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Dynamic Risk Measures: Time Consistency and Risk Measures from BMO Martingales

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Abstract Time consistency is a crucial property for dynamic risk measures. Making use of the dual representation for conditional risk measures, we characterize the time consistency by a cocycle condition for the minimal penalty function.
Taking advantage of this cocycle condition, we introduce a new methodology for the construction of time-consistent dynamic risk measures. Starting with BMO martingales, we provide new classes of time-consistent dynamic risk measures. These families generalize the Backward Stochastic Differential Equations. Quite importantly, starting with right continuous BMO martingales this construction leads naturally to paths with jumps.

Keywords dynamic risk measures · conditional risk measures · time consistency · BMO martingales

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1 Introduction

In recent years there has been an increasing interest in methods defining the risk of a financial position. Artzner et al [1] have introduced the concept of coherent risk measures on a probability space. More recently Föllmer and Schied [16],[17], and Frittelli and Rosazza Gianin [18], have addressed a more general issue, defining the notion of convex monetary measure of risk, not necessarily coherent.

Several authors have then extended the notion of monetary risk measures to a conditional or dynamic setting. The conditional risk measures were studied by Detlefsen and Scandolo [12] and Bion-Nadal [4]. Coherent dynamic risk measures have been developped by Delbaen [9] and Artzner et al [2]. Convex dynamic risk measures have been considered by Riedel [30], Frittelli and Rosazza Gianin [19], Klöppel and Schweizer [22], Cheridito,

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Delbaen and Kupper [6], Cheridito and Kupper [7], and Jobert and Rogers [20]. Other works concerning a dynamic setting are based on the Backward Stochastic Differential Equations (B.S.D.E.), approach called also conditional "g-expectation". Important works along these lines are by Peng [26] and [27], Coquet et al. [8], Rosazza Gianin [31], and Barrieu and El Karoui [3].
From a dynamic point of view, a key notion is that of time consistency, which means that for any $r \leq s \leq t$ the risk at time r, of a financial position defined at time t, can be indifferently evaluated directly or using an intermediate time s. Classical examples of time-consistent dynamic risk measures are the dynamic entropic risk measure associated with the exponential utility function, and the solutions of B.S.D.E. associated with a convex driver $g(t,z)$. In a continuous time dynamic setting F. Delbaen [9] has characterized the time consistency for coherent dynamic risk measures. A coherent dynamic risk measure defined from a set $\mathscr{Q}$ of probability measures is time consistent if and only if this set satisfies a stability property called m-stability. In a discrete time setting, the time consistency was characterized by Cheridito et al [6] by a condition on the acceptance set and also a "concatenation condition".

The main result of the present paper (Theorem 3.3) is the characterization of the time consistency for a dynamic risk measure by a "cocycle condition" for the minimal penalty. The importance of this characterization is that it is a simple condition, which can be easily checked. It is also a crucial property for the construction of new families of time-consistent dynamic risk measures. This result follows from the characterization of the composition rule for conditional risk measures $\rho_{1,3} = \rho_{1,2} \circ (-\rho_{2,3})$ in terms of a cocycle condition for the minimal penalty (Theorem 2.5). The key tools for the proof are the dual representation for conditional risk measures, and lattice properties. In the particular case of discrete time, we prove that a time-consistent dynamic risk measure can be simply viewed as a conditional risk measure on a larger space (Proposition 3.5).

Making use of this cocycle condition, the second most important contribution of this paper is the introduction of a new class of time-consistent dynamic risk measures constructed from BMO martingales. This new class generalizes the dynamic risk measures coming from B.S.D.E. and allows for jumps. For the construction of these new time-consistent dynamic risk measures, we introduce a new methodology based on the following result (Theorem 4.4): Any stable family of probability measures and any local penalty satisfying the cocycle condition lead to a time-consistent dynamic risk measure. This result generalizes to the general convex dynamic risk measures the result proved by Delbaen [9] in the coherent case (i.e. for the zero penalty) and mentioned above. The powerful notion of BMO martingales developped by Doléans-Dade and Meyer [13] in the general case and by Kazamaki [21] in the particular case of continuous martingales, is very well adapted to the construction of time-consistent dynamic risk measures. Starting with any finite family of BMO strongly orthogonal continuous martingales, we construct classes of time-consistent dynamic risk measures. In the particular case of independent Brownian motions we recover the dynamic risk measures coming from B.S.D.E.. More importantly, starting with right continuous BMO martingales with jumps (with in that case a restrictive condition on the BMO norm), the same construction leads naturally to dynamic risk measures with jumps.

2 Conditional risk measures

2.1 Dual representation

This subsection essentially recalls the main definitions and results needed for the study of time consistency. Let $\mathscr{F}_i \subset \mathscr{F}_j$ be two σ-algebras on Ω. Let P be a probability measure on $(\Omega, \mathscr{F}_j)$. Recall the following definition of conditional risk measure (cf [4] and [12]):

Definition 2.1 A risk measure $\rho_{i,j}$ on $(\Omega, \mathscr{F}_j, P)$ conditional to $(\Omega, \mathscr{F}_i, P)$ is a map $\rho_{i,j}$: $L^\infty(\Omega, \mathscr{F}_j, P) \to L^\infty(\Omega, \mathscr{F}_i, P)$ satisfying the following properties:

i) monotonicity:

$$\forall X, Y \in L^\infty(\Omega, \mathscr{F}_j, P), \quad if\ X \leq Y\ \ then\ \ \rho_{i,j}(X) \geq \rho_{i,j}(Y)$$

ii) translation invariance:

$$\forall Z \in L^\infty(\Omega, \mathscr{F}_i, P)\ ,\ \ \forall X \in L^\infty(\Omega, \mathscr{F}_j, P)\ ,\ \ \rho_{i,j}(X + Z) = \rho_{i,j}(X) - Z$$

A conditional risk measure can have additional properties:

iii) convexity:

$$\forall X, Y \in L^\infty(\Omega, \mathscr{F}_j, P)\ ,\ \ \forall \lambda \in [0,1]$$
$$\rho_{i,j}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_{i,j}(X) + (1-\lambda)\rho_{i,j}(Y)$$

iv) continuity from below (resp above): for any increasing (resp decreasing) sequence X_n of elements of $L^\infty(\Omega, \mathscr{F}_j, P)$ such that $X = \lim\ X_n\ P\ a.s.$, the sequence $\rho_{i,j}(X_n)$ has the limit $\rho_{i,j}(X)\ P\ a.s.$

v) normalization: $\rho_{i,j}(0) = 0$

Remark 2.2 The monotonicity and translation invariance property imply the following regularity property (cf [22] Section 2),

$$\forall X, Y \in L^\infty(\Omega, \mathscr{F}_j, P),\ \ \forall A \in \mathscr{F}_i\ \ \rho_{i,j}(X 1_A + Y 1_{A^c}) = 1_A \rho_{i,j}(X) + 1_{A^c} \rho_{i,j}(Y)$$

The continuity from below implies continuity from above (cf [17] and [12]).

Detlefsen and Scandolo have proved ([12] Theorem 1) that the existence of a dual representation in terms of probability measures is equivalent to the continuity from above. Dual representation results have been proved in [4] for conditional risk measures continuous from below in a general context of uncertainty. Klöppel and Schweizer have characterized the conditional risk measures admitting a representation in terms of probability measures all equivalent to the reference probability measure P.
Recall the dual representation result for a conditional risk measure $\rho_{i,j}$ continuous from below([4] and [12]):

$$\forall X \in L^\infty(\Omega, \mathscr{F}_j, P),\ \rho_{i,j}(X) = \text{essmax}_{Q \in \mathscr{M}_{i,j}} [(E_Q(-X|\mathscr{F}_i) - \alpha^m_{i,j}(Q)] \qquad (2.1)$$

where for any Q in $\tilde{\mathscr{M}}_{i,j} = \{Q\ on\ (\Omega, \mathscr{F}_j)\ |Q \ll P,\ Q_{|\mathscr{F}_i} = P\}$ the minimal penalty is defined by

$$\begin{aligned} \alpha^m_{i,j}(Q) &= \text{esssup}_{X \in L^\infty(\Omega, \mathscr{F}_j, P)} (E_Q(-X|\mathscr{F}_i) - \rho_{i,j}(X)) \\ &= \text{esssup}_{X \in \mathscr{A}_{i,j}} E_Q(-X|\mathscr{F}_i) \end{aligned} \qquad (2.2)$$

The acceptance set is

$$\mathscr{A}_{i,j} = \{X \in L^\infty(\Omega, \mathscr{F}_j, P) \mid \rho_{i,j}(X) \leq 0\}$$

and the dual set of probability measures is

$$\mathscr{M}_{i,j} = \{Q \in \tilde{\mathscr{M}}_{i,j} \mid \alpha^m_{i,j}(Q) \in L^\infty(\Omega, \mathscr{F}_i, P)\}.$$

In addition to these recalls, notice the following useful properties of the minimal penalty:

Lemma 2.3 *1. The penalty $\alpha^m_{i,j}$ is local, i.e. $\forall A \in \mathscr{F}_i$, $\forall Q_1, Q_2 \in \tilde{\mathscr{M}}_{i,j}$*
If $\frac{dQ_1}{dP} 1_A = \frac{dQ_2}{dP} 1_A$ then $1_A\ \alpha^m_{i,j}(Q_1) = 1_A\ \alpha^m_{i,j}(Q_2)$
2. For any probability measure Q in $\tilde{\mathscr{M}}_{i,j}$, $\alpha^m_{i,j}(Q)$ is bounded from below by $-||\rho_{i,j}(0)||$. Moreover there is a sequence X_n in $\mathscr{A}_{i,j}$ such that $\alpha^m_{i,j}(Q)$ is the increasing limit of $E_Q(-X_n|\mathscr{F}_i)$.

Proof. *1.* As $Q_{1|\mathscr{F}_i} = Q_{2|\mathscr{F}_i} = P$, $\frac{dQ_1}{dP} 1_A = \frac{dQ_2}{dP} 1_A$ means that

$$\forall X \in L^\infty(\Omega, \mathscr{F}_j, P),\ \ E_{Q_1}(X 1_A|\mathscr{F}_i) = E_{Q_2}(X 1_A|\mathscr{F}_i)\ P.a.s.$$

The local property of $\alpha^m_{i,j}$ follows easily from the equation (2.2).
2. From the translation invariance property, $\rho_{i,j}(0)$ belongs to $\mathscr{A}_{i,j}$ and this proves the first claim.
For $Q \in \tilde{\mathscr{M}}_{i,j}$ prove that $\{E_Q(-X|\mathscr{F}_i); X \in \mathscr{A}_{i,j}\}$ is a lattice upward directed. Let $Y, Z \in \mathscr{A}_{i,j}$, $B = \{\omega \in \Omega \mid E_Q(-Y|\mathscr{F}_i)(\omega) > (E_Q(-Z|\mathscr{F}_i)(\omega)\}$. From the regularity of $\rho_{i,j}$ (Remark 2.2), it follows that $X = Y 1_B + Z 1_{B^c}$ is in $\mathscr{A}_{i,j}$ and

$$E_Q(-X|\mathscr{F}_i) = \sup\left(E_Q(-Y|\mathscr{F}_i),\ E_Q(-Z|\mathscr{F}_i)\right)\ \ P\ a.s.$$

The lattice property is proved and the result follows then from [25] or from [17] Appendix A.5. □

2.2 Composition of conditional risk measures

Consider three σ-algebras $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$ on a space Ω.

Lemma 2.4 *Assume that $\rho_{2,3}$ is a risk measure on $(\Omega, \mathscr{F}_3, P)$ conditional to $(\Omega, \mathscr{F}_2, P)$ and $\rho_{1,2}$ a risk measure on $(\Omega, \mathscr{F}_2, P)$ conditional to $(\Omega, \mathscr{F}_1, P)$.*
Then $\rho(X) = \rho_{1,2}(-\rho_{2,3}(X))$ defines a risk measure on $(\Omega, \mathscr{F}_3, P)$ conditional to $(\Omega, \mathscr{F}_1, P)$.

The verification of this lemma is straightforward. □

Consider the composition rule $\rho_{1,3}(X) = \rho_{1,2}(-\rho_{2,3}(X))$ which means that the risk of a financial position can be computed either directly as $\rho_{1,3}(X)$, or using an intermediate instant of time as $\rho_{1,2}(-\rho_{2,3}(X))$. We address the following questions: How to characterize this composition rule using the dual representation, i.e. in terms of penalty functions? Or, given a conditional risk measure $\rho_{1,3}$, under which condition is it possible to factorize it through the σ-algebra $\mathscr{F}_2$? The answers are given by the following theorem (the notations are those of Section 2.1):

Theorem 2.5 *Let $(\rho_{i,j})_{1\leq i\leq j\leq 3}$ be convex risk measures continuous from below on $(\Omega,\mathscr{F}_j,P)$ conditional to $(\Omega,\mathscr{F}_i,P)$. The following properties are equivalent:*
i) $\rho_{1,3}(X)=\rho_{1,2}(-\rho_{2,3}(X))\ \ \forall X\in L^\infty(\Omega,\mathscr{F}_3,P)$
ii) $\mathscr{A}_{1,3}=\mathscr{A}_{1,2}+\mathscr{A}_{2,3}$
iii) cocycle condition:
(a) $(\mathscr{M}_{i,j})_{1\leq i<j\leq 3}$ satisfy the following stability property: $\forall Q\in\mathscr{M}_{2,3}$, $\forall R\in\mathscr{M}_{1,2}$, the probability measure S of Radon Nikodym derivative

$$\frac{dS}{dP}=\frac{dR}{dP}\frac{dQ}{dP} \tag{2.3}$$

is in $\mathscr{M}_{1,3}$.
(b) $\forall S\in\mathscr{M}_{1,3}$, $\forall R\in\mathscr{M}_{1,2}$, $\forall Q\in\tilde{\mathscr{M}}_{2,3}$ satisfying the relation (2.3), $\alpha^m_{2,3}(Q)$ is R-integrable and the penalty function α^m satisfies the cocycle condition:

$$\alpha^m_{1,3}(S)=E_R(\alpha^m_{2,3}(Q)|\mathscr{F}_1)+\alpha^m_{1,2}(R)\ \ P.a.s. \tag{2.4}$$

Remark 2.6 (1) The equivalence of i) and ii) was already proved by Cheridito et al [6].
(2) A characterization of time consistency is proved in [6] in terms of a concatenation condition. The advantage of the cocycle condition given here is that it is a very simple relation, and it is therefore easy to check if it is satisfied or not. It will allow (Section 4) for the construction of a new class of time-consistent dynamic risk measures generalizing the B.S.D.E. and allowing for jumps.
(3) Some condition was given in a discrete time setting in Proposition 8 of [12]. But instead of being an exact relation satisfied for any triplet (R,S,Q) as in equation (2.4), their relation involved a essinf taken over all the minimal penalties associated with a family of probability measures R. Furthermore in their result, the sum of this essinf and the conditional expectation of the minimal penalty is “some” penalty and not necessarily the minimal one.
(4) After a preliminary version of this paper has appeared as a working paper (CMAP preprint 596, March 2006), Föllmer and Penner [15] have proved a similar result for conditional risk measures continuous from above under the following restrictive conditions: the conditional risk measures are assumed to have a representation in terms of probability measures all equivalent to the reference probability measure (instead of absolutely continuous), and normalization is also assumed. The technical extension of these results to the most general case of convex conditional risk measures continuous from above is the subject of another paper [5].

Proof of the Theorem.
- For the proof of the statement *i) implies ii)*, we refer to [6].
- *ii) implies iii)*
First step. Let $S\in\tilde{\mathscr{M}}_{1,3}$ $R\in\mathscr{M}_{1,2}$ $Q\in\tilde{\mathscr{M}}_{2,3}$ satisfying the relation (2.3). From hypothesis ii) and equation (2.2)

$$\begin{aligned}\alpha^m_{1,3}(S)=&\ \text{esssup}_{Y\in\mathscr{A}_{1,2}}E_R(-Y|\mathscr{F}_1)+\text{esssup}_{Z\in\mathscr{A}_{2,3}}E_R(E_Q(-Z|\mathscr{F}_2)|\mathscr{F}_1)\\ =&\ \alpha^m_{1,2}(R)+\text{esssup}_{Z\in\mathscr{A}_{2,3}}E_R(E_Q(-Z|\mathscr{F}_2)|\mathscr{F}_1)\end{aligned} \tag{2.5}$$

The inequality

$$\text{esssup}_{Z\in\mathscr{A}_{2,3}}E_R(E_Q(-Z|\mathscr{F}_2)|\mathscr{F}_1)\leq E_R(\alpha^m_{2,3}(Q)|\mathscr{F}_1) \tag{2.6}$$

is straightforward.
Second step. Let $Q \in \mathscr{M}_{2,3}$ and $R \in \mathscr{M}_{1,2}$ $Q|_{\mathscr{F}_2} = P$, $R << P$ and $R|_{\mathscr{F}_1} = P$. Let S be the probability measure on $(\Omega, \mathscr{F}_3)$ defined by $\frac{dS}{dP} = \frac{dR}{dP}\frac{dQ}{dP}$. Then $S \in \tilde{\mathscr{M}}_{1,3}$.
From Lemma 2.3, $\alpha^m_{1,3}(S)$ is bounded from below. It follows then from the equations (2.5) and (2.6) that $\alpha^m_{1,3}(S)$ is essentially bounded. So, $S \in \mathscr{M}_{1,3}$. This proves *(a)*.
Last step, proof of *(b)*. Let $S \in \mathscr{M}_{1,3}$, $R \in \mathscr{M}_{1,2}$, $Q \in \tilde{\mathscr{M}}_{2,3}$ satisfying the relation (2.3). From the first step, in order to obtain the equality (2.4), it remains to prove that

$$E_R(\alpha^m_{2,3}(Q)|\mathscr{F}_1) \leq \text{esssup}_{Z \in \mathscr{A}_{2,3}} E_R(E_Q(-Z|\mathscr{F}_2)|\mathscr{F}_1) \tag{2.7}$$

From Lemma 2.3.*2*, there is a sequence $Z_n \in \mathscr{A}_{2,3}$ such that $\alpha^m_{2,3}(Q)$ is the increasing limit of $E_Q(-Z_n|\mathscr{F}_2)$. From the monotone convergence theorem it follows that $E_R(\alpha^m_{2,3}(Q)) = \lim(E_R(E_Q(-Z_n))$. By hypothesis *ii)*, $Z_n + \rho_{1,2}(0) \in \mathscr{A}_{1,3}$. Thus $E_R(\alpha^m_{2,3}(Q)) \leq E_R(\alpha^m_{1,3}(S)) + E_R(\rho_{1,2}(0)) < \infty$ (i.e. $\alpha^m_{2,3}(Q)$ is P and R integrable as the restriction of R to $\mathscr{F}_2$ is P and $\alpha^m_{2,3}(Q)$ is bounded from below by $-||\rho_{2,3}(0)||$). Applying now the monotone convergence theorem for conditional expectations ([10] chapter II page 57 and [11]), as $E_R(\alpha^m_{2,3}(Q)) < \infty$, we get $E_R(\alpha^m_{2,3}(Q)|\mathscr{F}_1) = \lim(E_R(E_Q(-Z_n|\mathscr{F}_2)|\mathscr{F}_1)$. This proves (2.7).
The cocycle condition (equation (2.4)) follows then from the equations (2.5), (2.6) and (2.7).
- *iii) implies i)* Let $X \in L^\infty(\Omega, \mathscr{F}_3, P)$. From the dual representation (2.1), there exist probability measures $R \in \mathscr{M}_{1,2}$ and $Q \in \mathscr{M}_{2,3}$ such that

$$\begin{aligned}\rho_{1,2}(-\rho_{2,3}(X)) &= E_R(\rho_{2,3}(X)|\mathscr{F}_1) - \alpha^m_{1,2}(R) \\ &= E_R(E_Q(-X|\mathscr{F}_2) - \alpha^m_{2,3}(Q)|\mathscr{F}_1) - \alpha^m_{1,2}(R)\end{aligned}$$

Applying hypothesis iii) we get the existence of a probability measure
$S_0 \in \mathscr{M}_{1,3}$ such that $\rho_{1,2}(-\rho_{2,3}(X)) = E_{S_0}(-X|\mathscr{F}_1) - \alpha^m_{1,3}(S_0)$.
From the representation of $\rho_{1,3}$, it follows that $\rho_{1,2}(-\rho_{2,3}(X)) \leq \rho_{1,3}(X)$.
Let us prove the converse inequality. From the dual representation of $\rho_{1,3}$, equation (2.1), given X, there exists $S \in \mathscr{M}_{1,3}$ such that

$$\rho_{1,3}(X) = E_S(-X|\mathscr{F}_1) - \alpha^m_{1,3}(S) \tag{2.8}$$

The restriction of S to $\mathscr{F}_2$ is absolutely continuous but not necessarily equivalent to P, therefore consider $A = \{\omega \in \Omega \;/\; (E_P(\frac{dS}{dP}|\mathscr{F}_2)(\omega) > 0\}$. Define now the probability measure Q on $(\Omega, \mathscr{F}_3)$ by its Radon Nikodym derivative

$$\frac{dQ}{dP} = \Big(\frac{\frac{dS}{dP}}{E_P(\frac{dS}{dP}|\mathscr{F}_2)}\Big)1_A + 1_{A^c}$$

Q is absolutely continuous with respect to P and its restriction to $\mathscr{F}_2$ is equal to P i.e. $Q \in \tilde{\mathscr{M}}_{2,3}$. (In general Q doesn't belong to $\mathscr{M}_{2,3}$, it is why we need the cocycle condition in that general situation *iii) (b)*). Denote $R \in \mathscr{M}_{1,2}$ the restriction of S to the σ-algebra $\mathscr{F}_2$. The triplet (S, Q, R) satisfies the relation (2.3) (because $\frac{dS}{dP}1_{A^c} = 0 \; P \; a.s.$). Thus

$$\rho_{1,3}(X) = E_R(E_Q(-X|\mathscr{F}_2)|\mathscr{F}_1)) - \alpha^m_{1,3}(S)$$

Applying the cocycle condition (2.4), this gives the inequality $\rho_{1,3}(X) \leq \rho_{1,2}(-\rho_{2,3}(X))$. So i) is proved. □

3 Time-consistent dynamic risk measures

3.1 Characterization of time consistency in terms of cocycle condition

In this section we consider a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in J},P)$, either in a continuous time or a discrete time setting. In the continuous time case $J=\mathbb{R}^+$, and the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}^+}$ is right continuous and $\mathscr{F}_0$ is assumed to be the σ-algebra generated by the P null sets so that $L^\infty(\Omega,\mathscr{F}_0,P)=\mathbb{R}$. In discrete time, $J=\mathbb{N}$, and $\mathscr{F}_0$ is a σ-algebra contained in the P null sets. We identify an essentially bounded $\mathscr{F}_t$-measurable function with its class in $L^\infty(\Omega,\mathscr{F}_t,P)$. The definition of dynamic risk measures given below, indexed by two dates, is close to the definition of non linear expectations introduced by Peng [27], and the notion of time-consistency first appeared in [27]. However the context of our work is that of a general filtration and not as in [27] the filtration generated by a d dimensional Brownian motion. As for the monetary and the conditional risk measures, we need the dynamic risk measures to be defined on the set of essentially bounded measurable random variables. Some authors (see [19] and [22]) have given such a definition, but considered a family indexed by a single time (corresponding to taking, in our definition below, the family (ρ_s) with the time t fixed). We note that, in the specific case of discrete time setting, a dynamic risk measure is defined in [12] as a family (ρ_n) of risk measures on $(\Omega,\mathscr{F},P)$ conditional to $(\Omega,\mathscr{F}_n,P)$. And finally, also in the discrete time case, in [6], a dynamic risk measure is defined for processes and not only for random variables.
We now give the relevant definition for our work.

Definition 3.1 A dynamic risk measure on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in J},P)$ is a family $(\rho_{s,t})_{0\le s\le t}$ of convex risk measures on $(\Omega,\mathscr{F}_t,P)$ conditional to $(\Omega,\mathscr{F}_s,P)$.
A dynamic risk measure is continuous from below (resp above) if each $\rho_{s,t}$ is continuous from below (resp above).

Definition 3.2 The dynamic risk measure $(\rho_{s,t})_{0\le s\le t}$ is time-consistent if

$$\forall r,s,t\in J\ \ r\le s\le t \ \textit{implies}\ \ \rho_{r,t}=\rho_{r,s}\circ(-\rho_{s,t}) \tag{3.1}$$

We deduce from Theorem 2.5 the following characterization of time consistency:

Theorem 3.3 *Let $(\rho_{s,t})_{0\le s\le t}$ be a dynamic risk measure continuous from below. It is time consistent if and only if the minimal penalty satisfies the cocycle condition (condition iii) of Theorem 2.5) for any instant of time $r\le s\le t$ in J.*

3.2 Dynamic risk measure in discrete time as a single conditional risk measure

In this section we restrict to discrete time i.e. to the case where $J=\mathbb{N}$. The aim of this section is to prove that in a discrete time setting, a time-consistent dynamic risk measure is nothing else that a conditional risk measure on a larger space.

Remark 3.4 The following statements are deduced from the composition rule
$\rho_{n,m}(X)=\rho_{n,n+1}(-\rho_{n+1,n+2}(...(-\rho_{m-1,m})(X)))$:
1. A time-consistent dynamic risk measure on $(\Omega,\mathscr{F},(\mathscr{F}_n)_{n\in\mathbb{N}},P)$ is uniquely determined by the family $(\rho_{n,n+1})_{n\in\mathbb{N}}$ of one step conditional risk measures.
2. With any family $(\rho_{n,n+1})_{n\in\mathbb{N}}$ of convex risk measures on $(\Omega,\mathscr{F}_{n+1},P)$ conditional to $(\Omega,\mathscr{F}_n,P)$ is canonically associated a time-consistent dynamic risk measure.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\in\mathbb{N}}, P)$ be a discrete time filtered probability space. Denote $\tilde{\Omega} = \Omega \times \mathbb{N}$ and $\tilde{\mathscr{F}}$ the σ-algebra generated by the sets $A_i \times \{i\}$ where $A_i \in \mathscr{F}_i$. Denote $\tilde{\mathscr{F}}^s$ the shifted algebra generated by the $A_i \times \{i\}, A_i \in \mathscr{F}_{i-1}$. Let $\tilde{P}$ be the probability measure on $\tilde{\mathscr{F}}$ defined by: $\tilde{P}(\cup_{i\in\mathbb{N}} A_i \times \{i\}) = \sum_{i\in\mathbb{N}} \frac{1}{2^{i+1}} P(A_i)$.

Proposition 3.5 *There is a canonical bijection between the set of time-consistent dynamic risk measures on* $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\in\mathbb{N}}, P)$ *and the set of convex risk measures on* $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ *conditional to* $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}^s, \tilde{P})$.

Proof. - Let $(\rho_{n,m})_{n\leq m}$ be a time-consistent dynamic risk measure on $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\in\mathbb{N}}, P)$. Define the risk measure $\Psi((\rho_{n,m})_{n\leq m}) = \rho$ on $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ conditional to $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}^s, \tilde{P})$ by

$$\rho(f)(\omega, i) = \rho_{i-1,i}(f_i)(\omega)$$

where $f_i(\omega) = f(\omega, i)$.
For any open set U in $\mathbb{R}$, $\rho(f)^{-1}(U) = \cup_{i\in\mathbb{N}}(\{\omega/\rho_{i-1,i}(f_i)(\omega) \in U\} \times \{i\}) = \cup_{i\in\mathbb{N}}(A_i \times \{i\})$ where A_i is $\mathscr{F}_{i-1}$ measurable. So $\rho(f)$ is $(\tilde{\Omega}, \tilde{\mathscr{F}}^s)$ measurable.
f is $\tilde{\mathscr{F}}^s$-measurable iff for any i, f_i is $\mathscr{F}_{i-1}$-measurable; so the translation invariance property of ρ follows from the translation invariance property of $\rho_{i-1,i}$ for all i.
Monotonicity and convexity of ρ easily follow from the same properties of the $\rho_{i-1,i}$ for all i.
So ρ is a convex risk measure on $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ conditional to $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}^s, \tilde{P})$.
- Conversely, consider a convex risk measure ρ on $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ conditional to $L^\infty(\tilde{\Omega}, \tilde{\mathscr{F}}^s, \tilde{P})$.
To each application $\mathscr{F}_i$-measurable g associate $\tilde{g}$ defined on $\tilde{\Omega}$ by $\tilde{g}(\omega, j) = 0$ if $j \neq i$ and $\tilde{g}(\omega, i) = g(\omega)$.
$\rho_{i-1,i}(g) = (\rho(\tilde{g}))_i$ defines a convex risk measure on $(\Omega, \mathscr{F}_i, P)$ conditional to $(\Omega, \mathscr{F}_{i-1}, P)$.
For any $i < j$, $\rho_{i,j}$ is then defined by the composition rule $\rho_{i,j} = \rho_{i,i+1}(-\rho_{i+1,i+2}(...(-\rho_{j-1,j})))$.
Define Φ by $\Phi(\rho) = (\rho_{i,j})_{i\leq j\in\mathbb{N}^*}$. Using the regularity property applied te the $\tilde{\mathscr{F}}^s$ measurable set $\tilde{\Omega} \times \{i\}$, it is not difficult to verify that Φ is the inverse function of Ψ. □

3.3 Examples of time-consistent dynamic risk measures

Backward Stochastic Differential Equations.

$(\mathscr{F}_t)_{t\in\mathbb{R}^+}$ is the augmented filtration of a d dimensional Brownian motion B_t. Assume that $g(t,z)$ is convex (in z), and satisfies the condition of quadratic growth. Let
$X \in L^\infty(\Omega, \mathscr{F}_T, P)$. The associated B.S.D.E.,

$$\begin{aligned} -dY_t &= g(t, Z_t)dt - Z_t^* dB_t \\ Y_T &= X \end{aligned}$$

has a solution which gives rise to a time consistent dynamic risk measure $\rho_{s,T}(-X) = Y_s$.
The paths of dynamic risk measures associated with B.S.D.E. are continuous.
For the B.S.D.E. and conditional g-expectations we refer to the works of Peng [26] and [27], and also to Barrieu and El Karoui [3], Klöppel and Schweizer [22], and Roosazza and Gianin [31].

Dynamic entropic risk measure with threshold.

Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}^+},P)$ be a filtered probability space. The study of the conditional risk measure associated with a loss function is done in [4], Section 5, generalizing to the conditional case the result proved in [17] for the monetary case.
When the loss function is exponential, this gives the dynamic entropic risk measure which has been studied by many authors starting with Rouge and El Karoui [29]. Detlefsen and Scandolo [12] have computed the minimal penalty. Barrieu and El Karoui [3] and Klöppel and Schweizer [22] have verified the time consistency.
Here we add thresholds. Let $(g_{s,t})_{0\leq s\leq t}$ be strictly positive $\mathscr{F}_s$-measurable functions, such that $\ln(g_{s,t})$ is essentially bounded ($g_{s,t}=1$ in the usual case). For $s\leq t$ define the set $\mathscr{A}_{s,t}$ of acceptable positions by

$$\mathscr{A}_{s,t}=\{Y\in L^\infty(\Omega,\mathscr{F}_t,P)\mid E(e^{-\alpha Y}|\mathscr{F}_s)\leq g_{s,t}\}$$

The dynamic entropic risk measure with threshold is defined by:

$$\begin{aligned}\rho_{s,t}(X) &= \text{essinf}\,\{Y\ \mathscr{F}_s-\text{measurable}\mid X+Y\in\mathscr{A}_{s,t}\}\\ &= \text{essmax}_{Q\in\mathscr{M}_{s,t}}(E_Q(-X|\mathscr{F}_s)-\alpha^m_{s,t}(Q))\end{aligned}$$

$\forall X\in L^\infty(\mathscr{F}_t)$, (cf [4], and [12] section 5), with

$$\alpha^m_{s,t}(Q)=\frac{1}{\alpha}(E_P(\ln(\frac{dQ}{dP})\frac{dQ}{dP}|\mathscr{F}_s)-\ln(g_{s,t})).$$

Proposition 3.6 *The dynamic entropic risk measure with thresholds $(g_{s,t})_{s\leq t}$ is time consistent if and only if the functions $g_{s,t}$ are $\mathscr{F}_0$ measurable and satisfy the relation $\ln(g_{r,t})=\ln(g_{r,s})+\ln(g_{s,t})$ a.s. for any $r\leq s\leq t$. In particular if we assume that there is a strictly positive real valued continuous function h such that $g_{s,t}=h(t-s)$ then the dynamic risk measure is time-consistent if and only if $g_{s,t}=e^{\lambda(t-s)}$ for some $\lambda\in\mathbb{R}$.*

Proof. Let $Q\in\tilde{\mathscr{M}}_{s,t}$, $R\in\mathscr{M}_{r,s}$ and $S\in\mathscr{M}_{r,t}$ be such that $\frac{dS}{dP}=\frac{dQ}{dP}\frac{dR}{dP}$,

$$\begin{aligned}E_R(\alpha^m_{s,t}(Q)|\mathscr{F}_r)+\alpha^m_{r,s}(R) &= \frac{1}{\alpha}[E_P\Big(\frac{dR}{dP}(E_P(\frac{dQ}{dP}\ln(\frac{dQ}{dP})|\mathscr{F}_s)|\mathscr{F}_r\Big)\\ &+(E_P(\frac{dR}{dP}\ln(\frac{dR}{dP})|\mathscr{F}_r)-E_P(\frac{dR}{dP}\ln(g_{s,t})|\mathscr{F}_r)-\ln(g_{r,s})].\end{aligned}$$

$\frac{dR}{dP}\ln(\frac{dR}{dP})$ is $\mathscr{F}_s$-measurable, $E(\frac{dQ}{dP}|\mathscr{F}_s)=1$ and $\frac{dS}{dP}=\frac{dQ}{dP}\frac{dR}{dP}$, so

$$E_R(\alpha^m_{s,t}(Q)|\mathscr{F}_r)+\alpha^m_{r,s}(R)=\alpha^m_{r,t}(S)+\frac{1}{\alpha}[\ln(g_{r,t})-\ln(g_{r,s})-E_P(\frac{dR}{dP}\ln(g_{s,t})|\mathscr{F}_r)]$$

Hence the dynamic risk measure is time-consistent if and only if

$$\ln(g_{r,t})-\ln(g_{r,s})-E_P(\frac{dR}{dP}\ln(g_{s,t})|\mathscr{F}_r)=0\ \ \forall R\in\mathscr{M}_{r,s}$$

It follows that $\ln(g_{s,t})$ has to be $\mathscr{F}_r$-measurable for any r so it is $\mathscr{F}_0$-measurable. The risk measure with thresholds $g_{s,t}$ is then time-consistent if and only if $\ln(g_{r,t})=\ln(g_{r,s})+\ln(g_{s,t})$ *a.s.* for any $r\leq s\leq t$.
The end of the proof is just the application of a classical result. □

Remark 3.7 In particular the usual dynamic entropic risk measure is time-consistent.

Cheridito and Kupper [7] have proved furthermore that loss functions lead to time-consistent dynamic risk measures if and only if the loss function is linear or exponential. Therefore to provide more examples of time consistent dynamic risk measures, we have to use a new methodology.

4 Time-consistent dynamic risk measures constructed from BMO martingales

4.1 Time consistency of a dynamic risk measure generated from a stable set of probability measures

In the preceding section we have characterized the time consistency of a dynamic risk measure in terms of a cocycle condition of the minimal penalty function. In this section we want to find a procedure in order to construct new time-consistent dynamic risk measures.
Delbaen has proved [9] that any stable set $\mathcal{Q}$ of probability measures gives rise to a coherent time-consistent dynamic risk measure.

$$\rho_{s,t}(X) = \text{esssup}_{Q\in\mathcal{Q}} E_Q(-X|\mathcal{F}_s)$$

We generalize here this result proving that any stable set of probability measures and any local penalty function satisfying the cocycle condition give rise to a time-consistent dynamic risk measure defined by

$$\rho_{s,t}(X) = \text{esssup}_{Q\in\mathcal{Q}}(E_Q(-X|\mathcal{F}_s) - \alpha^m_{s,t}(Q))$$

This result which doesn't assume that the penalty is the minimal one will be very useful in order to construct new families of time-consistent dynamic risk measures (Sections 4.3 and 4.4)
Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in J},P)$ be a filtered probability space. We define the stability condition for a set of probability measures all equivalent to P.

Definition 4.1 Let $\mathcal{Q}$ be a set of probability measures on $(\Omega,\mathcal{F})$ all equivalent to the probability measure P. $\mathcal{Q}$ is stable if

1. Stability by composition: $\forall r \le s \le t$, $\forall (Q,R) \in \mathcal{Q}^2$, there is a probability measure $S \in \mathcal{Q}$ such that $\forall X \in L^\infty(\Omega,\mathcal{F}_t,P)$,
$$E_S(X|\mathcal{F}_r) = E_R(E_Q(X|\mathcal{F}_s)|\mathcal{F}_r) \tag{4.1}$$
2. Stability by bifurcation: $\forall s \le t$, $\forall (Q,R) \in \mathcal{Q}^2$ $\forall A \in \mathcal{F}_s$ there is a probability measure $S \in \mathcal{Q}$ such that $\forall X \in L^\infty(\Omega,\mathcal{F}_t,P)$,
$$E_S(X|\mathcal{F}_s) = 1_A E_Q(X|\mathcal{F}_s) + 1_{A^c} E_R(X|\mathcal{F}_s) \tag{4.2}$$

Remark 4.2 (1) This condition of stability is close to the notion of m-stability introduced by Delbaen in [9]. The condition of stability given here is weaker than that of [9]: here it is formulated for deterministic times instead of stopping times, as the dynamic risk measure is indexed by deterministic times. Notice also that the conditions *1.* and *2.* enounced for deterministic times are not equivalent and are both needed in order to prove the time consistency.
(2) The conditions *1.* and *2.* in the preceding definition can be written in terms of Radon Nikodym derivatives. For example equation (4.1):

$$\frac{(\frac{dS}{dP})_t}{(\frac{dS}{dP})_r} = \frac{(\frac{dQ}{dP})_t}{(\frac{dQ}{dP})_s}\frac{(\frac{dR}{dP})_s}{(\frac{dR}{dP})_r}$$

where $(\frac{dS}{dP})_t$ means $E(\frac{dS}{dP}|\mathcal{F}_t)$.

Define the condition of locality and of cocycle for a penalty:

Definition 4.3 A penalty function α on a stable set $\mathcal{Q}$ of probability measures all equivalent to P is a family $(\alpha_{s,t})_{s\leq t}$ of functions defined on $\mathcal{Q}$ with values in $L^\infty(\Omega,\mathcal{F}_s,P)$.

1. Locality: α is local if, $\forall s\leq t$, $\forall Q,R\in\mathcal{Q}$, $\forall A\in\mathcal{F}_s$, the assertion $[\ 1_A E_Q(X|\mathcal{F}_s) = 1_A E_R(X|\mathcal{F}_s)\ \ \forall X\in L^\infty(\Omega,\mathcal{F}_t,P)\ ]$ implies
$$1_A\alpha_{s,t}(Q) = 1_A\alpha_{s,t}(R)$$
2. Cocycle Condition: $\forall r\leq s\leq t$, $\forall Q\in\mathcal{Q}$
$$\alpha_{r,t}(Q) = \alpha_{r,s}(Q) + E_Q(\alpha_{s,t}(Q)|\mathcal{F}_s) \tag{4.3}$$

As already seen (Lemma 2.3), the minimal penalty α^m is local.

Theorem 4.4 *Let $\mathcal{Q}$ be a stable set of probability measures on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in J},P)$ all equivalent to P. Let α be a local penalty on $\mathcal{Q}$ satisfying the cocycle condition. Assume also that for any $s\leq t$,* $\text{esssup}_{Q\in\mathcal{Q}}(-\alpha_{s,t}(Q))$ *is essentially bounded. Then the dynamic risk measure $(\rho_{s,t})_{0\leq r<s<t}$ defined by*
$$\rho_{s,t}(X) = \text{esssup}_{Q\in\mathcal{Q}}(E_Q(-X|\mathcal{F}_s)) - \alpha_{s,t}(Q)) \tag{4.4}$$
is time-consistent.

Proof. We adapt the proof of the implication *iii)* implies *i)* of Theorem 2.5 but here the penalty is not assumed to be the minimal one, and furthermore esssup in equation (4.4) is no more realized by one probability measure Q. The key point in this new proof is the lattice property proved in step one. Let $r\leq s\leq t$.
- Step one: We prove that for given X, $\{E_Q(-X|\mathcal{F}_s)-\alpha_{s,t}(Q)\ /Q\in\mathcal{Q}\}$ is a lattice upward directed. Let $(Q_1,Q_2)\in\mathcal{Q}^2$, let
$$A = \{\omega\in\Omega\ /\ E_{Q_1}(-X|\mathcal{F}_s)(\omega)-\alpha_{s,t}(Q_1)(\omega) > E_{Q_2}(-X|\mathcal{F}_s)(\omega)-\alpha_{s,t}(Q_2)(\omega)\}$$
From stability by bifurcation, there is $Q\in\mathcal{Q}$ such that
$$\forall Y\in L^\infty(\mathcal{F}_t)\ \ E_Q(Y|\mathcal{F}_s) = 1_A E_{Q_1}(Y|\mathcal{F}_s) + 1_{A^c}E_{Q_2}(Y|\mathcal{F}_s)$$
From the local property of $\alpha_{s,t}$,
$$1_A\alpha_{s,t}(Q) = 1_A\alpha_{s,t}(Q_1)\ \ \textit{and}\ \ 1_{A^c}\alpha_{s,t}(Q) = 1_{A^c}\alpha_{s,t}(Q_2)$$
This proves the lattice property.
- Step two: From the lattice property we deduce from [25], the existence of a sequence $R_n\in\mathcal{Q}$ such that $\rho_{r,s}(-\rho_{s,t}(X))$ is the increasing limit of $E_{R_n}(\rho_{s,t}(X)|\mathcal{F}_r)-\alpha_{r,s}(R_n)$, and a sequence $Q_k\in\mathcal{Q}$ such that $\rho_{s,t}(X)$ is the increasing limit of $E_{Q_k}(-X|\mathcal{F}_s)-\alpha_{s,t}(Q_k)$.
$\rho_{s,t}(X)$ is essentially bounded, thus from the monotone theorem for conditional expectations, for all n, $E_{R_n}(\rho_{s,t}(X)|\mathcal{F}_r)-\alpha_{r,s}(R_n)$ is the increasing limit of $E_{R_n}(E_{Q_k}(-X|\mathcal{F}_s)-\alpha_{s,t}(Q_k)|\mathcal{F}_r)-\alpha_{r,s}(R_n)$. From the stability by composition of $\mathcal{Q}$ and the cocycle condition, we get:
$$\rho_{r,s}(-\rho_{s,t}(X)) \leq \rho_{r,t}(X)$$
Conversely, for any $X\in L^\infty(\Omega,\mathcal{F}_t,P)$, there are R_n in $\mathcal{Q}$ such that $\rho_{r,t}(X)$ is the increasing limit of $E_{R_n}(-X|\mathcal{F}_r)-\alpha_{r,t}(R_n)$. From the cocycle condition
$$E_{R_n}(-X|\mathcal{F}_r)-\alpha_{r,t}(R_n) = E_{R_n}(E_{R_n}(-X|\mathcal{F}_s)-\alpha_{s,t}(R_n)|\mathcal{F}_r)-\alpha_{r,s}(R_n)$$
So
$$\rho_{r,t}(X) \leq \rho_{r,s}(-\rho_{s,t}(X))$$
which completes the proof. □

4.2 Stable Sets of probability measures

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}^+}, P)$ be a filtered probability space satisfying the usual conditions. In order to provide new examples of time-consistent dynamic risk measures, the first step is to construct stable sets of probability measures. To each set $\mathscr{Q}_1$ of probability measures equivalent to P we can associate a minimal stable set of probability measures containing $\mathscr{Q}_1$ (minimal with regard to inclusion). It is not difficult to verify that it is described in the following way:

Lemma 4.5 *Let $\mathscr{Q}_1$ be a set of probability measures all equivalent to P. The minimal stable set $\mathscr{Q}$ of probability measures containing $\mathscr{Q}_1$ is the set of probability measures Q such that there is a subdivision $0 = t_0 < t_1 < ...t_n$ and for any $i \in \{0,...n\}$ there is a finite family of disjoint $\mathscr{F}_{t_i}$ measurable sets $A_{i,j}$, $\cup_j A_{i,j} = \Omega$ and probability measures $Q_{i,j} \in \mathscr{Q}_1$ such that*

$$\frac{(\frac{dQ}{dP})_{t_{i+1}}}{(\frac{dQ}{dP})_{t_i}} = \sum_j \frac{(\frac{dQ_{i,j}}{dP})_{t_{i+1}}}{(\frac{dQ_{i,j}}{dP})_{t_i}} 1_{A_{i,j}} \text{ and } \forall t > t_n \ \frac{(\frac{dQ}{dP})_t}{(\frac{dQ}{dP})_{t_n}} = \sum_j \frac{(\frac{dQ_{n,j}}{dP})_t}{(\frac{dQ_{n,j}}{dP})_{t_n}} 1_{A_{n,j}} \tag{4.5}$$

where $(\frac{dQ}{dP})_t$ means $E(\frac{dQ}{dP}|\mathscr{F}_t)$.

The examples of stable sets of probability measures will be constructed from a stable set of martingales. The martingales considered are always null in zero. The stochastic exponential of the martingale M is the unique solution of the equation

$$\mathscr{E}(M)_t = 1 + \int_0^t \mathscr{E}(M)_{s^-} dM_s$$

(cf [28]) that is

$$\mathscr{E}(M)_t = \exp(M_t - \frac{1}{2}[M,M]^c_t)\Pi_{s\leq t}(1+\Delta M_s)e^{-\Delta M_s} \tag{4.6}$$

Notice that $\mathscr{E}(M)$ is positive if the jumps of M are strictly bounded from below by -1.

Definition 4.6 A set $\mathscr{M}$ of right continuous martingales is stable if $\forall\ 0 \leq s$, $\forall M,N \in \mathscr{M}$, $\forall A \in \mathscr{F}_s$, $(\tilde{M})_t$ defined by

$$\begin{aligned}(\tilde{M})_t &= (N_t - N_s)1_A + (M_t - M_s)1_{A^c} + M_s \ \ \forall s < t \\ (\tilde{M})_r &= M_r \ \forall r \leq s\end{aligned} \tag{4.7}$$

is a martingale in $\mathscr{M}$.

Remark 4.7 The set of continuous martingales on $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ (null in zero) is stable.

We get the following description of stable sets of martingales (analogous to stable set of probabilities):

Lemma 4.8 *Let $\mathscr{M}$ be a set of right continuous martingales. $\mathscr{M}$ is stable if for $0 = t_0 < ... < t_n$ and for any $i \in \{0,...n\}$, for any finite family of disjoint $\mathscr{F}_{t_i}$ measurable sets $(A_{i,j})_j$ such that $\cup_j A_{i,j} = \Omega$, for any $M_{i,j} \in \mathscr{M}$, $\tilde{M}$ defined by*

$$(\tilde{M})_t - (\tilde{M})_{t_i} = \sum_j ((M_{i,j})_t - (M_{i,j})_{t_i})1_{A_{i,j}} \ \ for\ \ t_i < t \leq t_{i+1}$$

$$(\tilde{M})_t - (\tilde{M})_{t_n} = \sum_j ((M_{n,j})_t - (M_{n,j})_{t_n})1_{A_{n,j}} \ \ for\ \ t_n < t$$

is in $\mathscr{M}$.

Lemma 4.9 *Let $\mathscr{M}$ be a stable set of right continuous martingales. Assume that for any $M \in \mathscr{M}$, $\mathscr{E}(M)$ is a uniformly integrable positive martingale. Then*

$$\mathscr{Q}(\mathscr{M}) = \{(Q_M)_{M \in \mathscr{M}}, \ \frac{dQ_M}{dP} = \mathscr{E}(M)\}$$

is a stable set of probability measures all equivalent to P.

Proof. As $(\mathscr{E}(M))$ is a uniformly integrable positive martingale, $\mathscr{E}(M)_\infty$ represents the density of a probability measure equivalent to P. The result follows then from lemmas 4.5 and 4.8. □

In [23] Lepingle and Memin give sufficient conditions for the uniform integability of $\mathscr{E}(M)$. Here we want furthermore to construct stable sets and for that, the theory of BMO martingales (Doléans-Dade and Meyer [13] and Kazamaki [21]) is very well adapted.

4.2.1 Examples of stable sets using continuous martingales.

We discuss two such examples.

Martingales with bounded quadratic variation.

Let $\mathscr{M}_1$ be defined by
$\mathscr{M}_1 = \{M \; continuous \; martingale, \; M_0 = 0 \mid [M,M]_\infty \in L^\infty(\Omega, \mathscr{F}, P)\}$ For any $M \in \mathscr{M}_1$, $\mathscr{E}(\mathscr{M})$ is a uniformly integrable positive martingale, thus from Lemma 4.9, $\mathscr{Q}(\mathscr{M}_1)$ is a stable set of probability measures.

BMO martingales.

In order to construct new stable sets of probability measures, the powerful theory of BMO martingales is very useful. For the theory of continuous BMO martingales we refer to [21]. Some results can be found in Appendix A.1.

Lemma 4.10 *Let $\mathscr{M}_2 = \{M \; continuous \; BMO \; martingale\}$. $\mathscr{Q}(\mathscr{M}_2)$ is a stable set of probability measures all equivalent to P.*

Proof. It is easy to verify that $\mathscr{M}_2$ is stable. From [21], for any $M \in \mathscr{M}_2$, $\mathscr{E}(M)$ is a uniformly integrable positive martingale. The result follows then from Lemma 4.9. □

4.2.2 Examples using the general case of right continuous BMO martingales.

For the theory of right continuous BMO martingales we refer to the works of Doléans-Dade and Meyer [13] and [14]. Some results are recalled in Appendix A.2. In view of these results, we have to construct a stable set $\mathscr{M}$ of right continuous BMO martingales of BMO norms uniformly bounded by $\frac{1}{8}$.

Lemma 4.11 *Let $M^1, ..., M^j$ be strongly orthogonal square integrable right continuous martingales. Let $(\Phi_i)_{1 \leq i \leq j}$ be non negative predictable processes. Assume that for any i, the stochastic integral $\Phi_i.M^i$ is a BMO martingale of BMO norm m^i. Let*

$$\mathscr{M} = \{ \sum_{1 \leq i \leq j} H_i.M^i \mid H_i \; predictable \; , \; |H_i| \leq \Phi_i \; a.s.\}$$

Any M in $\mathscr{M}$ is BMO and $||M||_{BMO} \leq (\sum_{1\leq i\leq j}(m^i)^2)^{\frac{1}{2}} = m$. Let

$$\mathscr{Q}(\mathscr{M}) = \{(Q_M)_{M\in\mathscr{M}};\ \frac{dQ_M}{dP} = \mathscr{E}(M)\}$$

- *If $m < \frac{1}{8}$, $\mathscr{Q}(\mathscr{M})$ is a stable set of probability measures equivalent to P.*
- *If any $(M^i)_{1\leq i\leq j}$ is continuous, the same result holds whithout any restriction on m.*

Proof. The set $\mathscr{M}$ of martingales is obviously stable. As the M^i are strongly orthogonal, it follows that each element of $\mathscr{M}$ is BMO and that $||M||^2_{BMO} \leq \sum_{1\leq i\leq j}(m^i)^2$.
Thus any element of M is BMO and $||M||_{BMO} \leq m$. From Theorem 2.3 of Kazamaki [21] in continuous case and from the result of Doléans Dade Meyer [13] recalled in Proposition A.4, $\mathscr{E}(M)$ is a uniformly integrable positive martingale for any M in $\mathscr{M}$. The result follows then from Lemma 4.9. □

4.3 Dynamic risk measure associated with a family of BMO continuous martingales

Denote $BMO(Q)$ the class of continuous BMO martingales with respect to the probability measure Q. When we want to stress that the reference probability is Q we use the notation $||\ ||_{BMO_p(Q)}$ instead of $||\ ||_{BMO_p}$. For any BMO continuous martingale M denote Q_M the probability measure with Radon Nykodym derivative $\frac{dQ_M}{dP} = \mathscr{E}(M)$. It is proved in [21] that the class $BMO(Q_M)$ is equal to the class $BMO(P)$. We prove here the more precise result:

Lemma 4.12 *For any $K > 0$ there exists $\tilde{K} > 0$ such that for any continuous $BMO(P)$ martingale M such that $||M||_{BMO_2(P)} \leq K$, for any X continuous $BMO(P)$ martingale, X is a $BMO(Q_M)$ martingale and*

$$||X||_{BMO_2(Q_M)} \leq \tilde{K}\ ||X||_{BMO_2(P)} \tag{4.8}$$

Proof. From Theorem 3.1. of [21] there is $p_0 \in]1,\infty[$ such that for $||M||_{BMO_2(P)} \leq K$, for $p \leq p_0$, $\mathscr{E}(M)$ satisfies the reverse Hölder inequality:

$$(R_p)\ \ E[(\mathscr{E}(M))^p_\infty|\mathscr{F}_T] \leq C_p(\mathscr{E}(M))^p_T$$

for any stopping time T (where C_p depends only on p).
Apply the conditional Hölder inequality (q is the conjugate exponent of p).

$$||X||_{BMO_1(Q_M)} \leq \sup_T ||(E((\frac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_T})^p|\mathscr{F}_T)||^{\frac{1}{p}}_\infty ||X||_{BMO_q(P)}$$

Applying now the inequalities (A-1), (4.8) and (R_p), we get

$$||X||_{BMO_2(Q_M)} \leq K_2K_q(C_p)^{\frac{1}{p}}||X||_{BMO_2(P)}$$

which is the announced result (4.8). □

We say that the BMO norms of a family $\mathscr{M}$ of continuous BMO martingales are uniformly bounded if $\{||M||_{BMO_1},\ M \in \mathscr{M}\}$ is bounded. From (A-1), it is equivalent to say that $\forall p \in [1,\infty[$, $\{||M||_{BMO_p},\ M \in \mathscr{M}\}$ is bounded.

We are now able to construct a new class of dynamic risk measures using continuous BMO martingales. In all the examples that we exhibit below, the fact that $\rho_{s,t}(0) = \text{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M))$ is bounded has to be guaranteed. For this, in some cases the existence of a bound on the coefficients and of a uniform bound on the BMO norms are assumed. In cases where esssup is realized (or if it is already known that $\rho_{s,t}(0)$ is bounded), a uniform bound on the BMO norms is not needed.

Proposition 4.13 *Let $\mathscr{M}$ be a stable set of continuous BMO martingales. Let (b_u) be a bounded predictable process. Define*

$$\rho_{s,t}(X) = \text{esssup}_{M\in\mathscr{M}}(E_{Q_M}(-X|\mathscr{F}_s) - \alpha_{s,t}(Q_M))$$

with

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\int_s^t b_u d[M,M]_u|\mathscr{F}_s)$$

i) If b is non negative and $0 \in \mathscr{M}$, $(\rho_{s,t})_{s\leq t}$ is a time-consistent normalized dynamic risk measure.
ii) If the BMO norms of elements of $\mathscr{M}$ are uniformly bounded, $(\rho_{s,t})_{s\leq t}$ is a time-consistent dynamic risk measure.

Proof. We verify first that $\text{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M))$ is essentially bounded.
- In case i), M is BMO so $\alpha_{s,t}(Q_M)$ is bounded. Furthermore $\alpha_{s,t}(Q_0) = 0$ and $\alpha_{s,t}(Q_M)$ is non negative, so $\rho_{s,t}(0) = \text{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M)) = 0$. $\rho_{s,t}$ is normalized. Notice that in that case the BMO norms don't have to be uniformly bounded.
- In case ii), The process b_s is bounded by C, so

$$||\alpha_{s,t}(Q_M)||_\infty \leq C||M||^2_{BMO_2(Q_M)}$$

From Lemma 4.12, as the BMO norms are uniformly bounded, it follows that

$$||\text{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M))||_\infty < \infty$$

It remains to prove that the penalty function α is local and satisfies the cocycle condition.
- α is local: Let $M_1, M_2 \in \mathscr{M}$, let $A \in \mathscr{F}_s$. Assume that for any $X \in L^\infty(\mathscr{F}_t)$, $E_{Q_{M_1}}(X1_A|\mathscr{F}_s) = E_{Q_{M_2}}(X1_A|\mathscr{F}_s)$. $(\frac{\mathscr{E}(M_1)_t}{\mathscr{E}(M_1)_s})1_A = (\frac{\mathscr{E}(M_2)_t}{\mathscr{E}(M_2)_s})1_A$. Let $u \in [s,t]$,

$$1_A((M_1)_u - (M_1)_s - \frac{1}{2}[M_1,M_1]_s^u) = 1_A((M_2)_u - (M_2)_s - \frac{1}{2}[M_2,M_2]_s^u)$$

From the uniqueness in the Doob Meyer decomposition, we deduce that $1_A((M_1)_u - (M_1)_s) = 1_A((M_2)_u - (M_2)_s)$. Thus $1_A\alpha_{s,t}(Q_{M_1}) = 1_A\alpha_{s,t}(Q_{M_2})$. So α is local.
- cocycle condition: Let $0 \leq r \leq s \leq t$, let $M,N,R \in \mathscr{M}$. Assume that for all X in $L^\infty(\mathscr{F}_t)$, $E_{Q_M}(X|\mathscr{F}_r) = E_{Q_R}(E_{Q_N}(X|\mathscr{F}_s)|\mathscr{F}_r)$. Then

$$\begin{aligned} E_{Q_R}(\alpha_{s,t}(Q_N)|\mathscr{F}_r) + \alpha_{r,s}(Q_R) &= E(\tfrac{\mathscr{E}(R)_s}{\mathscr{E}(R)_r}E(\tfrac{\mathscr{E}(N)_t}{\mathscr{E}(N)_s}\int_s^t b_u d[N,N]_u|\mathscr{F}_s)|\mathscr{F}_r) \\ &+E(\tfrac{\mathscr{E}(R)_s}{\mathscr{E}(R)_r}\int_r^s b_u d[R,R]_u|\mathscr{F}_r) \qquad (4.9) \end{aligned}$$

$Y_{r,s} = \frac{\mathscr{E}(R)_s}{\mathscr{E}(R)_r}(\int_r^s b_u d[R,R]_u)$ is $\mathscr{F}_s$-measurable. $\mathscr{E}(N)$ is a martingale. Thus

$$E(\frac{\mathscr{E}(N)_t}{\mathscr{E}(N)_s} Y_{r,s}|\mathscr{F}_r) = E(Y_{r,s}|\mathscr{F}_r) = \alpha_{r,s}(Q_R)$$

As in the proof of the locality of α, from the uniqueness in the Doob Meyer decomposition, it follows that $M_u - M_s = N_u - N_s \; \forall s \le u \le t$ and
$M_v - M_r = R_v - R_r \; \forall r \le v \le s$. Then

$$\int_r^t b_u d[M,M]_u = \int_r^s b_u d[R,R]_u + \int_s^t b_u d[N,N]_u$$

So α satisfies the cocycle condition. Hence from Theorem 4.4, $(\rho_{s,t})_{s\le t}$ is a time-consistent dynamic risk measure. □

Proposition 4.14 *Let $\mathscr{M}$ be a stable set of continuous BMO martingales uniformly bounded. Let A be a bounded predictable process. Then*

$$\rho_{s,t}(X) = \text{esssup}_{M\in\mathscr{M}}(E_{Q_M}(-X|\mathscr{F}_s) - \alpha_{s,t}(Q_M))$$

with $\alpha_{s,t}(Q_M) = E_{Q_M}(A.M_t - A.M_s|\mathscr{F}_s)$ defines a time-consistent dynamic risk measure.

Proof. This result can be proved easily directly following the same lines as the proof of Proposition 4.13. It can also be deduced from this proposition using Girsanov-Meyer Theorem [28],

$$E_{Q_M}(A.M_t - A.M_s|\mathscr{F}_s) = E_{Q_M}(\int_s^t A_u d[M,M]_u|\mathscr{F}_s)$$

□

We can construct variants of the preceding families. For example when

$$\mathscr{M} = \{\sum_{1\le i\le j} H_i.M^i \mid H_i \; predictable \; , \; |H_i| \le \Phi \;\; a.s.\}$$

as in Lemma 4.11 we can allow the process b of Proposition 4.13 or the process A of Proposition 4.14 to depend on H_i. These variants will be considered in the general case of right continuous BMO martingales (Propositions 4.19 and 4.20).

Link with B.S.D.E.

The following variant corresponds to the particular case where the stable set of martingales is the set of all BMO martingales obtained from a family of strongly orthogonal continuous martingales. This example generalizes the dynamic risk measures coming from B.S.D.E.

Proposition 4.15 *Let $(M^i)_{1\le i\le j}$ be strongly orthogonal continuous martingales. Then $\mathscr{M} = \{M = \sum_{1\le i\le j} H_i.M^i \mid H_i.M^l \text{ is BMO } \forall (i,l)\}$ is stable.*
Let $b_i(s,x_1,x_2,...,x_j)$ be Borel functions. Assume that there is a non negative predictable process ψ_i such that $\psi_i.M^i$ is BMO and $|b_i(s,x_1,x_2,...,x_j)| \le k((\psi_i)^2 + \sum_{1\le l\le j}|x_l|^2)$. Assume that

$$\rho_{s,t}(X) = \text{essmax}_{M\in\mathscr{M}}((E_{Q_M}(-X|\mathscr{F}_s) - \alpha_{s,t}(Q_M))$$

where for $M = \sum_{1\le i\le j} H_i.M^i$,

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\sum_{1\le i\le j}\int_s^t b_i(u,(H_1)_u,(H_2)_u,...,(H_j)_u)d[M^i,M^i]_u|\mathscr{F}_s)$$

Then the dynamic risk measure is time-consistent.

Proof. For any $M \in \mathscr{M}$,

$$||\alpha_{s,t}(Q_M)||_\infty \le k\sum_{1\le i\le j}(||\psi_i.M^i||^2_{BMO_2(Q_M)} + \sum_{1\le l\le j}||H_l.M^i||^2_{BMO_2(Q_M)})$$

By hypothesis $\rho_{s,t}$ is realized for one $M \in \mathscr{M}$. It follows then from Lemma 4.12, that $\text{essmax}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M))$ is essentially bounded. The proof ends as the proof of Proposition 4.13. □

Remark 4.16 Dynamic risk measures coming from B.S.D.E. are particular cases of dynamic risk measures constructed as in Proposition 4.15. The time consistency of these dynamic risk measures is already well known (dynamic programming). The risk measures associated with B.S.D.E. correspond to the case where the $(M^i)_{1\le i\le j}$ are independent Brownian motions and the filtration is the augmented filtration of the $(M^i)_{1\le i\le j}$. Consider the dynamic risk measure associated with a B.S.D.E. as in Section 3.3 (when the driver $g(t,z_1,...,z_j)$ is convex in $(z_1,...,z_j)$ and of quadratic growth). Barrieu and El Karoui ([3], section 7.3) have computed the dual representation, considering the set of continuous BMO martingales.

$$\rho_{s,t}(X) = \text{essmax}_{M\in\mathscr{M}}(E_{Q_M}(-X|\mathscr{F}_s) - \alpha_{s,t}(Q_M))$$

where $\mathscr{M} = \{\sum_{1\le i\le j} H_i.M^i|\ \sup_S ||E(\int_S^\infty H_i(u)^2du|\mathscr{F}_S)||_\infty < \infty\}$. This is exactly the set $\mathscr{M}$ considered in Proposition 4.15 when any M^i is a Brownian motion. The penalty function is

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\int_s^t G(u,(H_1)_u,(H_2)_u,...,(H_j)_u)du|\mathscr{F}_s)$$

where G is the Fenchel transform of g. When g is strongly convex G has quadratic growth as in the above proposition.

The new class of dynamic risk measures that we have constructed from continuous BMO martingales can thus be viewed as a generalization of the dynamic risk measures associated with B.S.D.E. However the examples constructed in this section stay inside the family of dynamic risk measures with continuous paths. We will provide now dynamic risk measures with jumps.

4.4 Dynamic risk measure associated with BMO martingales with jumps

We provide a new family of time-consistent dynamic risk measures using the stable sets constructed previously from right continuous BMO martingales (Section 4.2.2, Lemma 4.11). The importance of this family is that the paths associated with these dynamic risk measures may have jumps.
For two stopping times $S \le T$, denote $[M,M]^T_{S^-} = [M,M]_T - [M,M]_{S^-}$ We will make use of the two following technical lemmas:

Lemma 4.17 *Let M be a right continuous BMO martingale. Then for any stopping time T, $E(([M,M]^{\infty}_{T^-})^2|\mathscr{F}_T) \leq 2||M||^4_{BMO}$*

Proof. We apply Theorem 23 of Chapter V of [24] to the increasing process $[M,M]_t$, the constant positive random variable $||M||^2_{BMO}$ and the continuous increasing function $\phi(x) = 2x$.
Thus we get $E(([M,M]_\infty)^2|\mathscr{F}_0) \leq 2E([M,M]_\infty ||M||^2_{BMO})$. We end the proof as in the proof of Lemma 1 of [13] applying the preceding result to the martingale $M'_t = M_{T+t} - M_{T^-}$, and the σ-algebras $\mathscr{F}'_t = \mathscr{F}_{T+t}$. □

Lemma 4.18 *Let M be a right continuous BMO martingale with $|M||_{BMO} \leq K < \frac{1}{16}$. Then*

$$E((\frac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_{T^-}})^2|\mathscr{F}_T) \leq \frac{1}{1-16K} < \infty$$

Proof. Applying the formula of the stochastic exponential (4.6), we obtain

$$(\frac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_{T^-}})^2 = \exp[2(M_\infty - M_{T^-}) - ([M,M]^c_\infty - [M,M]^c_{T^-})]$$
$$(\Pi_{s \geq T}(1+\Delta M_s)e^{-\Delta M_s})^2$$

The jumps of M are bounded by $||M||_{BMO} < 1$. As in the proof of Theorem 1 in [13], it follows from the inequality $e^x \geq 1+x$ that any factor of the preceding product is between 0 and 1. Thus $(\frac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_{T^-}})^2 \leq \exp 2|M_\infty - M_{T^-}|$
Consider as in the end of the proof of Lemma 4.17 the martingale $M'_t = M_{T+t} - M_{T^-}$, and the σ-algebras $\mathscr{F}'_t = \mathscr{F}_{T+t}$. Let $M'^* = \sup_t |M'_t|$. It follows then from the John Nirenberg inequality (cf [13]) that

$$E(\exp(2M'^*)|\mathscr{F}'_0) \leq \frac{1}{1-16||M||_{BMO}}$$

and this gives the lemma. □

We are now able to construct two families of dynamic risk measures associated with stable sets of right continuous BMO martingales with BMO norms bounded by a constant $K < \frac{1}{16}$. The methodology is the same as in the case of continuous BMO martingales.

Proposition 4.19 *Let $(M^i)_{1 \leq i \leq j}$ ($(M^i)_0 = 0$) be a family of strongly orthogonal cadlag martingales. Let Φ be a locally bounded non negative predictable process such that for all i, $\Phi.M^i$ is BMO of BMO norm m^i. Let*

$$\mathscr{M} = \{\sum_{1 \leq i \leq j} H_i.M^i| \ |H_i| \leq \Phi\}$$

Let $a_i(s,x_1,x_2,...,x_j)$ be Borel functions with linear growth in (x_i); i.e. there is a constant $K>0$ such that $|a_i(s,x_1,x_2,...,x_j)| \leq K(\Phi + \sup_{1 \leq i \leq j}|x_i|)$. Define for $M = \sum_{1 \leq i \leq j} H_i.M^i$

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\sum_{1 \leq i \leq j} \int_s^t a_i(u,(H_1)_u,(H_2)_u,...,(H_j)_u)d(M^i)_u|\mathscr{F}_s)$$

Assume that $m = (\sum_{1 \leq i \leq j}(m_i)^2)^{\frac{1}{2}} < \frac{1}{16}$. Then

$$\rho_{s,t}(X) = \text{esssup}_{M \in \mathscr{M}}(E_{Q_M}(-X|\mathscr{F}_s) - \alpha_{s,t}(Q_M))$$

defines a time-consistent dynamic risk measure.

Proof. From Lemma 4.11, $\{Q_M,\, M \in \mathscr{M}\}$ is a stable set of probability measures equivalent to P. The process $a_i(u,(H_1)_u,(H_2)_u,...,(H_j)_u)$ is locally bounded predictable. The proof of locality and cocycle condition for the penalty is the same as in Proposition 4.13. Apply the conditional Cauchy Schwarz inequality. There is $C > 0$ such that for any $M \in \mathscr{M}$,

$$||\alpha_{s,t}(Q_M)||_\infty \le C \sum_{1\le i\le j} ||(E((\frac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_s})^2|\mathscr{F}_s)||_\infty^{\frac{1}{2}} ||E([\Phi.M^i,\Phi.M^i]_s^t|\mathscr{F}_s)||_\infty^{\frac{1}{2}} \tag{4.10}$$

From Lemma 4.18, it follows that $||\mathrm{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M))||_\infty$ is finite. The time consistency of $\rho_{s,t}$ follows then from Theorem 4.4. □

Proposition 4.20 *Let $\mathscr{M}$ be as in Proposition 4.19. Assume that $m < \frac{1}{16}$. Let $b_i(s,x_1,x_2,...,x_j)$ be Borel functions. For $M = \sum_{1\le i\le j} H_i.M^i$, define*

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\sum_{1\le i\le j} \int_s^t b_i(u,(H_1)_u,(H_2)_u,...,(H_j)_u)d[M^i,M^i]_u|\mathscr{F}_s)$$

Let

$$\rho_{s,t}(X) = \mathrm{esssup}_{M\in\mathscr{M}}((E_{Q_M}(-X|\mathscr{F}_s) - \alpha_{s,t}(Q_M))$$

- *If any b_i is non negative and $b_i(s,0,0,,0) = 0$, $(\rho_{s,t})$ is a normalized time-consistent dynamic risk measure.*
- *If any b_i is of quadratic growth i.e. there is positive real number k such that*

$$|b_i(s,x_1,x_2,...,x_k)| \le k(\Phi)^2 + \sum_{1\le i\le j} |x_i|^2),$$

  $\rho_{s,t}$ is a time-consistent dynamic risk measure.

Proof. The proof follows the same lines as that of the preceding proposition. The process $b_i(u,(H_1)_u,(H_2)_u,...,(H_j)_u)$ is predictable.

- If any b_i is non negative, $\alpha_{s,t}(Q_M) \ge 0$ and $\alpha_{s,t}(Q_0) = 0$. So $\mathrm{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M)) = 0$. $\rho_{s,t}(0) = 0$. The dynamic risk measure is normalized and time-consistent.
- If any b_i is of quadratic growth, let $M \in \mathscr{M}$. Apply the conditional Cauchy Schwarz inequality. From the hypothesis of quadratic growth, we get the existence of $\tilde{K}$ such that

$$||\alpha_{s,t}(Q_M)||_\infty \le \tilde{K}\sum_{1\le i\le j} ||(E(\tfrac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_s})^2|\mathscr{F}_s||_\infty^{\frac{1}{2}} ||E(([\Phi.M^i,\Phi.M^i]_s^t)^2|\mathscr{F}_s)||_\infty^{\frac{1}{2}} \tag{4.11}$$

  From Lemma 4.18,

$$E((\frac{\mathscr{E}(M)_\infty}{\mathscr{E}(M)_s})^2|\mathscr{F}_s) \le \frac{1}{1-16m} < \infty \tag{4.12}$$

  From Lemma 4.17,

$$E(([\Phi.M_i,\Phi.M_i]_s^t)^2|\mathscr{F}_s) \le 2(||\Phi.M_i||_{BMO})^4 \le 2m_i^4 \tag{4.13}$$

  Thus from equations (4.11), (4.12) and (4.13), $||\mathrm{esssup}_{M\in\mathscr{M}}(-\alpha_{s,t}(Q_M))||_\infty$ is finite, and the dynamic risk measure is time-consistent from Theorem 4.4. □

5 Conclusion

The time consistency is a crucial property for dynamic risk measures. Our main result is the characterization of time consistency by a cocycle condition for the minimal penalty function (Section 3 Theorem 3.3).
Since in discrete time, as shown in Section 3.2, a time-consistent dynamic risk measure is simply a conditional risk measure on a larger space, we then focused on the interesting case of continuous time.
Making use of the cocycle condition, we have introduced (Section 4) a new methodology in order to construct new families of time-consistent dynamic risk measures. The key point for this construction is Theorem 4.4: Any stable set of probability measures equivalent to a given probability measure, and any local penalty satisfying the cocycle condition give rise to a time-consistent dynamic risk measure.
We have constructed stable sets of probability measures Q_M from sets of martingales M, defining their Radon Nikodym derivative as the stochastic exponential $\mathscr{E}(M)$. Therefore $\mathscr{E}(M)$ has to be a positive uniformly integrable martingale. In order to satisfy this condition joint to the stability condition, the notion of BMO martingales is particularly well adapted. The set of all continuous BMO martingales gives rise to a stable set of probability measures. Starting with right continuous BMO martingales there is a restrictive condition on the BMO norms.
Given a stable set of probability measures, in order to construct general (not coherent) time-consistent dynamic risk measures the cocycle condition for the penalty is a crucial property. Taking advantage of the properties of BMO martingales we have constructed a new class of time-consistent dynamic risk measures which generalizes the risk measures coming from B.S.D.E.. Quite importantly, starting with right continuous BMO martingales with jumps, our construction leads to time-consistent dynamic risk measures with jumps.
These various examples will be very useful for dynamic pricing in incomplete markets. This will be the subject of a future work.

A Appendix: Some results on BMO martingales

A.1 Continuous BMO martingales

The reference for this subsection is [21]. Let $(M_t, \mathscr{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$ For $1 \leq p < \infty$, let

$$||M||_{BMO_p} = \sup_S ||E[|M_\infty - M_S|^p|\mathscr{F}_S]^{\frac{1}{p}}||_\infty$$

the sup being taken over all stopping times S.
There is (cf [21]) a positive constant K_p such that for any uniformly integrable continuous martingale:

$$||M||_{BMO_1} \leq ||M||_{BMO_p} \leq K_p ||M||_{BMO_1} \qquad \text{(A-1)}$$

Recall the following definition of continuous BMO martingales.

Definition A.1 (cf[21]) A uniformly integrable continuous martingale M is a BMO martingale if $||M||_{BMO_1} < \infty$.

Recall the following result ([21] Theorem 2.3.):
For any continuous BMO martingale M (null in zero), $\mathscr{E}(M)$ is a uniformly integrable positive martingale. (Notice that as M is continuous, $\mathscr{E}(M)_t = \exp(M_t - \frac{1}{2}[M,M]_t))$

A.2 Right Continuous BMO martingales

Definition A.2 (cf [13]) A right continuous uniformly integrable martingale M is BMO if there is a constant c such that for any stopping time S,

$$E([M,M]_\infty - [M,M]_{S^-}|\mathscr{F}_S) \leq c^2$$

The smallest c is by definition the BMO norm $||M||_{BMO}$

Remark A.3 (1) The size of the jumps is always bounded by $||M||_{BMO}$.
(2) When M is continuous, $||M||_{BMO}$ is equal to $||M||_{BMO_2}$ with the notations of Section 3.2.1.

Recall now the following result, which is a key result for the construction of time-consistent dynamic risk measures allowing for jumps. The proof of this result is included in the proof of Theorem 1 of Doléans-Dade and Meyer [13].

Proposition A.4 *(cf [13])*
Let M be a right continuous BMO martingale such that $||M||_{BMO} < \frac{1}{8}$, then $\mathscr{E}(M)$ is a strictly positive uniformly integrable martingale.

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