MESURES DE RISQUE DYNAMIQUES
DYNAMIC RISK MEASURES

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Outline

OUTLINE

INTRODUCTION

MONETARY RISK MEASURES:

Coherent risk measures: Artzner, Delbaen, Eber, Heath (1999)

Convex monetary risk measures: Föllmer and Schied (2002)and Frittelli and Rosazza Gianin (2002)

CONDITIONAL RISK MEASURES

on a probability space : Detlefsen Scandolo (2005)

in a context of uncertainty : Bion-Nadal (preprint 2004)

DYNAMIC RISK MEASURES

- Coherent Dynamic Risk Measures: Delbaen (2003) and Artzner, Delbaen, Eber, Heat, Ku (2004), and Riedel (2004)
- Convex dynamic risk measures considered in many recent papers, among them: Frittelli and Rosaza Gianin (2004), Klöppel, Schweizer (2005), Cheredito, Delbaen, Kupper (2006), Bion-Nadal (2006), Föllmer and Penner (2006)
- g expectations or Backward Stochastic Differential Equations : Peng (2004), Rosazza Gianin (2004) and Barrieu El Karoui (2005)

DYNAMIC RISK MEASURES

MAIN RESULTS FOR DYNAMIC RISK MEASURES

- Characterization of Time Consistency by a cocycle condition on the minimal penalty
- Regularity of paths for time consistent dynamic risk measures
- Families of dynamic risk measures constructed from right continuous BMO martingales generalizing B.S. D. E. and allowing for jumps.

APPLICATION TO DYNAMIC PRICING

Develop an axiomatic approach to associate to any financial product a dynamic ask price process and a dynamic bid price process taking into account liquidity risk and transaction costs. The ask price process is convex.
MAIN RESULT: A time consistent pricing procedure has No Free Lunch if and only if there is an equivalent probability measure R such that for any X the martingale $E_R(X|\mathcal{F}_t)$ is between the bid price process and the ask price process associated with X.
Furthermore the ask price process is then a R supermartingale with cadlag modification
This generalizes the result of Jouiny and Kallal which was obtained for sublinear ask prices.

OUTLINE

DYNAMIC RISK MEASURES

FRAMEWORK Consider a filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in R^+}, P)$. Assume that the filtration $(\mathcal{F}_t)$ is right continuous and that $\mathcal{F}_0$ is the σ-algebra generated by the P null sets of $\mathcal{F}_\infty$ so that $L^\infty(\Omega, \mathcal{F}_0, P) = \boldsymbol{R}$.
We work on stopping times. For a stopping time τ, recall that $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty | \forall t \in \boldsymbol{R}^+ \; A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$.

Dynamic Risk Measure

Definition

A Dynamic Risk Measure on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t \in R^+}, P)$ is a family of maps $(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau}$, ($\sigma \leq \tau$ are stopping times) defined on $L^\infty(\mathcal{F}_\tau)$ with values into $L^\infty(\mathcal{F}_\sigma)$ satisfying

1. monotonicity: *if* $X \leq Y$ *then* $\rho_{\sigma,\tau}(X) \geq \rho_{\sigma,\tau}(Y)$
2. translation invariance:

$$\forall Z \in L^\infty(\mathcal{F}_\sigma)\ ,\ \ \forall X \in L^\infty(\mathcal{F}_\tau)\ \ \rho_{\sigma,\tau}(X+Z) = \rho_{\sigma,\tau}(X) - Z$$

3. convexity:

$$\forall (X,Y) \in (L^\infty(\mathcal{F}_\tau))^2\ \ \forall \lambda \in [0,1]$$

$$\rho_{\sigma,\tau}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_{\sigma,\tau}(X) + (1-\lambda)\rho_{\sigma,\tau}(Y)$$

Continuity

Definition

A Dynamic Risk Measure $(\rho_{\sigma,\tau})$ is continuous from above (resp below) if for any decreasing (resp increasing) sequence X_n of elements of $L^\infty(\Omega, \mathcal{F}_\tau, P)$ converging to X, for any $\sigma \leq \tau$, the increasing (resp decreasing) sequence $\rho_{\sigma,\tau}(X_n)$ converges to $\rho_{\sigma,\tau}(X)$

Continuity from above is equivalent to the existence of a dual representation in terms of probability measures.
Continuity from below implies continuity from above.

Definition

A Dynamic Risk Measure $(\rho_{\sigma,\tau})$ is normalized if $\forall \sigma \leq \tau$, $\rho_{\sigma,\tau}(0) = 0$

Dual Representation

Proposition

Any Dynamic Risk Measure continuous from above has a dual representation : Let $\sigma \leq \tau$

$$\forall X \in L^{\infty}(\mathcal{F}_\tau) \;\; \rho_{\sigma,\tau}(X) = \; ess\ sup_{Q \in \mathcal{M}^1_{\sigma,\tau}(P)}[(E_Q(-X|\mathcal{F}_\sigma) - \alpha^m_{\sigma,\tau}(Q)]$$

where

$$\alpha^m_{\sigma,\tau}(Q) = Q\ ess\ sup_{X \in L^\infty(\Omega,\mathcal{F}_\tau,P)}[E_Q(-X|\mathcal{F}_\sigma) - \rho_{\sigma,\tau}(X)]$$

$$\mathcal{M}^1_{\sigma,\tau}(P) = \{Q \text{ on } (\Omega,\mathcal{F}_\tau)\ ,\ Q \ll P\ ,\ Q_{|\mathcal{F}_\sigma} = P \text{ and } E_P(\alpha^m_{\sigma,\tau}(Q)) < \ \infty\}$$

Dual Representation in case of continuity from below

Proposition

Any Dynamic Risk Measure continuous from below has the following dual representation

$$\forall X \in L^\infty(\mathcal{F}_\tau) \;\; \rho_{\sigma,\tau}(X) = \; ess\ max_{Q \in \mathcal{M}_{\sigma,\tau}(P)}[(E_Q(-X|\mathcal{F}_\sigma) - \alpha^m_{\sigma,\tau}(Q)]$$

where

$$\mathcal{M}_{\sigma,\tau}(P) = \{Q \ll P \;,\; Q_{|\mathcal{F}_\sigma} = P \text{ and } \alpha^m_{\sigma,\tau}(Q) \in L^\infty(\Omega, \mathcal{F}_\sigma, P\}$$

Time Consistency

Time Consistency

Definition

A Dynamic Risk Measure $(\rho_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is time consistent if

$$\forall 0 \leq \nu \leq \sigma \leq \tau \;\; \forall X \in L^\infty(\mathcal{F}_\tau) \;\; \rho_{\nu,\sigma}(-\rho_{\sigma,\tau}(X)) = \rho_{\nu,\tau}(X).$$

The notion of time consistency first appeared in the work of Peng.

- For coherent dynamic risk measures: Delbaen (2003)
- In a discrete time setting: Cheridito, Delbaen, Kupper: characterization by a condition on the acceptance sets and characterization by a concatenation condition.

Denote for $Q \ll P$

$$\mathcal{A}_{\nu,\tau}(Q) = \{Y \in L^\infty(\Omega, \mathcal{F}_\tau, P) \mid \rho_{\nu,\tau}(Y) \leq 0 \;\; Q\, a.s.\}$$

COCYCLE CONDITION

THEOREM

Let $(\rho_{\sigma,\tau})$ be a Dynamic Risk Measure continuous from above. The three following properties are equivalent. Let $\nu \leq \sigma \leq \tau$

I) *The dynamic risk measure is time consistent i.e.*

$$\rho_{\nu,\tau}(X) = \rho_{\nu,\sigma}(-\rho_{\sigma,\tau}(X)) \;\; \forall X \in L^{\infty}(\Omega, \mathcal{F}_{\tau})$$

II) *For any probability measure Q absolutely continuous with respect to P,*

$$\mathcal{A}_{\nu,\tau}(Q) = \mathcal{A}_{\nu,\sigma}(Q) + \mathcal{A}_{\sigma,\tau}(P)$$

III) *For any probability measure Q absolutely continuous with respect to P, the minimal penalty function satisfies the cocycle condition*

$$\alpha^m_{\nu,\tau}(Q) = \alpha^m_{\nu,\sigma}(Q) + E_Q(\alpha^m_{\sigma,\tau}(Q)|\mathcal{F}_{\nu}) \;\; Q\; a.s.$$

CADLAG MODIFICATION

Denote

$$\mathcal{M}^0_{0,\tau} = \{Q \ll P \mid \alpha^m_{0,\tau}(Q) = 0\}$$

THEOREM

Let $(\rho_{\sigma,\tau})_{\sigma\leq\tau}$ be a normalized time consistent dynamic risk measure continuous from above. Assume that $\mathcal{M}^0_{0,\tau} \neq \emptyset$. For any $X \in L^\infty(\mathcal{F}_\tau)$, for any $R \in \mathcal{M}^0_{0,\tau}$, there is a cadlag R-supermartingale process Y such that for any finite stopping time $\sigma \leq \tau$, $\rho_{\sigma,\tau}(X) = Y_\sigma$ R a.s.

The proof follows that of Delbaen for coherent dynamic risk measures and is based on theorems of Dellacherie Meyer.

LEMMA

Same hypothesis. Then the process $(\rho_{\sigma,\tau}(X))_\sigma$ is a R-supermartingale:

$$\forall \nu \leq \sigma \leq \tau \quad \rho_{\nu,\tau}(X) \geq E_R(\rho_{\sigma,\tau}(X)|\mathcal{F}_\nu) \ R \ a.s.$$

CADLAG MODIFICATION

LEMMA

Same hypothesis. Let $\sigma \leq \tau$, σ finite. Consider a decreasing sequence of finite stopping times $\sigma_n \leq \tau$ converging to σ. Then $E_R(\rho_{\sigma_n,\tau}(X))$ converges to $E_R(\rho_{\sigma,\tau}(X))$, and $\rho_{\sigma_n,\tau}(X)$ tends to $\rho_{\sigma,\tau}(X)$ in $L^1(\Omega, \mathcal{F}_\infty, R)$.

DEFINITION

A Normalized Dynamic Risk Measure $(\rho_{\sigma\tau})$ is called non degenerate if $\forall A \in \mathcal{F}_\infty$, $\rho_{0,\infty}(\lambda 1_A) = 0 \ \ \forall \lambda \in \boldsymbol{R}_+{}^*$ implies $P(A) = 0$.

COROLLARY

Let $(\rho_{\sigma,\tau})$ be time consistent, non degenerate and continuous from below. Then $\mathcal{M}^0_{0,\infty} \neq \emptyset$. Any $R \in \mathcal{M}^0_{0,\infty}$ is equivalent to P and $\forall X \in L^\infty(\mathcal{F}_\infty)$ there is a cadlag R-supermartingale process Y, such that for all finite σ,

$$\rho_{\sigma,\infty}(X) = Y_\sigma \in L^\infty(\Omega, \mathcal{F}_\sigma, P)$$

EXAMPLE:ENTROPIC RISK

ENTROPIC DYNAMIC RISK MEASURE

$$\mathcal{A}_{s,t} = \{Y \in L^\infty(\mathcal{F}_t) \mid E(e^{[-\alpha Y]}|\mathcal{F}_s) \leq 1\}$$

$$\begin{aligned} \rho_{s,t}(X) &= \mathrm{essinf}\{Y \in L^\infty(\mathcal{F}_s) \,|X+Y \in \mathcal{A}_{s,t}\} \\ &= \mathrm{essinf}\{Y \in L^\infty(\mathcal{F}_s) \;/\; E(e^{[-\alpha(X+Y)]}|\mathcal{F}_s) \leq 1\} \\ &= \frac{1}{\alpha} \ln E(e^{-\alpha X}|\mathcal{F}_s) \end{aligned}$$

$$\alpha^m_{s,t}(Q) = \frac{1}{\alpha}(E_P(\ln(\frac{dQ}{dP})\frac{dQ}{dP}|\mathcal{F}_s)$$

The entropic dynamic risk measure is time-consistent.

Also: Barrieu el Karoui, Cheredito Kupper, Detlefsen Scandolo.
Cheredito Kupper: Expected utilities lead to time-consistent dynamic risk measures iff the utility fonction is linear or exponential.

EXAMPLE:BSDE

BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

$\mathcal{F}_t$ is the augmented filtration of a d dimensional Brownian motion. Assume that $g(t,z)$ is convex (in z) and satisfies the condition of quadratic growth. The associated BSDE,

$$\begin{aligned} -dY_t &= g(t,Z_t)dt - Z_t^* dB_t \\ Y_T &= X \end{aligned}$$

has a solution which gives rise to a time-consistent dynamic risk measure $\rho_{s,T}(-X) = Y_s$

Peng. Also Barrieu El Karoui, Klöppel Schweizer, Roosazza Gianin...

The paths of dynamic risk measures associated with B.S.D.E. are continuous.

Outline

STABLE SET OF PROBABILITY MEASURES

Conditions on $\mathcal{Q}$ and α so that

$$\rho_{\sigma,\tau}(X) = \mathrm{esssup}_{\mathrm{Q}\in\mathcal{Q}}\{\mathrm{E_Q}(-\mathrm{X}|\mathcal{F}_\sigma)) - \alpha_{\sigma,\tau}(\mathrm{Q})\}$$

defines a time consistent Dynamic Risk Measure.

DEFINITION

A set $\mathcal{Q}$ of probability measures all equivalent to P is stable if for every stopping times, $\nu \leq \sigma \leq \tau$, for every $Q \in \mathcal{Q}$, for every $R \in \mathcal{Q}$, there is $S \in \mathcal{Q}$ such that

$$\forall X \in L^\infty(\mathcal{F}_\tau),\ E_S(X|\mathcal{F}_\nu) = E_R(E_Q(X|\mathcal{F}_\sigma)|\mathcal{F}_\nu)\ P.a.s.$$

i.e.

$$(\frac{dS}{dP})_\tau = \frac{(\frac{dQ}{dP})_\tau}{(\frac{dQ}{dP})_\sigma}(\frac{dR}{dP})_\sigma$$

where $(\frac{dS}{dP})_t$ means $E(\frac{dS}{dP}|\mathcal{F}_t)$

Conditions on the penalty function

Definition

A penalty function $\alpha_{\sigma,\tau}$ defined on $\mathcal{Q}$ with values in L^∞

- is local if for every stopping times $\sigma \leq \tau$, for every A $\mathcal{F}_\sigma$-measurable, if $E_{Q_1}(X1_A|\mathcal{F}_\sigma) = E_{Q_2}(X1_A|\mathcal{F}_\sigma)$ $P.a.s.$ $\forall X \in L^\infty(\Omega, \mathcal{F}_\tau, P)$, then $1_A\alpha_{\sigma,\tau}(Q_1) = 1_A\alpha_{\sigma,\tau}(Q_2)$ $P.a.s.$
- satisfies the cocycle condition if for every Q in $\mathcal{Q}$, for every stopping times $\nu \leq \sigma \leq \tau$,

$$\alpha_{\nu,\tau}(Q) = \alpha_{\nu,\sigma}(Q) + E_Q(\alpha_{\sigma,\tau}(Q)|\mathcal{F}_\nu)$$

SUFFICIENT CONDITION FOR TIME CONSISTENCY

The following theorem allows for the construction of many examples of time consistent Dynamic Risk Measures:

THEOREM

Let $\mathcal{Q}$ be a stable set of probabilities all equivalent to P. Assume that α is local, satisfies the cycle condition and $\text{esssup}_{\mathrm{Q}\in\mathcal{Q}}(-\alpha_{\sigma,\tau}(\mathrm{Q})) \in \mathrm{L}^\infty(\mathcal{F}_\sigma)$ *. Then the Dynamic Risk Measure $(\rho_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ defined by*

$$\rho_{\sigma,\tau}(X) = \text{esssup}_{\mathrm{Q}\in\mathcal{Q}}\{\mathrm{E_Q}(-\mathrm{X}|\mathcal{F}_\sigma)) - \alpha_{\sigma,\tau}(\mathrm{Q})\}$$

is time-consistent.

In the case of penalty identically equal to 0 on $\mathcal{Q}$ this result was first proved by Delbaen.

STOCHASTIC EXPONENTIAL

$(M)_t$ martingale, $M_0 = 0$. Stochastic exponential of M: unique solution of

$$\mathcal{E}(M)_t = 1 + \int_0^t \mathcal{E}(M)_{s^-}\, dM_s$$

$$\mathcal{E}(M)_t = exp(M_t - \frac{1}{2}([M,M]^c)_t)\Pi_{s\leq t}(1+\Delta M_s)e^{-\Delta M_s}$$

MARTINGALES WITH BOUNDED QUADRATIC VARIATION

$$\mathcal{Q}_1 = \{Q_M \; ; \; \frac{dQ_M}{dP} = \mathcal{E}(M) \mid M \; continuous \; P \; martingale \; ;$$

$$[M,M]_\infty \in L^\infty(\Omega, \mathcal{F}, P)\}$$

is a stable set of probability measures all equivalent to P.

BMO MARTINGALES

BMO MARTINGALES(Doleans-Dade Meyer)
a right continuous square integrable martingale M is BMO if there is a constant c such that for any stopping time S,

$$E([M,M]_\infty - [M,M]_{S-} | \mathcal{F}_S) \leq c^2$$

The smallest c is $||M||_{BMO}$
CASE OF CONTINUOUS BMO MARTINGALES

$$\mathcal{Q}_2 = \{Q_M \ ; \ \frac{dQ_M}{dP} = \mathcal{E}(M) \mid M \textit{ continuous martingale} \ ; ||M||_{BMO} < \infty\}$$

is a stable set of probability measures all equivalent to P.

STABLE SETS

RIGHT CONTINUOUS BMO MARTINGALES

For right continuous BMO martingales: restrictions on the BMO norms.

LEMMA

Let $M^1, ..., M^j$ be strongly orthogonal square integrable right continuous martingales in $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t}, P)$. Let Φ be a non negative predictable process such that $\Phi.M^i$ is a BMO martingale of BMO norm m^i. Let

$$\mathcal{M} = \{ \sum_{1\leq i\leq j} H_i.M^i,\ H_i \text{ predictable } |H_i| \leq \Phi\}$$

Any $M \in \mathcal{M}$ is a BMO martingale with BMO norm less than $(\sum_{1\leq i\leq j}(m^i)^2)^{\frac{1}{2}} = m$. If $m < \frac{1}{8}$, $\mathcal{Q}(\mathcal{M}) = \{(Q_M)_{M\in\mathcal{M}} \mid \frac{dQ_M}{dP} = \mathcal{E}(M)\}$ is a stable set of equivalent probability measures.
No restriction on m if each M^i is continuous.

TIME-CONSISTENT DYNAMIC RISK MEASURES FROM BMO RIGHT CONTINUOUS MARTINGALES

THEOREM

Let $\mathcal{M}$ be a set of BMO martingales; Assume that $\mathcal{Q}(\mathcal{M}) = \{Q_M \mid \frac{dQ_M}{dP} = \mathcal{E}(M),\ M \in \mathcal{M}\}$ is a stable set of probability measures all equivalent to P. Assume $\{||M||_{BMO} \mid M \in \mathcal{M}\}$ is bounded (by $\frac{1}{16}$ in right continuous case)
Let H a bounded predictible process.

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}((H.M)_\tau - (H.M)_\sigma|\mathcal{F}_\sigma)$$

$$\rho_{\sigma,\tau}(X) = \mathrm{esssup}_{Q_M \in \mathcal{Q}(\mathcal{M})}(E_{Q_M}(-X|\mathcal{F}_\sigma) - \alpha_{\sigma,\tau}(Q_M))$$

defines a time consistent dynamic risk measure.

Dynamic Risk Measures from BMO martingales

Theorem

Let $\mathcal{Q}(\mathcal{M}) = \{Q_M \mid \frac{dQ_M}{dP} = \mathcal{E}(M),\ M \in \mathcal{M}\}$ *be a stable set of probability measures such that any* $M \in \mathcal{M}$ *is BMO.*

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}(\int_\sigma^\tau b_u d[M,M]_u)|\mathcal{F}_\sigma)$$

- *If b bounded and* $\{||M||_{BMO}\}$ *bounded (by* $\frac{1}{16}$ *in right continuous case)*
- *or if b is non negative and* $0 \in \mathcal{M}$

$$\rho_{\sigma,\tau}(X) = \mathrm{esssup}_{Q_M \in \mathcal{Q}(\mathcal{M})}(E_{Q_M}(-X|\mathcal{F}_\sigma) - \alpha_{\sigma,\tau}(Q_M))$$

defines a time-consistent dynamic risk measure.

Dynamic Risk Measures from BMO martingales generalize BSDE

Proposition

Let (M^i), $(M_0^i = 0)$ be strongly orthogonal continuous martingales. Let $\mathcal{M} = \{\sum_{1 \leq i \leq j} H_i.M^i \mid H_i.M^l \text{ BMO } \forall (i,l)\}$. Let $b_i(s, x_1, \ldots, x_k)$ Borel functions of quadratic growth. For $M = \sum_{1 \leq i \leq j} H_i.M^i$, define

$$\alpha_{s,t}(Q_M) = E_{Q_M}\Big(\sum_{1 \leq i \leq j} \int_s^t b_i(u, (H_1)_u, (H_2)_u, \ldots, (H_j)_u) d[M^i, M^i]_u | \mathcal{F}_s\Big)$$

Assume that

$$\rho_{s,t}(X) = \mathrm{essmax}_{M \in \mathcal{M}}((E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$$

Then $\rho_{s,t}$ is a time-consistent dynamic risk measure.

Dynamic Risk Measures from BMO martingales generalize BSDE

N. El Karoui and P. Barrieu have computed the dual representation associated withe BSDE when g is strictly convex of quadratic growth. $(M^i)_{1\leq i\leq j}$ are independent Brownian motions. The filtration is the augmented filtration of the $(M^i)_{1\leq i\leq j}$. Thus $\mathcal{M}$ as in proposition is $\mathcal{M} = \{\sum_{1\leq i\leq k} H_i.M^i| \ \sup_S ||E(\int_S^\infty H_i(u)^2 du|\mathcal{F}_S)||_\infty < \infty\}$. The penalty function is

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\int_s^t G(u,(H_1)_u,(H_2)_u,...,(H_k)_u)du|\mathcal{F}_s)$$

where G is the Fenchel transform of g.

Risk Measures from right continuous BMO martingales

Proposition

Let $\mathcal{M} = \{\sum_{1\le i\le j} H_i.M^i |\ |H_i| \le \Phi\}$ $m = (\sum ||\Phi.M^i||^2_{BMO})^{\frac{1}{2}} < \frac{1}{16}$. *Define*

$$\alpha_{s,t}(Q_M) = E_{Q_M}(\sum_{1\le i\le j} \int_s^t b_i(u,(H_1)_u,(H_2)_u,\dots,(H_j)_u)d[M^i,M^i]_u|\mathcal{F}_s)$$

$$\rho_{s,t}(X) = \text{esssup}_{M\in\mathcal{M}}((\mathrm{E}_{\mathrm{Q_M}}(-\mathrm{X}|\mathcal{F}_\mathrm{s}) - \alpha_{\mathrm{s,t}}(\mathrm{Q_M}))$$

- *If any* b_i *is non negative and* $b_i(s,0,0,,0) = 0$, *or*
- *If any* b_i *is of quadratic growth*

$\rho_{s,t}$ *is a time-consistent dynamic risk measure.*

OUTLINE

Economic Model

In financial markets, for any traded security you observe a book of orders.
For a financial product X: bid and ask prices associated with nX ($n \in N$).
For n large the ask price of nX is greater than n times the ask price of X.

Dynamic Pricing

Financial position at time τ: $X \in L^\infty(\mathcal{F}_\tau)$
Goal: Assign to X a dynamic ask price process $(\Pi_{\sigma,\tau}(X))_{\sigma \leq \tau}$ and a dynamic bid price process $(-\Pi_{\sigma,\tau}(-X))_{\sigma \leq \tau}$ (selling X is the same as buying $-X$).
Taking into account the impact of diversification, and liquidity risk , this leads to the convexity of the ask price.
Develop an axiomatic approach taking inspiration from that of Jouiny Kallal for sublinear pricing.

Time Consistent Pricing Procedure

Definition

A Time Consistent Pricing Procedure, TCPP, $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is a family of maps $(\Pi_{\sigma,\tau}) : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\sigma)$ continuous from below, satisfying

1. monotonicity: *if* $X \leq Y$ *then* $\Pi_{\sigma,\tau}(X) \leq \Pi_{\sigma,\tau}(Y)$
2. translation invariance: $\forall Z \in L^\infty(\mathcal{F}_\sigma)$, $\forall X \in L^\infty(\mathcal{F}_\tau)$
$$\Pi_{\sigma,\tau}(X+Z) = \Pi_{\sigma,\tau}(X) + Z$$
3. convexity: $\forall (X,Y) \in (L^\infty(\mathcal{F}_\tau))^2$
$$\Pi_{\sigma,\tau}(\lambda X + (1-\lambda)Y) \leq \lambda\Pi_{\sigma,\tau}(X) + (1-\lambda)\Pi_{\sigma,\tau}(Y)$$
4. normalization: $\Pi_{\sigma,\tau}(0) = 0$
5. time consistency: $\forall X \in L^\infty(\mathcal{F}_\tau)$, $\quad \Pi_{\nu,\sigma}(\Pi_{\sigma,\tau}(X)) = \Pi_{\nu,\tau}(X)$

Time Consistency

Remark

1. $\rho_{\sigma,\tau}(X) = \Pi_{\sigma,\tau}(-X)$ *is a normalized time consistent Dynamic Risk Measure continuous from above.*
2. *For any stopping time* τ *for any* $X \in L^\infty(\Omega, \mathcal{F}_\tau)$

$$-\Pi_{\sigma,\tau}(-X) \leq \Pi_{\sigma,\tau}(X)$$

No Free Lunch Pricing Procedure

Definition

Set financial products atteignable at zero cost via self financing simple strategies:

$$\mathcal{K}_0 = \{X = X_0 + \sum_{1 \leq i \leq n} (Z_i - Y_i),\ (X_0, Z_i, Y_i) \in L^\infty(\mathcal{F}_\infty) \ \mid$$
$$\Pi_{0,\infty}(X_0) \leq 0;\ \Pi_{\tau_i,\infty}(Z_i) \leq -\Pi_{\tau_i,\infty}(-Y_i)\ \forall 1 \leq i \leq n\}$$

where $0 \leq \tau_1 \leq ... \leq \tau_n < \infty$ are stopping times.

Definition

The TCPP has No Free Lunch if $\overline{\mathcal{K}} \cap L^\infty_+(\Omega, \mathcal{F}_\infty, P) = \{0\}$ where $\overline{\mathcal{K}}$ is the weak* closure of

$$\mathcal{K} = \{\lambda X, (\lambda, X) \in \boldsymbol{R}^+ \times \mathcal{K}_0\}$$

First Fundamental Theorem

Theorem

Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ *be a TCPP. The following conditions are equivalent:*
i) The Dynamic Pricing Procedure has No Free Lunch.
ii) There is a probability measure R equivalent to P with zero penalty $\alpha^m_{0,\infty}(R) = 0$
iii) There is a probability measure R equivalent to P such that

$$\forall X \in L^\infty(\Omega, \mathcal{F}_\tau, P) \;\; \forall \sigma \leq \tau \quad -\Pi_{\sigma,\tau}(-X) \leq E_R(X|\mathcal{F}_\sigma) \leq \Pi_{\sigma,\tau}(X) \quad (1)$$

Cadlag Pricing Processes

Corollary

Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ *be a No Free Lunch TCPP. For any* $X \in L^\infty(\Omega, \mathcal{F}_\infty, P)$, *for any probability measure R equivalent to P with zero penalty, there is a cadlag modification of* $(\Pi_{\sigma,\tau}(X))_{\sigma\leq\tau}$ *which is a R-supermartingale process (resp. a cadlag modification of* $-(\Pi_{\sigma,\tau}(-X))_{\sigma\leq\tau}$ *which is a R-submartingale process), and*

$$-\Pi_{\sigma,\tau}(-X) \leq E_R(X|\mathcal{F}_\sigma) \leq \Pi_{\sigma,\tau}(X) \quad (2)$$

for any stopping time $\sigma \leq \tau$.

STRONG ADMISSIBILITY

DEFINITION

A TCPP $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is strongly admissible with respect to the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the observed bid (resp ask) prices $({C^l}_{bid}, {C^l}_{ask})_{1\leq l\leq p}$ if

- it extends the dynamics of the process $(S^k)_{0\leq k\leq d}$

$$\text{if } \; S^k_\tau \in L^\infty(\mathcal{F}_\tau) \; \text{ then } \; \Pi_{\sigma,\tau}(nS^k_\tau) = nS^k_\sigma$$

- it is compatible with the observed bid and ask prices for the $(Y^l)_{1\leq l\leq p}$

$$\forall 1 \leq l \leq p \quad {C^l}_{bid} \leq -\Pi_{0,\tau_l}(-Y^l) \leq \; \Pi_{0,\tau_l}(Y^l) \leq {C^l}_{ask}$$

CHARACTERIZATION OF STRONG ADMISSIBILITY

PROPOSITION

A TCPP is strongly admissible with respect to the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the observed bid (resp ask) prices $({C^l}_{bid}, {C^l}_{ask})_{1\leq l\leq p}$ if and only if

- *Any probability measure $R \ll P$ such that $\alpha^m_{0,\infty}(R) < \infty$ is a local martingale measure with respect to any process S^k.*
- *For any probability measure $R \ll P$, for any stopping time τ,*

$$\alpha^m_{0,\tau}(R) \geq \sup(0, \sup_{\{l \mid \tau_l \leq \tau\}} (C^l_{bid} - E_R(Y^l), E_R(Y^l) - C^l_{ask}) \quad (3)$$

Let Q_0 be an equivalent local martingale measure for S^k such that

$$\forall l \;\; C^l_{bid} \leq E_{Q_0}(Y^l) \leq C^l_{ask}$$

Admissible Pricing from BMO martingales

Theorem

Let $\mathcal{M} = \{M = \sum_{1\leq i\leq j} H_i.M_i,\ H_i \leq \Phi\}$, *set of BMO martingales either continuous, or right continuous and of BMO norm less than* $m < \frac{1}{16}$
$\mathcal{Q}(\mathcal{M}) = \{Q_M \mid \frac{dQ_M}{dQ_0} = \mathcal{E}(M),\ M \in \mathcal{M}\}$ *Let* b_s *be a bounded non negative predictable process. Let*

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}\Big(\int_\sigma^\tau b_u d[M,M]_u\Big)|\mathcal{F}_\sigma)$$

$$\Pi_{\sigma,\tau}(X) = ess\ inf_{Q_M\in\mathcal{Q}(\mathcal{M})}(E_{Q_M}(X|\mathcal{F}_\sigma) + \alpha_{\sigma,\tau}(Q_M))$$

defines a No Free Lunch time TCPP.
If any M_i *is strongly orthogonal to any martingale* S^k, *there is* $B \in \mathbf{R}^+$ *such that the procedure is strongly admissible with respect to* (S^k, Y^l) *if* $b_u \geq B$.

OUTLINE

Conclusion

- Time consistency is a key property for Dynamic Risk Measures. It is characterized by the cocycle condition of the minimal penalty function.
- For normalized dynamic risk measures, the process associated with any financial instrument is a supermartingale with respect to any equivalent measure with zero penalty. It has a cadlag modification.
- Using the theory of right continuous BMO martingales, we construct time-consistent dynamic risk measures , generalizing those coming from B.S.D.E.. Starting with right continuous BMO martingales with jumps this leads naturally to dynamic risk measures with jumps.

CONCLUSION

- A TCPP $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ associates to any financial instrument a ask (resp bid) price process $\Pi_{\sigma,\tau}(X)$ (resp $-\Pi_{\sigma,\tau}(-X)$).
- First Frundamental Theorem: A TCPP has No Free Lunch If and only if there is a probability measure R equivalent to P such that for any financial instrument X, the martingale process $E_R(X|\mathcal{F}_\sigma)$ is between the bid price process $-\Pi_{\sigma,\tau}(-X)$ and the ask price process $\Pi_{\sigma,\tau}(X)$.
- Using BMO martingales we are able to construct No Free Lunch time-consistent pricing procedures extending the dynamics of reference assets and compatible with the observed bid and ask prices for reference options.

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