^{1,0}}{\left( \frac{\partial C_{BS}(\sigma)}{\partial \sigma} \right)_{\sigma = \Sigma^{0,0}}} = \frac{\nu_1}{2 (u + w)^{3/2}} \left( \rho_{11} u + \rho_{12} v \right) \left( \frac{u + w}{2} T + \log \left( \frac{K}{S_0} \right) \right) + \log \left( \frac{K}{S_0} \right).
\]

\[
\Sigma^{0,1} = \frac{P^{0,1}}{\left( \frac{\partial C_{BS}(\sigma)}{\partial \sigma} \right)_{\sigma = \Sigma^{0,0}}} = \frac{\nu_2}{2 (u + w)^{3/2}} \left( \rho_{22} w + \rho_{21} v \right) \left( \frac{u + w}{2} T + \log \left( \frac{K}{S_0} \right) \right).
\]
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## Smile approximation

For instance, at order \((\sqrt{\varepsilon}, \sqrt{\delta})\), an explicit and concise formulae for the smile can be deduced

\[
\hat{\Sigma}_t(T, K) = \sqrt{\text{Tr}(V_t)} + \frac{\sqrt{\varepsilon} P^{1,0}(t, Y_t, V_{t1}^{11}, V_{t2}^{21}, V_{t2}^{22}) + \sqrt{\delta} P^{0,1}(t, Y_t, V_{t1}^{11}, V_{t1}^{12}, V_{t2}^{22})}{\frac{\partial C_{BS}^t}{\partial \sigma}(T, K, \sqrt{\text{Tr}(V_t)})}
\]

\[
= \sqrt{\text{Tr}(V_t)}
\]

\[
+ \frac{q_1(\rho_{11} V_{t1}^{11} + \rho_{12} V_{t1}^{12}) + q_2(\rho_{22} V_{t2}^{22} + \rho_{21} V_{t1}^{12})}{2(\text{Tr}(V_t))^{\frac{3}{2}}} \left[ \log(K e^{-r(T-t)}) - Y_t + \frac{\text{Tr}(V_t)(T-t)}{2} \right].
\]

Furthermore, an expression at order \((\sqrt{\varepsilon}, \sqrt{\delta})\) of the skew at the forward money \((K = F_t = e^{Y_t + r(T-t)})\) is obtained and one can check that this formulae corresponds to the one obtained by the Durrlemann theorem

\[
\frac{\partial \Sigma_t}{\partial K}(T, F_t) \sim \frac{q_1(\rho_{11} V_{t1}^{11} + \rho_{12} V_{t1}^{12}) + q_2(\rho_{22} V_{t2}^{22} + \rho_{21} V_{t1}^{12})}{2(\text{Tr}(V_t))^{\frac{3}{2}}} F_t.
\]
Considering the case $\tau_1 \ll (T - t) \ll \tau_2$: a first component of the volatility with a fast mean reverting and a second with a slow evolution. The introduction of two scales of volatility allows to obtain a persistent skew. That is what we will try to prove using the singular perturbations method.
The two-scale volatility case is $\tau_1 \ll T_t \ll \tau_2$ i.e $m_2 \ll \frac{1}{T-t} \ll m_1$

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = -\frac{1}{\varepsilon} M_1 - \delta M_2,$$

$$Q = \frac{1}{\sqrt{\varepsilon}} \nu_1 M_1 + \sqrt{\delta} \nu_2 M_2, \quad Q^2 = \frac{1}{\varepsilon} \nu_1^2 M_1 + \delta \nu_2^2 M_2,$$

$$V_0 = \begin{pmatrix} u & \nu \\ \nu & w \end{pmatrix}, \quad \theta = i\gamma.$$
A step by step procedure

**Order $\frac{1}{\varepsilon}$:**

$$0 = -A^{0,0}(\tau)M_1 - M_1 A^{0,0}(\tau) + 2A^{0,0}(\tau)Q_1^2 A^{0,0}(\tau).$$

Hence, $A^{0,0}$ the form is deduced

$$A^{0,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & a_{2,2}^{0,0}(\tau) \end{pmatrix}.$$ 

By pursuing the procedure at all orders, all the coefficients $A^{i,j}$ can be calculated.
Analysis of the Riccati function at order 1

\[ A^{0,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\theta(\theta-1)}{2} \tau \end{pmatrix}, \quad A^{1,0}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]
\[ A^{0,1}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\theta^2(\theta-1)}{2} \rho_{22} \nu_2 \tau^2 \end{pmatrix}. \]

\[
C^{0,0} = rT \theta + \beta \int_0^\tau \text{Tr}(Q_1^2 A^{20}(s)) \, ds = rT \theta + \frac{\theta(\theta-1)}{2} \beta \nu_1^2 \tau.
\]
\[
C^{1,0} = \beta \int_0^\tau \text{Tr}(Q_1^2 A^{30}(s)) \, ds = \frac{\theta^2(\theta-1)}{4} \beta \nu_1^3 \rho_{11} \tau.
\]
\[
C^{0,1} = \beta \int_0^\tau \text{Tr}(Q_1^2 A^{21}(s)) \, ds = 0.
\]
Price approximation at order 1

\[ P = P^{0,0} + \sqrt{\varepsilon} P^{1,0} + \sqrt{\delta} P^{0,1} + o \left( \max \left( \sqrt{\varepsilon}, \sqrt{\delta} \right) \right). \]

with

\[ P^{0,0} = C_{BS}(\sigma). \]
\[ \sigma = \sqrt{\frac{\beta \nu^2}{2}} + w. \]
\[ P^{1,0} = \frac{\beta \rho_{11} \nu_1^3 \tau}{4} \left( \frac{\partial^3 C_{BS}}{\partial y^3} - \frac{\partial^2 C_{BS}}{\partial y^2} \right). \]
\[ P^{0,1} = \frac{\rho_{22} \nu_2 \tau^2 w}{2} \left( \frac{\partial^3 C_{BS}}{\partial y^3} - \frac{\partial^2 C_{BS}}{\partial y^2} \right). \]
Smile approximation (order 1)

At order \((\sqrt{\varepsilon}, \sqrt{\delta})\), the expression of the smile is easy to analyze

\[
\hat{\Sigma}_t(T, K) = \sqrt{\frac{\beta \nu^2}{2}} + V_t^{22} + \frac{\sqrt{\varepsilon} P^{1,0}(t, Y_t, V_t^{22}) + \sqrt{\delta} P^{0,1}(t, Y_t, V_t^{22})}{\partial C^B_S(t, K, \sqrt{\frac{\beta \nu^2}{2}} + V_t^{22})} \]

\[
= \sqrt{\frac{\beta \nu^2}{2}} + V_t^{22} + \frac{1}{\beta \nu^2} \left[ \frac{q_1^3}{4m_1} \beta \rho_{11} + \frac{q_2}{2} \rho_{22} (T - t) V_t^{22} \right] \left[ \frac{\log(K e^{-r(T-t)} - Y_t)}{\sqrt{\beta \nu^2}} V_t^{22}(T - t) \right] + \sqrt{\frac{\beta \nu^2}{2}} + V_t^{22} \]
Skew ATM approximation (order 1)

\[
\frac{\partial \Sigma_t}{\partial K} \sim \frac{1}{F_t} \frac{1}{\left(\beta \frac{\nu_1^2}{2} + V_t^{22}\right)^3} \left[ \frac{q_1^3}{4m_1^2(T-t)} \beta \rho_{11} + \frac{q_2}{2} \rho_{22} V_t^{22} \right]
\]

\[
\sim \frac{1}{F_t} \frac{1}{\left(\beta \frac{\nu_1^2}{2} + V_t^{22}\right)^3} \left[ \nu_1^3 \frac{\sqrt{\varepsilon}}{T-t} \beta \rho_{11} + \frac{\nu_2}{2} \sqrt{\delta} \rho_{22} V_t^{22} \right].
\]

This formulae underlines that the skew splits up into two components: the first one, proportional to \(\rho_{11}\), coming from the fast mean-reversion volatility component, and the second one, proportional to \(\rho_{22}\), is persistent and proceeds from the slow variation component of the volatility.
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The matrix of mean reversion $M$ represents the speed the volatility process to return to its mean.

The matrix $Q$ corresponds to the volatility of volatility which is the most significant parameter of the stochastic volatility model concerning the evolution of the volatility and the dispersion of the volatility around its expected value.
Set of parameters

\[ \beta = 4 \quad r = 0 \quad V_0 = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{pmatrix} \]

\[ M = \begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix} \quad Q = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix} \]

The matrix \( Q \) was chosen so that the volatility does not reach quickly the asymptotic volatility \( \Sigma_0 = \sqrt{V_0^{11} + V_0^{22}} = 20\% \) and \( \Sigma_\infty = \sqrt{\beta (\nu_1^2 + \nu_2^2)} \sim 17, 8\% \).

Characterititic times of the process \( V \) are denoted \( \tau_1 = \frac{1}{2m_1} = 10y \) and \( \tau_2 = \frac{1}{2m_2} = 10y \) and the maturity is \( T = 6 \text{ months} \). Then, this framework deals with a ”short maturity” case.
Short Term Smile by FFT

Figure: Short term smile by FFT ($T = 6m \ll \tau_1 = \tau_2 = 10y$)
Short Term Smile Order 1

Figure: Short term smile at order $(\sqrt{\epsilon}, \sqrt{\delta})$ ($T = 6m \ll \tau_1 = \tau_2 = 10y$)
Short Term Smile Order 2

Figure: Short term smile at order $(\varepsilon, \delta)$ $(T = 6m \ll \tau_1 = \tau_2 = 10y)$
Figure: Error for short term smile at order $(\sqrt{\varepsilon}, \sqrt{\delta})$

$(T = 6m \ll \tau_1 = \tau_2 = 10y)$
Figure: Error for short term smile at order $(\varepsilon, \delta)$

$(T = 6m \ll \tau_1 = \tau_2 = 10y)$
The approximation gives very good results and the approximation at order 2 is accurate enough to be used as a calibration tool: one can see that between the strike 80 and 120, the error is below $5 \times 10^{-5}$. This error is totally acceptable by practitioners in a calibration prospect.
Robustness of this approximation

\[ \beta = 4 \quad r = 0 \quad V_0 = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{pmatrix} \]

\[ M = \begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix} \quad Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \]

We consider now the unrealistic case where the volatility of volatility matrix \( Q \) is outstanding so that our volatility explodes. In this case, we have a great asymptotic volatility (\( \Sigma_\infty \sim 89\% \))
Figure: Short term smile by FFT ($T = 6m \ll \tau_1 = \tau_2 = 10y$)
Figure: Convergence in the case: $\rho_{11} = \rho_{22} = -0.6$, $\rho_{12} = \rho_{21} = 0$
Short Term Smile Case 2

Figure: Convergence in the case: $\rho_{11} = \rho_{22} = -0.1, \rho_{12} = \rho_{21} = 0$
Short Term Smile Case 3

Figure: Convergence in the case: $\rho_{11} = \rho_{22} = -0.6$ $\rho_{12} = \rho_{21} = -0.3$
Figure: Convergence in the case: $\rho_{11} = -0.1$ $\rho_{22} = -0.6$ $\rho_{12} = \rho_{21} = 0$
Robustness of the approximation

The perturbation method becomes more accurate at order 2: the approximation reaches the good level ATM and the shape of the curves is reproduced more precisely. Indeed, the smirk in the fourth case is faithfully reproduced and the smile level ATM is really closed to the one obtained by FFT 24%.
Short Term Smile Surface

Figure: Smile surface for short maturities
Consider a maturity $T = 3y$ what allows the case of a medium maturity with $\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y$. 

$$\begin{align*}
\beta &= 4 \\
r &= 0 \\
V_0 &= \begin{pmatrix}
0.02 & 0 \\
0 & 0.02
\end{pmatrix} \\
M &= \begin{pmatrix}
-2 & 0 \\
0 & -0.02
\end{pmatrix} \\
Q &= \begin{pmatrix}
0.05 & 0 \\
0 & 0.02
\end{pmatrix}
\end{align*}$$

The matrix $Q$ is chosen so that the volatility has gone from 20% until 15.8% in 3 years.
Figure: Multiscale volatility smiles by FFT

\( \tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y \)
Multiscale smile order 1

Figure: Multiscale volatility smiles at order \((\sqrt{\epsilon}, \sqrt{\delta})\)  
\((\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y)\)
Multiscale smile order 2

Figure: Multiscale volatility smiles at order \((\varepsilon, \delta)\)

\((\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y)\)
Analysis of the approximation

At order 1, the perturbation method furnishes a good shape for the skew but the level of the smile at the money is far from the good one (15%). However, at order 2, the approximated smile is accurate and the level is really close from the good one (15, 7%).

As expected, the skew comes from the component $\rho_{22}$ and the slope of the smile curve is almost insensible to a change of $\rho_{11}$ given that $\tau_1 \ll T$. 
Error of the approximation at order 1

**Figure:** Error for multiscale volatility smile at order \((\sqrt{\varepsilon}, \sqrt{\delta})\)

\((\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y)\)
Error of the approximation at order 2

Figure: Error for multiscale volatility smile at order \((\varepsilon, \delta)\)
\((\tau_1 = 3m \ll T = 3y \ll \tau_2 = 25y)\)
Multiscale smile Surface

Figure: Smile surface for a multiscale volatility
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Conclusion

1. Wishart model is tractable analytically
Conclusion

1. Wishart model is tractable analytically

2. Good idea of parameters influence
Conclusion

1. Wishart model is tractable analytically

2. Good idea of parameters influence

3. Approximation of asymptotic smiles (short term and multiscale smiles) for calibration
Prospect

- Study of the theoretical error of our approximation
Prospect

1. Study of the theoretical error of our approximation

2. Application to the interest rate market or longevity market
Prospect

1. Study of the theoretical error of our approximation

2. Application to the interest rate market or longevity market

3. Influence of all parameters (general approximation)