ON THE VAN DER WAALS THEORY OF SURFACE TENSION.

T. BODINEAU

Abstract. In this paper, the works on the justification of the van der Waals theory of surface tension in the context of the Kac Ising models are reviewed. The second part of the paper is devoted to a coarse grained definition of the surface tension for Kac Ising models which is appropriate for the $\mathbb{L}^1$ approach of phase coexistence.

1. Introduction

The microscopic model with long range interactions introduced by Kac [KUH] was motivated by the van der Waals theory of liquid/vapor phase transition. This model provided a justification from a microscopic point of view of the phase diagram obtained from the van der Waals theory (including Maxwell's rule). The program launched by Kac was completed in the celebrated paper by Lebowitz and Penrose [LP]. Shifting from the bulk properties to surface properties, several works were devoted during the last 10 years to a rigorous justification of the van der Waals theory of surface tension [vdW]. In this paper, we survey these results and explain the interplay between the mean field representation of surface tension and its microscopic counterpart.

Beyond the justification of the van der Waals theory, the Kac model played also a key role in the recent developments on the phase coexistence. The ambivalence of the model which has been devised as a bridge between the microscopic systems and the continuous models was certainly one of the reason for which progress have been made. For example, the setting of the geometric measure theory was introduced in order to analyze the Γ-convergence of the mean field functionals [ABCP, BCP]. The renormalization scheme, which was developed to study bulk properties for the Kac Ising model with finite range interactions appeared also to be an appropriate tool for the localizations of the mesoscopic interfaces as well as to control the phase of the small contours and to obtain exponential tightness estimates [BBBP, BBP]. A detailed account of the precise techniques would go far beyond the scope of this paper. We will focus on the particular issue of the surface tension and refer the reader to [BIV1] for a comprehensive discussion on the phenomenon of phase coexistence.

This paper is divided into two parts. The first one is a review of the works on the van der Waals theory of surface tension in the context of the Kac Ising model. The second part is an attempt to define a robust notion of surface tension for finite range interactions Kac Ising model. The main motivation is to devise a procedure which enables to prove the existence of the surface tension as the thermodynamic limit of quantities defined only

Date: May 2, 2001.

I am very grateful to E. Presutti, D. Ioffe and Y. Velenik for many interesting discussions and suggestions.

in terms of a coarse grained information. Let us come back to the problem of phase coexistence in order to stress the importance of a tractable concept of surface tension.

The $\mathbb{L}^1$-theory (see [BIV1] for a review) is a general strategy to study the phase separation. Its implementation relies only on coarse grained estimates and thus is not model dependent. Nevertheless, this requires the existence of a priori coarse grained estimates and of an appropriate definition of the surface tension which can be related to the coarse graining. As a consequence the $\mathbb{L}^1$ approach has been implemented mainly for models which can be analyzed in terms of FK measures [Ce, B1, CePi1, CePi2, BIV2]. The generalization to more realistic models (for example models in the Pirogov Sinai theory) would require a different approach of the coarse graining and of the surface tension. We refer to [B2] for a discussion on this issues (in particular for the coarse graining). In [B2], the Wulff construction was derived in the context of the Kac Ising model without using the FK representation. Nevertheless, the treatment of the surface tension was still relying on ferromagnetic inequalities and on the spin flip symmetry. Contrary to the usual microscopic definition of surface tension, the derivation of the surface tension proposed in the second part of the paper is based only upon coarse grained estimates : as a consequence, this derivation is compatible with the $\mathbb{L}^1$ approach of phase coexistence and it does not use ferromagnetic inequalities.

The understanding of surface tension for models with non symmetric pure phases remains an open problem. In the case of the Dobrushin interface, we refer the reader to the recent paper by Holicky, Kotecky, Zahradnik [HKZ] for a study of the surface tension and of the horizontally invariant Gibbs states in a very general context.

We hope that the strategy implemented for the Kac Ising model could provide a step towards the study of more complex systems. An interesting future prospect is the study of phase coexistence for particles in the continuum interacting with Kac potentials. In a recent breakthrough Lebowitz, Mazel, Presutti [LMP] proved the liquid/vapor transition for this model. This system has a coarse grained structure similar to the Kac Ising model, but the repulsive/attractive interactions prevent to use inequalities and the pure phases are non symmetric.

2. Bulk properties of the Kac Ising model

2.1. **Kac Ising model.** The Kac Ising model is a ferromagnetic spin system with interactions tuned by a scaling parameter γ. Let J be a non-negative smooth function supported by $[-1,1]$ and such that $\int_{\mathbb{R}^d} J(|r|)\,dr = 1$. For any $\gamma > 0$, the Kac potentials are defined by

$$\forall i \in \mathbb{Z}^d, \qquad J_\gamma(i) = \gamma^d J(\gamma|i|)\,.$$

For simplicity, γ is chosen as 2^{-g} with $g \in \mathbb{N}$.

Let Λ be a finite domain of $\mathbb{Z}^d$, for $d \geqslant 2$. Each site i in Λ indexes a spin σ_i which takes values ± 1. The spin configurations $\{\sigma_i\}_{i\in\Lambda}$ have a statistical weight determined by the Hamiltonian

$$H^{\bar\sigma}_{\gamma,\Lambda}(\sigma) = -\frac{1}{2}\sum_{i,j\in\Lambda} J_\gamma(i-j)\sigma_i\sigma_j - \sum_{i\in\Lambda, j\in\Lambda^c} J_\gamma(i-j)\sigma_i\bar\sigma_j\,,$$

where $\bar\sigma = \{\bar\sigma_i\}_{i\in\Lambda^c}$ are the boundary conditions outside Λ.

The Gibbs measure associated to the spin system with boundary conditions $\bar\sigma$ is

$$\forall \sigma = \{\sigma_x\}_{x\in\Lambda}, \qquad \mu^{\bar\sigma}_{\beta,\gamma,\Lambda}(\sigma) = \frac{1}{Z^{\bar\sigma}_{\beta,\gamma,\Lambda}} \exp\left(-\beta H^{\bar\sigma}_{\gamma,\Lambda}(\sigma)\right),$$

where β is the inverse of the temperature ($\beta = \frac{1}{T}$) and $Z^{\bar\sigma}_{\beta,\gamma,\Lambda}$ is the partition function. If the boundary conditions are uniformly equal to 1 (resp -1), the Gibbs measure will be denoted by $\mu^+_{\beta,\gamma,\Lambda}$ (resp $\mu^-_{\beta,\gamma,\Lambda}$).

For any $\gamma > 0$, a phase transition occurs above the critical parameter $\beta_c(\gamma)$. It is proved by Cassandro, Presutti [CaPr] and Bovier, Zahradnik [BZ] that

$$\lim_{\gamma\to 0} \beta_c(\gamma) = \beta_c^{\mathrm{mf}} = 1\,,$$

where β_c^{mf} denotes the inverse critical temperature of the mean field system. For any $\beta > 1$, this implies the existence of two distinct Gibbs measures $\mu^+_{\beta,\gamma}$ and $\mu^-_{\beta,\gamma}$ for sufficiently small γ. In particular, the phase transition is characterized by a spontaneous magnetization

$$\forall \beta > 1, \exists \gamma_\beta > 0, \forall \gamma < \gamma_\beta, \qquad \mu^+_{\beta,\gamma}(\sigma_0) = m^*_{\beta,\gamma} > 0\,. \tag{2.1}$$

When γ vanishes, $m^*_{\beta,\gamma}$ converges to the mean field magnetization m^*_β.

The complete characterization of the translation invariant pure phases was obtained by Butta, Merola and Presutti

Theorem 2.1. [BMP] *For any $\beta > 1$, there is $\gamma_\beta > 0$ such that for any $\gamma < \gamma_\beta$, any translation invariant Gibbs measure is a convex combination of $\mu^+_{\beta,\gamma}$ and $\mu^-_{\beta,\gamma}$.*

2.2. **Coarse graining.** The analysis of phase transition and the description of the pure phases are obtained as consequences of a renormalization procedure which provides a complete characterization of the of the bulk properties. The renormalization procedure which has been developed in [CaPr, BZ] enables to control the local order parameter on mesoscopic scales of the order γ^{-1}. At the scale γ^{-1}, the renormalized system behaves like an Ising model with effective inverse temperature proportional to γ^{-d} and a Peierls type estimate can be derived.

This result has been enhanced by Butta, Merola, Presutti [BMP], who proved the exponential relaxation to equilibrium of the Gibbs measure with boundary conditions which are only statistically pure. The bulk estimates are recalled below; they will play a crucial role in the analysis of the thermodynamic limit of the surface tension.

For any integer K and x in $\mathbb{Z}^d$, the box of side length K centered in x is defined as

$$\mathbb{B}_K(x) = x + \left]-\frac{K}{2}, \frac{K}{2}\right]^d.$$

The local magnetization in $\mathbb{B}_K(x)$ is denoted by

$$\mathcal{M}_K(x) = \frac{1}{K^d} \sum_{i\in\mathbb{B}_K(x)} \sigma_i\,. \tag{2.2}$$

We introduce two intermediate scales depending on the range of interaction $\gamma^{-1} = 2^g$. Let $L = 2^{-\ell}\gamma^{-1} = 2^{g-\ell}$ and $H = 2^h\gamma^{-1} = 2^{g+h}$, where $\ell, h \in \mathbb{N}$. In the following, ℓ will be fixed and γ will go to 0 thus $\ell \leqslant g$. By construction each box $\mathbb{B}_H$ is partioned into

smaller boxes $\mathbb{B}_L$. Given $\varepsilon > 0$ and $\gamma > 0$, the phase labels η_H^ε at the mesoscopic scale H are defined as follows : the box $\mathbb{B}_H(x)$ centered in x in $H\mathbb{Z}^d$ is labelled by

$$\eta_H^\varepsilon(x) = \begin{cases} \pm 1, & \text{if } |\mathcal{M}_L(y) \mp m_\beta^*| \leqslant \varepsilon, \quad \forall \mathbb{B}_L(y) \subset \mathbb{B}_H(x)\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

The parameter ε quantifies the deviation of the averaged magnetization from the mean field magnetization m_β^*.

A region Λ (measurable wrt the partition at the scale H) has + boundary conditions in averaged if the phase labels $\eta_{N,H}^{\varepsilon/2}$ associated to σ_{Λ^c} are uniformly equal to 1. This set of boundary conditions is denoted by $G^{+,\varepsilon}(\Lambda) \subset [-1,1]^{\Lambda^c}$. In the same way, the set of $-$ boundary conditions in averaged is denoted by $G^{-,\varepsilon}(\Lambda)$.

For the renormalized contours (defined in terms of phase labels), the following Peierls type estimate has been proven in [CaPr, BZ]

Theorem 2.2. *There exist functions $\varepsilon^*(\beta) > 0$, $\ell(\beta,\varepsilon), h(\beta,\varepsilon) \in \mathbb{N}$, $c(\beta,\varepsilon) > 0$, $c'(\beta,\varepsilon) > 0$ such that the following holds. Let $\gamma > 0, \beta > 1$ and $\varepsilon < \varepsilon^*$. Then for any region Λ (measurable wrt the partition at the H-scale) and any generalized contour Γ in Λ*

$$\forall \sigma_{\Lambda^c} \in G^{+,\varepsilon}(\Lambda), \qquad \mu_{\beta,\gamma,\Lambda}\left(\Gamma \,\middle|\, \sigma_{\Lambda^c}\right) \leqslant c'(\beta,\varepsilon)\, \exp\left(-\frac{c(\beta,\varepsilon)}{\gamma^d}\,|\Gamma|\right)\,, \tag{2.3}$$

where $|\Gamma|$ denotes the number of H-boxes in the contour Γ.

Furthermore, the Gibbs measures with almost pure boundary conditions relax exponentially fast to the corresponding pure phase.

Theorem 2.3. [BMP] *For any $\beta > 1$, $\forall \varepsilon \in]0, \varepsilon^*(\beta)[$ there exists $\gamma(\beta,\varepsilon) > 0, c(\varepsilon) > 0$ such that the following holds. For any subset Δ of $\mathbb{Z}^d$, for any $\mathcal{K} \subset \Delta$, one has $\forall \gamma \leqslant \gamma(\beta,\varepsilon), \forall (\sigma_{\Delta^c}, \sigma'_{\Delta^c}) \in G^{+,\varepsilon}(\Delta)$*

$$\left|\mu_{\beta,\gamma,\Delta}\big(\sigma_{\mathcal{K}}\big|\sigma_{\Delta^c}\big) - \mu_{\beta,\gamma,\Delta}\big(\sigma_{\mathcal{K}}\big|\sigma'_{\Delta^c}\big)\right| \leqslant |\mathcal{K}|\ \exp\big(-c(\varepsilon)\,\gamma^2\, dist(\mathcal{K},\mathcal{S})\big)\,,$$

where $\mathcal{S}$ is the subset of Δ^c where σ and σ' differ.

3. On the validity of the van der Waals theory of surface tension

In this section, we review several derivations of the van der Waals surface tension theory which are obtained from different microscopic limits : the mean field $N \sim \frac{1}{\gamma}$; the scaling $N \sim \gamma^{-1-a}$ (with a small) and the limit $N \to \infty, \gamma \to 0$.

3.1. The mean field approach.

The main technical feature of the Kac model is his ability to interpolate between a microscopic structure (as presented in Section 2) and some continuous limit.

Let us consider $\mathbb{D}_L = [-L, L] \subset \mathbb{R}^d$, for some $L > 0$. This domain is partioned into cubes of side length γ and therefore it contains $(L/\gamma)^d$ cells. These cubes are indexed by the subset $D_{L,\gamma}$ of $\mathbb{Z}^d$. As the number of particles is scaled proportionally to the range of the interaction, the limiting regime corresponds to the continuous mean field picture. In particular, one can define a functional $\mathcal{F}_L$ such that for m in $L^\infty(\mathbb{D}_L, [-1,1])$

$$\mathcal{F}_L(m) = \int_{x\in\mathbb{D}_L} dx\big(f_\beta(m(x)) - f_\beta(m_\beta)\big) + \frac{1}{4}\int\int_{\mathbb{D}_L\times\mathbb{D}_L} dx\,dx'\,J(|x-x'|)\big(m(x)-m(x')\big)^2\,,$$

where f_β is a double well potential with distinct minimizers $\pm m_\beta$ if $\beta > 1$. As γ vanishes, the Gibbs measure can be approximated as follows

$$\mu_{\beta,\gamma,D_{L,\gamma}}(\,\cdot\,) \sim \exp\Big(-\beta\gamma^{-d}\big(\mathcal{F}_L(\cdot) + o(\gamma)L^d\big)\Big)\,. \tag{3.1}$$

We refer to the book of Presutti [Pr] for a detailed account on this approximation procedure.

The previous approximation procedure does not hold if $L = \infty$, nevertheless the functional makes sense and we simply denote it by $\mathcal{F}$. In order to define the surface tension, we restrict to $d = 1$ and consider $\mathcal{I}_\beta$ the subset of functions $m \in L^\infty(\mathbb{R}, [-1, 1])$ such that

$$\lim_{x\to+\infty} m(x) = m^*_\beta \qquad \text{and} \qquad \lim_{x\to-\infty} m(x) = -m^*_\beta\,.$$

According to the van der Waals theory of surface tension

$$\tau^*_\beta = \inf_{m\in\mathcal{I}_\beta} \mathcal{F}(m)\,. \tag{3.2}$$

A thorough analysis of τ^*_β was accomplished in the case of a d dimensional cylinder by De Masi, Orlandi, Presutti, Triolo (see [DOPT1]). The infimum is attained on a unique (modulo translations) function $\bar m$ which satisfies the mean field equation [DOPT1]

$$\bar m = \tanh(\beta J \star \bar m)\,,$$

where $J \star \bar m$ is the convolution of the functions J and $\bar m$. The instanton $\bar m$ should be understood as the optimal profile for a domain wall on a mesoscopic scale. Further results on the stability of the instanton are also derived in [DOPT2].

The model obtained in the continuous limit is isotropic therefore the van der Waals surface tension is defined as τ^*_β for any directions. We refer the reader to [AlBe, Pr] for a treatment of the anisotropic case.

For particles in the continuum, a similar mean field picture holds [LMP]. Nevertheless the study of the surface tension and more precisely of the minimizers is much more difficult due to the attractive/repulsive nature of the interactions. As a consequence, the instanton is no longer an increasing function and artifacts (oscillations ...) appear in some regions of the phase diagram. We refer to Gayrard, Presutti, Triolo [GPT] for a complete investigation of these phenomena.

3.2. **Relation with Γ-convergence.** Fix the dimension $d \geqslant 2$. For domains $\mathbb{D}_L$, where L is properly scaled wrt γ, the approximation formula (3.1) remains valid. Following [ABCP], we consider the scaling $L = \gamma^{-a}$, for some $a > 0$ small enough.

In order to discuss the limiting procedure, some notation have to be introduced. The sets $D_{L,\gamma} = \{-\gamma^{-1-a}, \dots, \gamma^{-1-a}\}^d$ are embed in $\widehat{\mathbb{D}} = [-1, 1]^d$ which is divided in a grid with mesh γ/L. The local parameter (2.2) is now defined as a piecewise constant function on $\widehat{\mathbb{D}}$

$$\forall x \in \widehat{\mathbb{D}}, \qquad \mathcal{M}_{L,K}(x) = \frac{1}{(K/\gamma)^d} \sum_{i\in\widehat{\mathbb{B}}_{\frac{K/\gamma}{L/\gamma}}(x)} \sigma_i\,,$$

where $\widehat{\mathbb{B}}_\varepsilon(x) = \{y \in \widehat{\mathbb{D}} \mid |y - x| \leqslant \varepsilon\}$ for $\varepsilon > 0$. The results recalled in section 2 ensure that for K large enough, $\mathcal{M}_{L,K}$ concentrates to m^*_β in $\mathbb{L}^1(\widehat{\mathbb{D}})$ with probability $\mu^+_{\beta,\gamma,D_{L,\gamma}}$ converging to 1 as γ goes to 0.

Instead of defining the surface tension for every directions, we proceed as in [ABCP] and state a more global result on the interfacial energy of an arbitrary crystal. Let V be a set in $\widehat{\mathbb{D}}$ with smooth boundary. The occurrence of a crystal of shape V with one phase surrounded by the other phase is represented by the spin configurations in $\{-1,1\}^{D_{L,\gamma}}$ such that

$$\|\mathcal{M}_{L,K} - m^*_\beta 1_V + m^*_\beta 1_{V^c}\|_1 \leqslant \varepsilon\,,$$

where the parameter ε controls the accuracy of the description in $\mathbb{L}^1(\widehat{\mathbb{D}})$. More general crystals can be considered : to any function of bounded variation u in $\widehat{\mathbb{D}}$ taking values ± 1, one can associate the volume $V = \{u = -1\}$ for which a generalized notion of boundary can be defined ($\partial V = \partial\{u = -1\}$).

The interfacial energy of a crystal, i.e. the surface tension integrated along the boundary can be computed from the probabilistic cost of the spin configurations constrained to be close to the crystal.

Theorem 3.1. *[ABCP] For any function u of bounded variation in $\widehat{\mathbb{D}}$ taking values ± 1, there is a sequence ε_γ which vanishes as γ goes to 0 and such that for $L = \gamma^{-a}$*

$$\lim_{\gamma\to 0} -\frac{\gamma^d}{L^{d-1}} \log \mu^+_{\beta,\gamma,D_{L,\gamma}}\big(\|\mathcal{M}_{L,K} - m^*_\beta u\|_1 \leqslant \varepsilon_\gamma\big) = \beta\tau^*_\beta P(u)\,,$$

where τ^*_β is the van der Waals surface tension (3.2) *and $P(u)$ is the perimeter of the interface $\partial\{u = -1\}$.*

Let us briefly comment on the proof. On a macroscopic level, a rescaled version of the van der Waals functional is defined as

$$\forall v \in L^\infty(\widehat{\mathbb{D}}, [-1,1]), \qquad \tilde{\mathcal{F}}_\varepsilon(v) = \frac{1}{\varepsilon}\mathcal{F}_{1/\varepsilon}\big(v(\varepsilon\cdot)\big)\,.$$

The main point in the derivation of the theorem above is the proof of the Γ-convergence of the sequence of functionals $\tilde{\mathcal{F}}_\varepsilon$ to the functional $\tau^*_\beta P(\cdot)$ as ε vanishes. The strategy involves a microscopic reduction to the variational problem as well as arguments of geometric measure theory.

For the same scaling, a similar result was obtained by [AlBe] in the case of anisotropic interactions. In the latter case, the limiting surface tension is anisotropic.

3.3. **The thermodynamic limit for vanishing γ.** The thermodynamic limit as goes to infinity while γ is kept fixed has been investigated in [BBBP]. For any function of bounded variation u, we define two approximations of the interfacial energy by

$$\begin{aligned} F_{\beta,\gamma}(u) &= \lim_{\varepsilon\to 0}\lim_{K\to\infty}\liminf_{L\to\infty} -\frac{\gamma^d}{L^{d-1}} \log \mu^+_{\beta,\gamma,D_{L,\gamma}}\big(\|\mathcal{M}_{L,K} - m^*_\beta u\|_1 \leqslant \varepsilon\big)\,,\\ G_{\beta,\gamma}(u) &= \lim_{\varepsilon\to 0}\lim_{K\to\infty}\limsup_{L\to\infty} -\frac{\gamma^d}{L^{d-1}} \log \mu^+_{\beta,\gamma,D_{L,\gamma}}\big(\|\mathcal{M}_{L,K} - m^*_\beta u\|_1 \leqslant \varepsilon\big)\,. \end{aligned}$$

In [BBBP], it was proved that both quantities converge to the van der Waals surface tension as γ vanishes.

$$\lim_{\gamma\to 0} F_{\beta,\gamma}(u) = \lim_{\gamma\to 0} G_{\beta,\gamma}(u) = \beta\tau^*_\beta P(u)\,.$$

As L diverges and γ is kept fixed, the approximation of the Gibbs measure by the functional (3.1) is no longer valid. Therefore, the core of the proof is to define a mesoscopic notion of interface and to localize it in regions where the estimates of subsection 3.2 can be applied.

In fact, it was then derived in [B2] that

$$F_{\beta,\gamma}(u) = G_{\beta,\gamma}(u) = \beta \int_{\partial\{u=-1\}} \tau_{\beta,\gamma}(\vec{n}_x)\, d\mathcal{H}_x^{(d-1)}\,,$$

where $\tau_{\beta,\gamma}$ is the anisotropic surface tension for finite range interactions (see definition 4.1). Combined with the previous results this implies that $\tau_{\beta,\gamma}$ converges to the van der Waals surface tension τ_β^* when the range of the interactions diverges.

4. A coarse grained approach of surface tension

4.1. **A microscopic definition.** For finite range models, the lattice structure induces, in general, an anisotropic surface tension. For simplicity, let us recall the usual microscopic definition of surface tension in the direction $\vec{e}_d = (0,\dots,0,1)$.

Let $\Lambda_{N,M} = \{i = (i_1,\dots,i_d) \in \mathbb{Z}^d \mid \forall k < d, \quad |i_k| \leqslant N, \quad |i_d| \leqslant M\}$. The mixed boundary conditions $\bar{\sigma}$ are defined as

$$\forall i \in \Lambda_{N,M}^c, \qquad \bar{\sigma}_i = \begin{cases} +1, & \text{if} \quad i_d \geqslant 0, \\ -1, & \text{if} \quad i_d < 0. \end{cases}$$

The partition function with mixed boundary conditions is denoted by $Z^{\pm}_{\beta,\gamma,\Lambda_{N,M}}(\vec{e}_d)$ and the one with boundary conditions uniformly equal to 1 by $Z^{+}_{\beta,\gamma,\Lambda_{N,M}}$.

Definition 4.1. *The surface tension in the direction $\vec{e}_d$ is defined by*

$$\tau_{\beta,\gamma}(\vec{e}_d) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{N^{d-1}} \log \frac{Z^{\pm}_{\beta,\gamma,\Lambda_{N,M}}}{Z^{+}_{\beta,\gamma,\Lambda_{N,M}}}\,. \tag{4.1}$$

The reader is referred to [MMR] for a discussion on the properties of surface tension. In particular, it is proved that the height M of the box can be scaled with N and a straightforward extension of this result implies that the limits with respect to N and M can be interchanged (see also [BLP] for a similar statement).

The derivation of the limit (4.1) relies on ferromagnetic correlations inequalities (FKG). Thus the representation of the surface tension is not robust; in the sense that any modification of the boundary conditions would prevent a direct application of the FKG inequality. Notice that in the vicinity of an interface a much more complex microscopic structure would be observed : the boundary conditions are in averaged close to the pure phases.

In the analysis of the equilibrium crystal shapes, the difficulty of considering the richer structure of the boundary conditions was overcome by using the FK representation (see [B1], [CePi1]; [CePi2] for the Potts model and [BIV2] for the boundary surface tension). The FK measure is less sensitive to the boundary conditions because there exists only a unique limiting measure independent of the choice of the boundary conditions, even in the range of temperature for which there is a breaking of symmetry for the Ising model. As a consequence, an alternative definition of the surface tension (independent of the choice of the boundary conditions) was implemented in the FK representation. The drawback of such an approach is that the FK representation is limited to a restricted class of models. Notice also that the analysis of the surface tension via the FK correspondence heavily relies on the symmetry of the model and on ferromagnetic inequalities.

4.2. **A mesoscopic definition.** We would like to propose an alternative definition of the surface tension which relies only on coarse grained estimates. The Kac Ising model is one of the rare instances where a renormalization procedure in terms of the local magnetization has been fully developed. The mesoscopic approach of the surface tension which is presented below is implemented in the context of the Kac Ising model. Nevertheless, we hope that such a strategy could be extended to a broader class of models.

In the following, the parameters γ and H are fixed such that the hypotheses of Theorems 2.2 and 2.3 are satisfied. The bulk estimates of section 2.2 are the key ingredients to control the thermodynamics of the surface tension. To simplify the notation, the dependency on the numerical constants wrt γ and H will be omitted.

We fix $\vec{n} \in \mathbb{S}^{d-1}$ such that $(\vec{n} \cdot \vec{e}_d) > \frac{1}{\sqrt{d}}$. Define

$$\Lambda_{N,M}(\vec{n}) = \left\{ i \in \mathbb{Z}^d \mid \qquad \forall k < d, \quad |i_k| \leqslant NH, \quad |i \cdot \vec{n}| \leqslant MH \right\}.$$

By abuse of notation, $\Lambda_{N,M}(\vec{n})$ denotes also the measurable version of the set above wrt the H-partitions. For any $M' \geqslant M$, we define

$$\begin{aligned} \Lambda^+_{N,M',M}(\vec{n}) &= \Lambda_{N,M'}(\vec{n}) \setminus \Lambda_{N,M}(\vec{n}) \bigcap \{ i \cdot \vec{n} \geqslant 0 \}, \\ \Lambda^-_{N,M',M}(\vec{n}) &= \Lambda_{N,M'}(\vec{n}) \setminus \Lambda_{N,M}(\vec{n}) \bigcap \{ i \cdot \vec{n} < 0 \}. \end{aligned}$$

In order to localize an interface inside the slab $\Lambda_{N,M}(\vec{n})$, we introduce the event $\mathcal{S}_{N,M}(\vec{n})$ of spin configurations which contain a surface of $+$ good (resp $-$ good) blocks crossing $\Lambda^+_{N,M,M/2}(\vec{n})$ (resp $\Lambda^-_{N,M,M/2}(\vec{n})$) in the direction orthogonal to $\vec{n}$ (see figure 1).

For M, N large enough, let $\mathcal{G}^\varepsilon_{N,M}(\vec{n})$ be the set of triplets $(\Delta, \sigma^+, \sigma^-)$ such that

- Δ is a measurable set with respect to H-partitions and $\Lambda_{N,M/2}(\vec{n}) \subset \Delta \subset \Lambda_{N,M}(\vec{n})$.
- σ^+ is a spin configuration in $\Delta^c \bigcap \{ i \cdot \vec{n} \geqslant 0 \}$ such that all the blocks connected to Δ have η^ε_H phase labels equal to 1.
- σ^- is a spin configuration in $\Delta^c \bigcap \{ i \cdot \vec{n} < 0 \}$ such that all the blocks connected to Δ have η^ε_H phase labels equal to -1.

Let

$$F(N,M) = - \inf_{(\Delta,\sigma^+,\sigma^-) \in \mathcal{G}^\varepsilon_{N,M}(\vec{n})} \frac{1}{N^{d-1}} \log \frac{Z^{\sigma^+,\sigma^-}_\Delta \left(\mathcal{S}_{N,M/4}(\vec{n}) \right)}{Z^{\sigma^+,\tilde{\sigma}^-}_\Delta},$$

where $\tilde{\sigma}^-$ is deduced from σ^- by spin flip and $Z^{\sigma^+,\sigma^-}_\Delta \left(\mathcal{S}_{N,M/4}(\vec{n}) \right)$ is the constrained partition function defined by

$$Z^{\sigma^+,\sigma^-}_\Delta \left(\mathcal{S}_{N,M/4}(\vec{n}) \right) = \sum_{\sigma \in \{-1,1\}^\Delta} 1_{\{\sigma \in \mathcal{S}_{N,M/4}(\vec{n})\}} \exp\left(- \beta H_{\gamma,\Delta}(\sigma \,|\, \sigma^+, \sigma^-) \right).$$

In the same way, we set

$$G(N,M) = - \sup_{(\Delta,\sigma^+,\sigma^-) \in \mathcal{G}^\varepsilon_{N,M}(\vec{n})} \frac{1}{N^{d-1}} \log \frac{Z^{\sigma^+,\sigma^-}_\Delta \left(\mathcal{S}_{N,M/4}(\vec{n}) \right)}{Z^{\sigma^+,\tilde{\sigma}^-}_\Delta}.$$

In the following, we consider asymptotic such that $M \ll N$. Therefore the boundary conditions on the sides of Δ parallel to $\vec{n}$ will play no role and can be chosen arbitrarily.

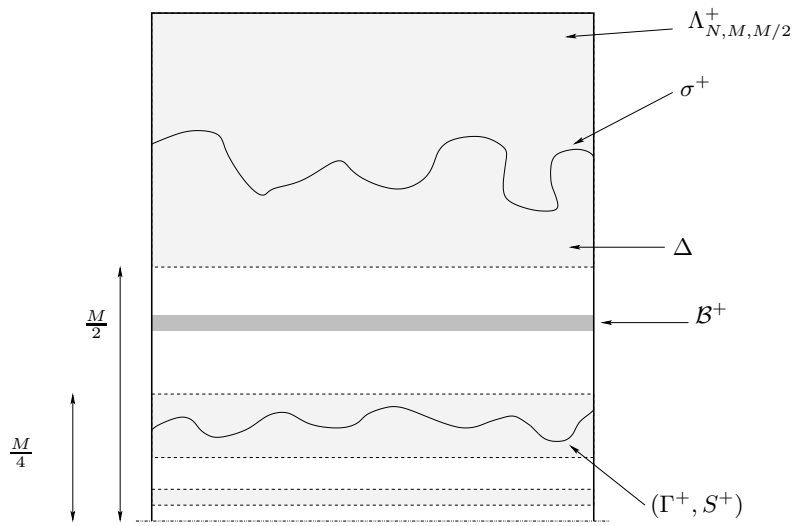

FIGURE 1. Hierarchy of the constraints $\mathcal{S}_{N,M}$ in the domain $\Lambda^+_{N,M}$.

Theorem 4.1. *For any $\vec{n} \in \mathbb{S}^{d-1}$, the surface tension in the direction $\vec{n}$ can be obtained as follows*

$$\tau_\beta(\vec{n}) = \lim_{m\to\infty} \lim_{N\to\infty} F(N, 4^m) = \lim_{m\to\infty} \lim_{N\to\infty} G(N, 4^m)\,, \tag{4.2}$$

where we restrict to values $M = 4^m$.

Furthermore, we check that

$$\tau_\beta(\vec{n}) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} F(N, \varepsilon N) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} G(N, \varepsilon N)\,, \tag{4.3}$$

where εN is of the form 4^m.

The assumption on the choice of the parameters M can be relaxed. We set

$$\tilde{F}(N, M) = -\inf_{(\Delta,\sigma^+,\sigma^-)\in\mathcal{G}^\varepsilon_{N,M}(\vec{n})} \frac{1}{N^{d-1}} \log \frac{Z^{\sigma^+,\sigma^-}_\Delta\left(\mathcal{S}_{N,m}(\vec{n})\right)}{Z^{\sigma^+,\tilde{\sigma}^-}_\Delta}\,,$$

where m is the integer part of $\log(M/4^3)$.

Then

Theorem 4.2. *For any $\vec{n} \in \mathbb{S}^{d-1}$, the surface tension in the direction $\vec{n}$ can be obtained as follows*

$$\tau_\beta(\vec{n}) = \lim_{M\to\infty} \lim_{N\to\infty} \tilde{F}(N, M) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \tilde{F}(N, \varepsilon N)\,. \tag{4.4}$$

Remark 4.1. *It can be proven that this mesoscopic definition of the surface tension coincides with the microscopic definition 4.1.*

In order to prove the convergence (4.2), we will proceed as follows. First, we deduce the existence of the thermodynamic limit $F(N, M)$ as N goes to infinity. For a fixed M, this follows from a standard argument of subadditivity which ensures the convergence of a $d-1$ dimensional partition function. Then the parameter M is taken to infinity. The next step is to compare $G(N, M)$ and $F(N, M)$ in order to identify their limits. This procedure follows the argument introduced in [B2] and requires only the bulk estimates

of subsection 2.2. At the first sight, the order of the limits (first $N \to \infty$ then $M \to \infty$) might be considered as unphysical because this does not take into account the (arbitrarily large) fluctuations of the interface. In a final step, we derive (4.3) which implies that the limiting procedure (4.2) gives rise to the right physical quantities. In particular, the analysis of the equilibrium crystal shapes and the identification of the interfacial energy can be obtained from (4.3). We refer the reader to [B2] for a concrete implementation of formula (4.3) in the context of the Wulff construction.

Proof of Theorem 4.1 :

The vector $\vec{n}$ is chosen with rational coordinates. For general directions $\vec{n}$, the value of the surface tension can be deduced by continuity. In the following, the vector $\vec{n}$ is fixed and the dependency on $\vec{n}$ is omitted in the notation.

Step 1 :

Let us check that

$$\forall M \geqslant 0, \qquad F(M) = \lim_{N\to\infty} F(N,M)\,. \tag{4.5}$$

This will follow from the standard argument used for the derivation of the thermodynamic limit of the free energy (see [Ru]).

Let R be a constant large enough such that for all N, the set $\Lambda_{2N+R,M}$ can be split into 2^{d-1} translates of the set $\Lambda_{N,M}$ and another set $\Lambda'_{2N+R,M}$ which contains less than $N^{d-2}RMH^d$ sites. Let $(\Delta,\sigma^+,\sigma^-)$ be the triplet for which the infimum is attained in the definition of $F(2N+R,M)$. The intersections of Δ with the 2^{d-1} translates of $\Lambda_{N,M}$ are denoted by $\Delta^1,\dots,\Delta^{2^{d-1}}$.

$$\begin{aligned} -(2N+R)^{d-1}F(2N+R,M) &= \log \frac{Z_\Delta^{\sigma^+,\sigma^-}\left(\mathcal{S}_{2N+R,M/4}\right)}{Z_\Delta^{\sigma^+,\tilde\sigma^-}}\,, \\ &\geqslant \sum_{i=1}^{2^{d-1}} \log \frac{Z_{\Delta^i}^{\sigma_i^+,\sigma_i^-}\left(\mathcal{S}_{N,M/4}\right)}{Z_{\Delta^i}^{\sigma_i^+,\tilde\sigma_i^-}} - C_\beta\, RN^{d-2}MH^d\,, \end{aligned}$$

where $\sigma_i^\pm$ is the restriction of $\sigma^\pm$ to the boundary of the set Δ^i and C_β is a constant depending on the interaction potential.

This implies that

$$-(2N+R)^{d-1}F(2N+R,M) \geqslant -(2N)^{d-1}F(N,M) - C_\beta\, RN^{d-2}MH^d\,.$$

As F is uniformly bounded from below and above, subadditivity implies that the limit (4.5) holds.

Step 2 :

We are going to check that the limit of the sequence $F(4^m)$ exists as m diverges. For any spin configuration in $\mathcal{S}_{N,M/4} = \mathcal{S}_{N,m-1}$, the crossing surfaces of $+$ good blocks and $-$ good blocks which are the closest to $\Lambda^c_{N,M/4}$ are denoted by Γ^+,Γ^- (see figure 1). For a given triplet $(\Delta,\sigma^+,\sigma^-)$, we have

$$Z_\Delta^{\sigma^+,\sigma^-}\left(\mathcal{S}_{N,M/4}\right) \geqslant Z_\Delta^{\sigma^+,\sigma^-}\left(\mathcal{S}_{N,m-1}\bigcap\mathcal{S}_{N,m-2}\right).$$

Decomposing the partition function with respect to (Γ^+,Γ^-), we get

$$Z_\Delta^{\sigma^+,\sigma^-}\left(\mathcal{S}_{N,m-1}\bigcap\mathcal{S}_{N,m-2}\right)=\sum_{\Gamma^+,\Gamma^-}\sum_{\substack{S^+\Rightarrow\Gamma^+\\ S^-\Rightarrow\Gamma^-}}Z^{\sigma^+,S^+}\,Z^{S^+,S^-}\left(\mathcal{S}_{N,m-2}\right)Z^{S^-,\sigma^-}\,,$$

where S^+ and S^- are the spin configurations supported by Γ^+ and Γ^-. By construction, they contain phase labels uniformly equal to 1 and -1.

Using the spin flip symmetry, we get

$$Z_\Delta^{\sigma^+,\sigma^-}\left(\mathcal{S}_{N,m-1}\bigcap\mathcal{S}_{N,m-2}\right)\geqslant\left(\sum_{\Gamma^+,\Gamma^-}\sum_{\substack{S^+\Rightarrow\Gamma^+\\ S^-\Rightarrow\Gamma^-}}Z^{\sigma^+,S^+}\,Z^{S^+,\tilde{S}^-}\,Z^{\tilde{S}^-,\tilde{\sigma}^-}\right)\inf_{\Gamma^+,\Gamma^-}\inf_{S^+,S^-}\frac{Z^{S^+,S^-}\left(\mathcal{S}_{N,m-2}\right)}{Z^{S^+,\tilde{S}^-}}\,. \tag{4.6}$$

Let $\tilde{\mathcal{S}}_{N,m-1}=\tilde{\mathcal{S}}_{N,M/4}$ be the set of spin configurations which contain two crossing surfaces of $+$ good blocks in $\Lambda^+_{N,\frac{M}{4},\frac{M}{8}}$ and $\Lambda^-_{N,\frac{M}{4},\frac{M}{8}}$. By using the definition of F, inequality (4.6) can be rewritten as

$$-F(N,M)\geqslant\inf_{(\Delta,\sigma^+,\sigma^-)\in\mathcal{G}^\varepsilon_{N,M}(\vec{n})}\frac{1}{N^{d-1}}\log\mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\left(\tilde{\mathcal{S}}_{N,m-1}\right)-F(N,M/4)\,. \tag{4.7}$$

It remains to check that uniformly over the boundary conditions

$$-C_\beta 2^{-m}\leqslant\frac{1}{N^{d-1}}\log\mu^{\sigma^+,\tilde{\sigma}^-}\left(\tilde{\mathcal{S}}_{N,m-1}\right)\leqslant 0\,. \tag{4.8}$$

Once this is done, combining (4.7) and (4.8), we see that

$$F(m)\leqslant F(m-1)+\exp(-c_1 m)\,.$$

Thus for any integers (p,m), we have

$$F(m+p)\leqslant F(m)+c_2\exp(-c_1 m)\,.$$

Taking p to infinity and then m

$$\limsup_{p\to\infty}F(p)\leqslant\liminf_{m\to\infty}F(m)\,.$$

As F is uniformly bounded from below and above, the limit $\tau_\beta(\vec{n})$ exists.

Let $\bar{\mathcal{S}}_{N,m-1}$ be the event such that there exists a $+$ surface crossing the slab $\Lambda^+_{N,\frac{M}{4},\frac{M}{8}}(\vec{n})$. In order to check (4.8), it is enough to prove that $\bar{\mathcal{S}}_{N,m-1}$ occurs with a probability larger than $\exp(-N^{d-1}C_\beta 2^{-m})$, in the $+$ pure phase.

The slab $\Lambda^+_{N,\frac{M}{4},\frac{M}{8}}(\vec{n})$ is partitioned into disjoint translates of $D_M=\Lambda_{M^2,\frac{M}{8}}(\vec{n})$ which are denoted by D^i_M. The rest contains at most $N^{d-1}\frac{R}{8M}H^d$ sites. Let $\mathcal{A}$ be the set of spin configurations such that the blocks in $\Lambda^+_{N,\frac{M}{4},\frac{M}{8}}(\vec{n})\setminus\cup_i D^i_M$ are equal to 1. As $\mathcal{A}$ is supported by at most $\frac{R}{M}N^{d-1}H^d$ sites

$$\mu_\Delta^{\sigma^+,\tilde{\sigma}^-}(\mathcal{A})\geqslant\exp\left(-\frac{c_\beta R}{M}N^{d-1}H^d\right)\,. \tag{4.9}$$

Let $\mathcal{D}_M^i$ be the set of spin configurations such that there exists a + surface crossing D_M^i in the direction orthogonal to $\vec{n}$. By the very construction

$$\bigcap_i \mathcal{D}_M^i \bigcap \mathcal{A} \subset \bar{\mathcal{S}}_{N,m-1}\,.$$

Thus

$$\mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\big(\bar{\mathcal{S}}_{N,m-1}\big) \geqslant \mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\left(\bigcap_i \mathcal{D}_M^i \bigcap \mathcal{A}\right)\,.$$

The events $\mathcal{D}_M^i$ are not independent, but after conditioning, the Peierls estimates imply

$$\mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\left((\mathcal{D}_M^1)^c \,\Big|\, \bigcap_{i\geqslant 2} \mathcal{D}_M^i \bigcap \mathcal{A}\right) \leqslant M^{2(d-1)} \exp\left(-\frac{c}{\gamma^d} M\right)\,.$$

By iterating this inequality, we obtain

$$\begin{aligned}
\mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\left(\bigcap_i \mathcal{D}_M^i \bigcap \mathcal{A}\right) &\geqslant \left(1-\mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\left((\mathcal{D}_M^1)^c \,\Big|\, \bigcap_{i\geqslant 2} \mathcal{D}_M^i \bigcap \mathcal{A}\right)\right) \mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\left(\bigcap_{i\geqslant 2} \mathcal{D}_M^i \bigcap \mathcal{A}\right)\,,\\
&\geqslant \left(1-M^{2(d-1)}\exp\big(-\frac{c}{\gamma^d}M\big)\right)^{\frac{N^{d-1}}{M^{2(d-1)}}} \mu_\Delta^{\sigma^+,\tilde{\sigma}^-}\big(\mathcal{A}\big)\,.
\end{aligned}$$

Combining this inequality with (4.9), we see that

$$\frac{1}{N^{d-1}} \log \mu^{\sigma^+,\tilde{\sigma}^-}\big(\tilde{\mathcal{S}}_{N,m-1}\big) \geqslant -\frac{C}{M} - \exp(-\frac{c'}{\gamma^d}M)\,, \tag{4.10}$$

where C, c' are constant depending on β, γ, H. Noticing that $1/M = 2^{-m}$, inequality (4.8) is complete.

Step 3 :

In order to check (4.2), let us prove that

$$\forall (N,M) \in \mathbb{N}, \qquad 0 \leqslant F(N,M) - G(N,M) \leqslant C_2 \exp(-C_1\gamma^2 M)\,, \tag{4.11}$$

where C_1, C_2 are positive constants. The event $\mathcal{S}_{N,M/4}$ decouples the interface from the boundary conditions and thus (4.11) can be derived by using only bulk estimates. More precisely, the boundary conditions are screened and the system relaxes to the pure phases in $\Lambda_{N,M/2} \setminus \Lambda_{N,M/4}$.

The proof follows the interpolation scheme introduced in [B2]. We consider two slabs of width 10 blocks (see figure 1)

$$\mathcal{B}^+ = \Lambda^+_{N,\frac{3M}{8},\frac{3M}{8}+10} \qquad \text{and} \qquad \mathcal{B}^- = \Lambda^-_{N,\frac{3M}{8},\frac{3M}{8}-10}\,.$$

The spin configurations inside $(\mathcal{B}^+, \mathcal{B}^-)$ are interpolated by the constant configuration $(m^*_\beta, -m^*_\beta)$

$$\begin{aligned}
\forall s \in [0,1], \forall i \in \mathcal{B}^+, \qquad \sigma_i(s) &= s\sigma_i + (1-s)m^*_\beta\,,\\
\forall i \in \mathcal{B}^-, \qquad \sigma_i(s) &= s\sigma_i - (1-s)m^*_\beta\,.
\end{aligned}$$

Outside $(\mathcal{B}^+, \mathcal{B}^-)$, the spins are unchanged. For a given triplet $(\Delta, \sigma^+, \sigma^-)$, let $Z^{\sigma^+,\sigma^-}_{\Delta,s}(\mathcal{S}_{N,m-1})$ be the partition function depending on the spins $\sigma(s)$. The configurations of the partition

function in the denominator are interpolated in $(\mathcal{B}^+, \mathcal{B}^-)$ by the configuration (m^*_β, m^*_β) and the corresponding partition function is denoted by $Z^{\sigma^+,\tilde{\sigma}^-}_{\Delta,s}$.

Let $(\Delta, \sigma^+, \sigma^-)$ and (D, ω^+, ω^-) be two boundary conditions in $\mathcal{G}^\varepsilon_{N,M}(\vec{n})$, we introduce

$$\Phi(s) = \log \frac{Z^{\omega^+,\omega^-}_{D,s}(\mathcal{S}_{N,M/4})}{Z^{\omega^+,\tilde{\omega}^-}_{D,s}} - \log \frac{Z^{\sigma^+,\sigma^-}_{\Delta,s}(\mathcal{S}_{N,M/4})}{Z^{\sigma^+,\tilde{\sigma}^-}_{\Delta,s}} \,. \tag{4.12}$$

When $s = 0$, the spin configurations in $\mathcal{B}^+$ and $\mathcal{B}^-$ decouple the configurations into three independent systems.

$$\Phi(0) = \log \left(\frac{Z^{\omega^+,m^*}}{Z^{\omega^+,m^*}} \frac{Z^{m^*,-m^*}_{\Lambda_{N,3M/8}}(\mathcal{S}_{N,M/4})}{Z^{m^*,m^*}_{\Lambda_{N,3M/8}}} \frac{Z^{-m^*,\omega^-}}{Z^{m^*,\tilde{\omega}^-}} \right) - \log \left(\frac{Z^{\sigma^+,m^*}}{Z^{\sigma^+,m^*}} \frac{Z^{m^*,-m^*}_{\Lambda_{N,3M/8}}(\mathcal{S}_{N,M/4})}{Z^{m^*,m^*}_{\Lambda_{N,3M/8}}} \frac{Z^{-m^*,\sigma^-}}{Z^{m^*,\tilde{\sigma}^-}} \right) .$$

The symmetries of the model imply the cancellation of all the terms and thus $\Phi(0) = 0$. Assertion (4.11) will follow if one can prove that

$$|\Phi'(s)| \leqslant c_2 N^{d-1} \exp(-c_1 \gamma^2 M) \,, \tag{4.13}$$

where the constants are independent of the choice of the boundary conditions.

To simplify the notation, we set

$$\nu_s = \mu^{\sigma^+,\sigma^-}_{\beta,\gamma,\Delta,s} \;; \quad \tilde{\nu}_s = \mu^{\sigma^+,\tilde{\sigma}^-}_{\beta,\gamma,\Delta,s} \;; \quad \nu'_s = \mu^{\omega^+,\omega^-}_{\beta,\gamma,D,s} \;; \quad \tilde{\nu}'_s = \mu^{\omega^+,\tilde{\omega}^-}_{\beta,\gamma,D,s} \,,$$

where the subscript s means that the measure has been modified by the interpolation introduced above.

$$\begin{aligned} \Phi'(s) \;&= \textstyle\sum_{i\in\mathcal{B}^+, j\in\Delta} J_\gamma(i,j) \;\; \big[\big(\nu'_s\big(\sigma_j(\sigma_i - m^*_\beta)|\mathcal{S}_{N,M/4}\big) - \tilde{\nu}'_s\big(\sigma_j(\sigma_i - m^*_\beta)\big)\big) \\ &\qquad\qquad\qquad - \big(\nu_s\big(\sigma_j(\sigma_i - m^*_\beta)|\mathcal{S}_{N,M/4}\big) - \tilde{\nu}_s\big(\sigma_j(\sigma_i - m^*_\beta)\big)\big)\big] \\ &\quad + \textstyle\sum_{i\in\mathcal{B}^-, j\in\Delta} J_\gamma(i,j) \;\; \big[\big(\nu'_s\big(\sigma_j(\sigma_i + m^*_\beta)|\mathcal{S}_{N,M/4}\big) - \tilde{\nu}'_s\big(\sigma_j(\sigma_i - m^*_\beta)\big)\big) \\ &\qquad\qquad\qquad - \big(\nu_s\big(\sigma_j(\sigma_i + m^*_\beta)|\mathcal{S}_{N,M/4}\big) - \tilde{\nu}_s\big(\sigma_j(\sigma_i - m^*_\beta)\big)\big)\big] \;. \end{aligned}$$

At this point, the bulk estimates are crucial. In particular, Theorem 2.3 leads to

$$\sum_{i\in\mathcal{B}^+, j\in\Delta} J_\gamma(i,j) \left| \tilde{\nu}'_s\big(\sigma_j(\sigma_i - m^*_\beta)\big) - \tilde{\nu}_s\big(\sigma_j(\sigma_i - m^*_\beta)\big) \right| \leqslant D \frac{N^{d-1}}{\gamma^d} \exp(-c\gamma^2 M) \,.$$

This estimate holds uniformly over the boundary conditions in $\mathcal{G}^\varepsilon_{N,M}(\vec{n})$.

Remark 4.2. *As pointed to us by E. Presutti, the Theorem 2.3 holds also for the interpolated measures* $(\mu^{\sigma^+,\tilde{\sigma}^-}_{\beta,\gamma,N,s})_{s\in[0,1]}$.

The terms containing mixed boundary conditions cannot be estimated by a direct application of Theorem 2.3. Nevertheless the event $\mathcal{S}_{N,M/4}$ screens the effect of the boundary conditions. Thus, by conditioning the measure wrt the spin configurations S^+ supported by the + surface Γ^+ which crosses $\Lambda^+_{N,M/4,M/8}$, we get for all (i,j) in $\mathcal{B}^+$

$$\begin{aligned} \big|\nu_s\big(\sigma_i\sigma_j|\mathcal{S}_{N,M/4}\big) \;&-\; \mu^+_{\beta,\gamma,s}(\sigma_i\sigma_j)\big| \\ &\leqslant \frac{1}{\nu_s(\mathcal{S}_{N,M/4})} \left| \nu_s\big(1_{\mathcal{S}_{N,M/4}}(S^+) \;\; \big[\mu^{\sigma^+,S^+}_{\beta,\gamma,s}(\sigma_i\sigma_j) - \mu^+_{\beta,\gamma,s}(\sigma_i\sigma_j)\big]\,\big)\right| \\ &\leqslant \exp(-c\gamma^2 M) \,, \end{aligned}$$

where $\mu^+_{\beta,\gamma,s}$ is the measure on the infinite stripe $\{i, \quad |i_j| \leqslant N,\ j = 2, \dots, d\}$. As the same estimate holds for ν'_s, inequality (4.13) is complete.

Step 4 :

The final step is the derivation of (4.3). We are going to check that for any integer (k, M) and $\varepsilon > 0$

$$F\left(\frac{M}{\varepsilon}, M\right) \geqslant F\left(k(\frac{M}{\varepsilon} + R), M\right) + o(\varepsilon)\,, \tag{4.14}$$

where R is a constant. Taking k, then M to infinity, (4.2) and (4.14) will imply that

$$\liminf_{M\to\infty} F\left(\frac{M}{\varepsilon}, M\right) \geqslant \lim_{M\to\infty} F(M) + o(\varepsilon) = \tau_\beta(\vec{n}) + o(\varepsilon)\,. \tag{4.15}$$

The same procedure applied to G leads to the converse inequality

$$\limsup_{M\to\infty} G\left(\frac{M}{\varepsilon}, M\right) \leqslant \lim_{M\to\infty} G(M) + o(\varepsilon) = \tau_\beta(\vec{n}) + o(\varepsilon)\,. \tag{4.16}$$

Furthermore, (4.11) implies that

$$0 \leqslant F\left(\frac{M}{\varepsilon}, M\right) - G\left(\frac{M}{\varepsilon}, M\right) \leqslant C_2 \exp(-C_1\gamma^2 M)\,. \tag{4.17}$$

Therefore, combining (4.15), (4.16), (4.17), we obtain (4.3).

The derivation of (4.14) follows closely the argument of step 1. The slab $\Lambda_{k(\frac{M}{\varepsilon}+R),M}$ is partioned into k^d translates of $\Lambda_{\frac{M}{\varepsilon},M}$ and another set containing at most $[k(\frac{M}{\varepsilon} + R)]^{d-2}kRMH^d$ sites. Using notations of step 1, we get

$$-\left(k(\frac{M}{\varepsilon} + R)\right)^{d-1} F\left(k(\frac{M}{\varepsilon} + R), M\right) \geqslant -\left(k\frac{M}{\varepsilon}\right)^{d-1} F\left(\frac{M}{\varepsilon}, M\right) - C_\beta\, H^d RMk\left(k\frac{M}{\varepsilon}\right)^{d-2}.$$

This leads to

$$F\left(\frac{M}{\varepsilon}, M\right) \geqslant F\left(k(\frac{M}{\varepsilon} + R), M\right) - C_\beta\, RH^d\varepsilon - \frac{C'R}{kM}\varepsilon\,.$$

Thus (4.14) holds.

Proof of Theorem 4.2 :

For any M, there is m such that $4^m \leqslant M < 4^{m+1}$. Then a proof similar to the derivation of inequality (4.11) implies that

$$\forall (N, M) \in \mathbb{N}, \qquad \left|F(N, 4^{m-2}) - \tilde{F}(N, M)\right| \leqslant C_2 \exp(-C_1\gamma^2 M)\,,$$

This comes from the fact that F and $\tilde{F}$ depend both on the same constraint $\mathcal{S}_{N,4^{m-3}}$. Thus, Theorem 4.2 follows from Theorem 4.1.

References

[AlBe] G. Alberti, G. Bellettini, *Asymptotic behavior of a non local anisotropic model for phase transition*, *J. Math. Ann.*, **310**, No. 3, 527-560, (1998).

[ABCP] G. Alberti, G. Bellettini, M. Cassandro, E. Presutti, *Surface tension in Ising system with Kac potentials*, J. Stat. Phys. **82**, 743–796 (1996).

[BCP] G. Bellettini, M. Cassandro, E. Presutti, *Constrained minima of non local free energy functionals*, J. Stat. Phys. **84**, 1337–1349 (1996).

[BBBP] O. Benois, T. Bodineau, P. Butta, E. Presutti, *On the validity of van der Waals theory of surface tension*, Mark. Proc. and Rel. Fields **3**, 175–198 (1997).

[BBP] O. Benois, T. Bodineau, E. Presutti, *Large deviations in the van der Waals limit*, Stoch. Proc. and Appl. **75**, 89–104 (1998).

[B1] T. Bodineau, *The Wulff construction in three and more dimensions*, Comm. Math. Phys. **207**, 197–229 (1999).

[B2] T. Bodineau, *Phase coexistence for the Kac Ising models*, preprint (2000).

[BIV1] T. Bodineau, D. Ioffe, Y. Velenik, *Rigorous probabilistic analysis of equilibrium crystal shapes*, J. Math. Phys. **41**, No.3, 1033–1098 (2000).

[BIV2] T. Bodineau, D. Ioffe, Y. Velenik, *Winterbottom construction for finite range ferromagnetic models : a $\mathbb{L}^1$-approach*, preprint (2000).

[BLP] J. Bricmont, J. Lebowitz, C. Pfister, *On the surface tension of lattice models*, Annals N.Y. academy of sciences **337**, No.3, 1033–1098 (1980).

[BZ] A. Bovier, M. Zahradnik, *The low-temperature phase of Kac Ising models*, J. Stat. Phys. **87**, No.1-2, 311–332 (1997).

[BMP] P. Buttà, I. Merola, E. Presutti, *On the validity of the van der Waals theory in Ising systems with long range interactions*, Mark. Proc. and Rel. Fields **3**, No.1, 63–88 (1997).

[CaPr] M. Cassandro, E. Presutti, *Phase transitions in Ising systems with long but finite range*, Mark. Proc. and Rel. Fields **2**, 241–262 (1996).

[Ce] R. Cerf, *Large deviations for three dimensional supercritical percolation*, Astérisque **267** (2000).

[CePi1] R. Cerf, A. Pisztora, *On the Wulff crystal in the Ising model*, Ann. Probab. **28**, no. 3, 947–1017 (2000).

[CePi2] R. Cerf, A. Pisztora, *Phase coexistence in Ising, Potts and percolation models*, Preprint (2000).

[DOPT1] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Stability of the interface in a model of phase separation*, Proceedings Royal Soc. Edinburgh **124A**, 1013–1022 (1994).

[DOPT2] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Uniqueness and global stability of the instanton in non local evolution equations*, Rendiconti di Matematica **14**, 693–723 (1994).

[GPT] V. Gayrard, E. Presutti, L. Triolo, *preprint* (2001).

[HKZ] Holicky, R. Kotecky, M. Zahradnik, *Phase diagram of horyzontally invariant Gibbs states for lattice models*, preprint (2001).

[LMP] J. Lebowitz, Mazel, E. Presutti, *Liquid-vapor phase transitions for systems with finite-range interactions.*, J. Stat. Phys. **94**, No.5-6, 955–1025 (1999).

[KUH] M. Kac, G. Uhlenbeck, P.C. Hemmer *On the Van der Waals theory of vapor-liquid equilibrium:*
I *Discussion of a one dimensional model.* J. Math. Phys., **4**, 216–228 (1963);
II. *Discussion of the distribution functions.* J. Math. Phys., **4**, 229–247 (1963);
III. *Discussion of the critical region.* J. Math. Phys., **5**, 60–74 (1964);

[LP] J. Lebowitz, O. Penrose, *Rigorous treatment of the Van der Waals Maxwell theory of the liquid vapour transition.* J. Math. Phys., **7**, 98–113, (1966)

[MMR] A. Messager, S. Miracle-Sole, J. Ruiz, *Surface tension, step free energy and facets in the equilibrium crystal*, J. Stat. Phys. **79**, (1995).

[Pr] E. Presutti, *in preparation* (2001).

[Ru] D. Ruelle, *Statistical mechanics. Rigorous results.*, Reprint of the 1989 edition. World Scientific Publishing Co., Inc., River Edge, NJ; Imperial College Press, London, (1999).

[vdW] J.D. Van der Waals, Z.f Phys. Chem. **13** (1894); J. Rowlinson, translation of J.D. Van der Waals *The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density*, J. Stat. Phys. **20**, 197 (1979).

Département de mathématiques, Université Paris 7, case 7012, 2 place Jussieu, Paris 75251, France

E-mail address: bodineau@gauss.math.jussieu.fr