OPTIMAL CONTROL OF SEMILINEAR MULTISTATE SYSTEMS WITH
STATE CONSTRAINTS*

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Abstract. This paper deals with state-constrained optimal control problems governed by a semilinear
multistate equation. The authors prove the existence of solutions and derive optimality conditions.

Key words. optimal control, subdifferential calculus, optimality conditions, elliptic operators, semilinear
equations, multistate systems

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1. Introduction. This paper is concerned with state-constrained optimal control
problems governed by a semilinear elliptic operator. As we make no monotonicity
assumption, the state equation may be unsolvable or may have several solutions. These
kinds of ill-posed systems may arise in connection with bifurcation theory; some models
arising in enzymatic reactions, plasma physics, and chemistry have this property (see
some examples in Crandall and Rabinowitz [11] and Lions [15]). However, this paper
studies only a model problem. Our aim is to obtain existence results and to derive the
optimality system.

There exists a vast literature on the control of well-posed state-constrained systems.
The subdifferential calculus of convex analysis is a useful tool for dealing with linear
state equations (see Mackenroth [16], [17], Bonnans and Casas [7], and Casas [8],
[9]). In the nonlinear case, Bonnans and Casas [4]-[6] derived the optimality system
using the results of Clarke [10].

The control of nonmonotone elliptic systems, but without state constraints, has
been studied by Lions [15] (see also Komornik [14]). The optimality system is derived
there by penalizing the state equation and passing to the limit in the optimality
conditions of the penalized problem.

The novelty of this paper lies in the simultaneous presence of state constraints
and of an ill-posed system. Our method consists of approximating the problem by
removing the nonlinearity from the state equation and penalizing a part of the state
constraints. We formulate the problem and obtain an existence result in § 2, derive the
optimality system in § 3, and study several examples in § 4.

2. Formulation of the control problem. Let Ω be an open bounded subset of \( \mathbb{R}^n \)
\((n \leq 3)\) with \( C^2 \) boundary \( \Gamma \). Let us consider the following system:

\[
\begin{align*}
Ay + \phi(y) &= u & \text{in } \Omega, \\
y &= 0 & \text{on } \Gamma,
\end{align*}
\]

(2.1)

where

\[
Ay = -\sum_{i,j=1}^{n} \frac{\partial x_j}{\partial y_i} (a_{ij}(x)\partial x_i y) + a_0(x) y,
\]

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$a_0 \in L^\infty(\Omega), \quad a_0(x) \geq 0 \quad \text{a.e. } x \in \Omega,$

\[ a_{ij} \text{ is Lipschitz on } \bar{\Omega} \quad (1 \leq i, j \leq n), \]

\[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha_0 \| \xi \|^2, \quad \alpha_0 > 0 \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \Omega, \]

\[ \phi : \mathbb{R} \to \mathbb{R} \quad \text{is } C^1. \]

Let $K$ be a nonempty, convex, closed subset of $L^2(\Omega)$, $\sigma$ be greater than or equal to 2, $N$ be nonnegative, and $y_d$ in $L^\sigma(\Omega)$ be given, and let $J : L^\sigma(\Omega) \times L^2(\Omega) \to \mathbb{R}$ be the functional

\[ J(y, u) = \frac{1}{\sigma} \int_{\Omega} |y(x) - y_d(x)|^\sigma \, dx + \frac{N}{2} \int_{\Omega} u^2(x) \, dx. \]

Let $Z$ be a separable Banach space, $B$ be a closed convex subset of $Z$ with nonempty interior, and $a$ be given in $\mathbb{R}^m$ ($m \geq 0$; we identify $\mathbb{R}^0$ and $\{0\}$). Define $Y = H^2(\Omega) \cap H^1_0(\Omega)$, where $H^2(\Omega)$ and $H^1_0(\Omega)$ are the usual Sobolev spaces (see Adams [1], Nečas [18]). Let $C_0(\Omega)$ be the space of real continuous functions on $\bar{\Omega}$ vanishing on $\Gamma$, endowed with the supremum norm $\| \cdot \|_\infty$. It is known that $Y$ is compactly embedded in $C_0(\Omega)$ for $n \leq 3$. The dual of $C_0(\Omega)$ is the space $M(\Omega)$ of real and regular Borel measures on $\Omega$, endowed with the norm

\[ \| \mu \|_{M(\Omega)} = |\mu|(\Omega), \]

where $|\mu|$ is the total variation measure of $\mu$ (Rudin [19]). Finally, let $T : C_0(\Omega) \to \mathbb{R}^m$ and $L : C_0(\Omega) \to Z$ be linear continuous mappings. In order to derive the optimality conditions, we will suppose that

\[ T(Y) = \mathbb{R}^m \quad \text{and} \quad L(Y) = Z. \]

We consider the following control problem:

\[ \text{minimize } J(y, u) \]

P

subject to (2.1), $u \in K$, $y \in Y$, $Ty = a$, $Ly \in B$.

\[ \text{Remark 1. The assumptions on } \Omega \text{ and } A \text{ imply (Nečas [18]) that for each } f \text{ in } L^2(\Omega) \text{ there exists a unique solution } y \in Y \text{ of the Dirichlet problem} \]

\[ Ay = f \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma, \]

and moreover, there exists $C_1$ independent of $f$ such that

\[ \| y \|_{H^2(\Omega)} \leq C_1 \| f \|_{L^2(\Omega)}. \]

In fact all our results still hold if we assume that $\Omega$ is bounded, $Y$ is compactly embedded in $C_0(\Omega)$, and (2.6) holds. This is the case, for instance, if $A$ is symmetric and satisfies (2.2) and $\Omega$ is bounded and convex (Grisvard [13]).

\[ \text{Remark 2. The existence of several states associated to the same control has been obtained, e.g., with cubic nonlinearities [11]. The inclusion of } Y \text{ in } C_0(\Omega) \text{ for } n \leq 3 \text{ (Adams [1]) implies that } A + \phi \text{ maps } Y \text{ into } L^2(\Omega) \text{; hence all elements of } Y \text{ are associated to a control. For parabolic systems the situation is essentially different (Bonnans [3]).} \]
Let us now give some examples of control problem that fall into the previous formulation.

\[
\text{(P1)} \quad \begin{align*}
& \text{minimize} \quad J(y, u) \\
& \text{subject to} \quad (2.1), \quad u \in K, \quad y \in Y, \quad y(x_i) = a_i, \quad 1 \leq i \leq m.
\end{align*}
\]

Here \( \{x_i\} \) are given in \( \Omega \) and we may take \( B = Z = C_0(\Omega) \), \( L \) is the identity in \( C_0(\Omega) \), and \( Ty = \{y(x_i)\} \).

\[
\text{(P2)} \quad \begin{align*}
& \text{minimize} \quad J(y, u) \\
& \text{subject to} \quad (2.1), \quad u \in K, \quad y \in Y, \quad \int_{\Omega} |y(x)| \, dx \leq \delta \quad \text{with} \quad \delta > 0.
\end{align*}
\]

Here \( m = 0 \), \( T = 0 \), \( Z = L^1(\Omega) \), \( B \) is the closed ball with center zero and radius \( \delta \), and \( L \) is the canonical injection from \( C_0(\Omega) \) into \( L^1(\Omega) \).

\[
\text{(P3)} \quad \begin{align*}
& \text{minimize} \quad J(y, u) \\
& \text{subject to} \quad (2.1), \quad u \in K, \quad y \in Y, \quad \int_{\Omega} y(x) \, dx = a, \\
& |y(x)| \leq \delta \quad \forall x \in \Omega, \quad \text{with} \quad \delta > 0.
\end{align*}
\]

Here \( m = 1 \) and \( Ty = \int_{\Omega} y(x) \, dx, \) \( Z = C_0(\Omega) \), \( B \) is the closed ball with radius \( \delta \) and center zero, and \( L \) is the identity. These three examples obviously satisfy (2.5).

We now give a result concerning the existence of a solution to problem (P). For this we need a relation between \( \sigma \) and the nonmonotone part of \( \phi \).

**Theorem 1.** Suppose (2.2) and (2.3) hold, and suppose the following:

(i) There exists \((y, u)\) satisfying the constraints of (P) (i.e., (P) is feasible).

(ii) Either \( N > 0 \) or \( K \) is bounded in \( C_0(\Omega) \).

(iii) We may write \( \phi(t) = \phi_1(t) + \phi_2(t) \), with \( \phi_i \) continuous, \( i = 1, 2 \), \( \phi_1(t) \) nondecreasing, and such that for some \( C > 0 \)

\[ |\phi_2(t)| \leq C(1 + |t|^{\alpha/2}). \]

Then problem (P) has (at least) one solution.

**Proof.** As (P) is feasible, there exists a minimizing sequence \( \{(y_n, u_n)\} \) in \( Y \times K \). Because of (ii), \( \{u_n\} \) is bounded in \( L^2(\Omega) \). We are going to prove that \( \{Ay_n\} \) is bounded in \( L^2(\Omega) \), and for this we may assume that \( \phi_1 \) is differentiable. Otherwise, we would approximate \( \phi_1 \) by a standard convolution technique and then pass to the limit. We also may assume without loss of generality that \( \phi_1(0) = 0 \).

The form of \( J \) implies that \( \{y_n\} \) is bounded in \( L^\infty(\Omega) \); hence with (iii), \( \phi_2(y_n) \) is bounded in \( L^2(\Omega) \), as is \( f_n = -\phi_2(y_n) + u_n = Ay_n + \phi_1(y_n) \). As \( \phi_1(y_n) \) is in \( C_0(\Omega) \), \( Ay_n \) belongs to \( L^2(\Omega) \). Computing the scalar product of \( f_n \) with \( Ay_n \) in \( L^2(\Omega) \), and integrating the nonlinear term by parts, we obtain

\[
\|Ay_n\|_{L^2(\Omega)}^2 + \int_{\Omega} \phi_1(y_n) \sum_{i,j=1}^n a_{ij}(x) \frac{\partial y_n}{\partial x_i} \frac{\partial y_n}{\partial x_j} \, dx \\
+ \int_{\Omega} a_0(x)\phi_1(y_n(x))y_n(x) \, dx \leq \|f_n\|_{L^2(\Omega)}\|Ay_n\|_{L^2(\Omega)}.
\]

The second and third term of the left-hand side are nonnegative because of (2.2), the monotonicity of \( \phi_1 \), and the equality \( \Phi_1(0) = 0 \). Hence \( \|Ay_n\| \) is bounded in \( L^2(\Omega) \),
with (2.6), this implies that \( \{y_n\} \) is bounded in \( Y \). As \( Y \) is compactly embedded in \( C_0(\Omega) \) for \( n \leq 3 \), selecting a subsequence if necessary, we may assume that

\[
y_n \to \bar{y} \quad \text{weakly in } Y, \quad \text{strongly in } C_0(\Omega),
\]

\[
Ay_n \to A\bar{y} \quad \text{weakly in } L^2(\Omega),
\]

\[
u_n \to \bar{u} \quad \text{weakly in } L^2(\Omega).
\]

This implies \( T\bar{y} = a, L\bar{y} \in B, \) and \( \phi(y_n) \to \phi(\bar{y}) \) in \( C_0(\Omega) \); hence \( Ay_n \) weakly converges in \( L^2(\Omega) \) toward \( \bar{u} - \phi(\bar{y}) \); hence \( (\bar{y}, \bar{u}) \) satisfies (2.1). As \( K \) is closed and convex, hence weakly closed, \( \bar{u} \) is in \( K \). Finally, the convexity and continuity of \( J \) implies its weak lower semicontinuity; the result follows.

3. The optimality system. For any set \( C \), denote its indicatrix by \( I_C \), defined by

\[
I_C(x) = \begin{cases} 
0 & \text{if } x \in C, \\
+\infty & \text{otherwise.}
\end{cases}
\]

We denote the subdifferential of a convex function \( f \) by \( \partial f \) (see Barbu and Precupanu [2], Ekeland and Temam [12]). The spaces \( W_0^{1,2}(\Omega) \) and \( W^{1,2}(\Omega) \) are the usual Sobolev spaces (Adams [1]). We denote by \( T^* \) the adjoint operator of \( T \) and by \( R(T^*) \) its range. The aim of this section is to prove the following result.

**THEOREM 2.** Let \((\bar{y}, \bar{u})\) be a solution of (P). We assume that (2.2)-(2.5) hold and that

\[
(3.1) \quad \partial(I_B \circ L)(\bar{y}) \cap R(T^*) = \{0\}.
\]

Then there exists \( \bar{p} \in W_0^{1,2}(\Omega) \) for all \( s < n/(n-1) \), \( \bar{\lambda} \) in \( \mathbb{R}^m \), \( \bar{\mu} \) in \( Z' \), and \( \alpha \geq 0 \) such that

\[
(3.2) \quad \bar{\alpha} + \|\bar{p}\|_{W_0^{1,2}(\Omega)} > 0,
\]

\[
(3.3) \quad A^*\bar{p} + \phi'(\bar{y})\bar{p} = \bar{\alpha}|\bar{y} - y_d|^{\gamma-2}(\bar{y} - y_d) + T^*\bar{\lambda} + L^*\bar{\mu},
\]

\[
(3.4) \quad \langle \bar{\mu}, z - L\bar{y} \rangle \leq 0 \quad \forall z \in B,
\]

\[
(3.5) \quad \int_\Omega (\bar{p} + \bar{\alpha}N\bar{u})(v - \bar{u}) \, dx \geq 0 \quad \forall v \in K.
\]

**Remark 3.** Since \( B \) has a nonempty interior, we deduce from (2.5) that \( R(L) \cap \bar{B} \neq \emptyset \). This implies (see Barbu and Precupanu [2], Ekeland and Temam [12]) that \( \partial(I_B \circ L) \cdot (\bar{y}) = L^*\partial I_B(L\bar{y}) \).

**Remark 4.** We will verify that hypothesis (3.1) holds in our three examples. However, if (3.1) does not hold, then by Remark 3 there exists \((\bar{\lambda}, \bar{\mu})\) in \( \mathbb{R}^m \times \partial I_B(L\bar{y}) \) such that \( \|\bar{\lambda}\| + \|\bar{\mu}\| > 0 \) and \( T^*\bar{\lambda} + L^*\bar{\mu} = 0 \). In other words, if all hypotheses of Theorem 2 are satisfied except perhaps (3.1), there exist \( \bar{p}, \bar{\lambda}, \bar{\mu}, \bar{\alpha} \) as in Theorem 1, not all null, satisfying (3.3)-(3.5).

In order to prove Theorem 2, we need to establish some preliminary results.

**LEMMA 1.** Let \( W \) be a Banach space and \( D \) be a convex subset of \( W \) (not necessarily closed) with nonempty interior. Let \( \{(w_n, \eta_n)\} \) be a sequence in \( W \times W' \) such that \( w_n \in D, \) \( \eta_n \in \partial I_D(w_n) \). If \( \lim \inf \|\eta_n\| > 0 \), then zero is not a weak-star limit point of \( \{\eta_n\} \).

**Proof.** Assume that the conclusion does not hold. Let \( w_0 \) be given in \( \bar{D} \). There exists \( r > 0 \) such that \( \|w\| \leq r \) implies that \( w_0 + w \) is in \( D \); hence

\[
\langle \eta_n, w_0 + w - w_n \rangle \leq 0,
\]

and this implies

\[
r\|\eta_n\| = \sup_{\|w\| \leq r} \langle \eta_n, w \rangle \leq \langle \eta_n, w_n - w_0 \rangle.
\]
The strong convergence of \( w_n \) allows us to pass to the limit and we get
\[
\lim \inf r_n \leq 0,
\]
which gives a contradiction. □

**Lemma 2.** Let \( W \) be a Banach space, and \( f \) (respectively, \( g \)) be a Gâteaux-differentiable (respectively, convex) mapping from \( W \) into \( \mathbb{R} \) (respectively, \( \mathbb{R}^+ \)). Let \( \bar{x} \) be a solution of the following problem:
\[
\min f(x) + g(x), \quad x \in W.
\]
Then
\[
\langle \nabla f(\bar{x}), x - \bar{x} \rangle + g(x) - g(\bar{x}) \geq 0 \quad \forall x \in W,
\]
or, equivalently,
\[
\nabla f(\bar{x}) + \partial g(\bar{x}) \ni 0.
\]

**Proof.** A straightforward application of the definition of the subdifferential [12] allows us to verify the equivalence of the two statements of the conclusion. Now consider \( x' = \bar{x} + t(x - \bar{x}) \) for \( t \in [0, 1] \). We have, using the convexity of \( g: f(x') + g(x') \leq f(x') + (1 - t)g(\bar{x}) + t g(x) \); hence, as \( \bar{x} \) is a solution of the problem above,
\[
0 \leq f(x') + g(x') - (f(x') + g(x')) \leq f(x') - f(x) + t(g(x) - g(\bar{x})).
\]
Dividing by \( t \) and passing to the limit, we obtain the result. □

We now consider the following approximate problem. Let the state equation be
\[
Ay = u + w \quad \text{in} \quad \Omega,
\]
\[
y = 0 \quad \text{on} \quad \Gamma.
\]
The control is now \((u, w)\) in \( L^2(\Omega) \times L^2(\Omega) \). We define
\[
J_\varepsilon(y, u, w) = J(y, u) + \frac{1}{2\varepsilon} \int_\Omega (w + \phi(y))^2 \, dx
\]
\[
+ \frac{1}{2\varepsilon} \| Ty - a \|^2 + \frac{1}{2} \int_\Omega (u - \bar{u})^2 \, dx + \frac{1}{2} \int_\Omega (w + \phi(y))^2 \, dx.
\]
The approximate problem is
\[
\text{minimize} \quad J_\varepsilon(y, u, w)
\]
\[
\text{subject to} \quad (3.6), \quad u \in K, \quad w \in L^2(\Omega), \quad y \in Y, \quad Ly \in B.
\]

**Theorem 3.** Let \((\bar{y}, \bar{u})\) be a solution of \((P)\). We assume that (2.2)–(2.5) hold. Then we have the following:

(i) Problem \((P_\varepsilon)\) has at least one solution.

(ii) To each solution \((y_\varepsilon, u_\varepsilon, w_\varepsilon)\) of \((P_\varepsilon)\) is associated \( p_\varepsilon \) in \( W_0^{1,s}(\Omega) \) for all \( s < n/(n - 1) \), \( \mu_\varepsilon \in Z' \), and \( \lambda_\varepsilon \in \mathbb{R}^m \) such that
\[
A^* \mu_\varepsilon = (y_\varepsilon - y_d) - (y_\varepsilon - y_d) + T^* \lambda_\varepsilon + L^* \mu_\varepsilon + \frac{1}{\varepsilon} \phi'(y_\varepsilon)(w_\varepsilon + \phi(y_\varepsilon)),
\]
\[
p_\varepsilon = 0 \quad \text{on} \quad \Gamma,
\]
\[
\langle \mu_\varepsilon, z - Ly_\varepsilon \rangle \leq 0 \quad \forall z \in B
\]
\[
\int_\Omega (p_\varepsilon + Nu_\varepsilon + u_\varepsilon - \bar{u})(v - u_\varepsilon) \, dx \geq 0 \quad \forall v \in K,
\]
\[
p_\varepsilon + \frac{1}{\varepsilon} [w_\varepsilon + \phi(y_\varepsilon)] + w_\varepsilon + \phi(y_\varepsilon) = 0.
\]
**Proof.** (i) The triple \((\bar{y}, \bar{u}, -\phi(\bar{y}))\) is feasible for \((P,\lambda)\). Any minimizing sequence is bounded in \(L^n(\Omega) \times L^2(\Omega) \times L^2(\Omega)\), and hence by (3.6) in \(Y \times L^2(\Omega) \times L^2(\Omega)\). Taking a subsequence if necessary and using the compactness of \(Y\) in \(C_0(\Omega)\) (n \(\geq 3\)) to pass to the limit in the nonlinear terms, we get the result as in the proof of Theorem 1.

(ii) Denote by \(y_{u,w}\) the solution of (3.6) and by \(\theta(u, w)\) the mapping \((u, w) \rightarrow J_\varepsilon(y_{u,w}, u, w)\). It is easy to verify that \(\theta\) is \(C^1\) and that

\[
\theta'_u(u, w) = q + Nu + u - \bar{u},
\]

\[
\theta'_w(u, w) = q + \frac{1}{\varepsilon} (w + \phi(y_{u,w})) + w + \phi(\bar{y}),
\]

where \(q\) is the solution of \((A^*\text{ being the formal transpose of } A)\):

\[
A^*q = |y_{u,w} - y_d|^{r-2}(y_{u,w} - y_d) + \frac{1}{\varepsilon} \phi'(y_{u,w})(w + \phi(y_{u,w})) + \frac{1}{\varepsilon} T^*(Ty_{u,w} - a) \quad \text{in } \Omega,
\]

\[
q = 0 \quad \text{on } \Gamma.
\]

Let \((y_\varepsilon, u_\varepsilon, w_\varepsilon)\) be a solution of \((P,\varepsilon)\) and \(q_\varepsilon\) the associated adjoint-state. Let us define

\[
\hat{L}: L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R},
\]

\[
(u, w) \rightarrow Ly_{u,w},
\]

\[
\hat{K} = K \times L^2(\Omega),
\]

\[
g(u, w) = I_\varepsilon(\hat{L}(u,w)) + I_\hat{K}(u, w).
\]

Problem \((P,\varepsilon)\) is equivalent to

\[
\min \theta(u, w) + g(u, w), (u, w) \in L^2(\Omega) \times L^2(\Omega).
\]

Now applying Lemma 2, we get

\[
\nabla \theta(u_\varepsilon, w_\varepsilon) + \partial g(u_\varepsilon, w_\varepsilon) \geq 0.
\]

The mapping \(w \rightarrow y_{u,w}(\text{with } u \text{ fixed})\) is an isomorphism from \(L^2(\Omega)\) onto \(Y\). Hence by (2.1) there exists \((u, w)\) in \(\hat{K}\) with \(\hat{L}(u, w)\) in \(\hat{B}\). This allows us [12] to apply the rules of subdifferential calculus to the mapping \(g\) and we get the equality

\[
\partial g(u_\varepsilon, w_\varepsilon) = \hat{L}^*\partial I_\varepsilon(Ly_\varepsilon) + \partial I_{\hat{K}}(u_\varepsilon, w_\varepsilon).
\]

Hence there exists \(\mu_\varepsilon\) in \(\partial I_\varepsilon(Ly_\varepsilon)\) such that

\[
\nabla \theta(u_\varepsilon, w_\varepsilon) - \hat{L}^*\mu_\varepsilon + \partial I_{\hat{K}}(u_\varepsilon, w_\varepsilon) \geq 0,
\]

or equivalently,

\[
(\theta'_u(u_\varepsilon, w_\varepsilon), u - u_\varepsilon) + (\theta'_w(u_\varepsilon, w_\varepsilon), w - w_\varepsilon) + \langle \mu_\varepsilon, Ly_{u,w} - Ly_\varepsilon \rangle \geq 0 \quad \forall (u, w) \in K \times L^2(\Omega).
\]

Let \(r_\varepsilon\) be the solution of

\[
A^*r_\varepsilon = L^*\mu_\varepsilon \quad \text{in } \Omega,
\]

\[
r_\varepsilon = 0 \quad \text{on } \Gamma.
\]

We get

\[
(\theta'_u(u_\varepsilon, w_\varepsilon) + r_\varepsilon, u - u_\varepsilon) \geq 0 \quad \forall u \in K,
\]

\[
\theta'_w(u_\varepsilon, w_\varepsilon) + r_\varepsilon = 0.
\]
We obtain the result with \( p_e = q_e + r_e \) and \( \lambda_e = (1/\varepsilon)(T y_e - a) \). As \( A^* p_e \) is in \( M(\Omega) \), \( p_e \) is in \( W_0^{1,\varepsilon}(\Omega) \) for all \( s < n/(n-1) \) (see [9], [21]).

**Lemma 3.** Let \( \{ (y_e, u_e, w_e) \} \) be a sequence of solutions of \( (P_e) \). Then

\[
0 = \lim_{\varepsilon \to 0} \| y_e - \bar{y} \|_{L^2(\Omega)} = \lim_{\varepsilon \to 0} \| u_e - \bar{u} \|_{L^2(\Omega)} = \lim_{\varepsilon \to 0} \| w_e + \phi(\bar{y}) \|_{L^2(\Omega)}.
\]

**Proof.** From the inequality \( J_e(y_e, u_e, w_e) \leq J_e(y, u, \bar{y} - \phi(\bar{y})) = J(y, u, \bar{y}) \) and the form of \( J \), we deduce that \( \{(y_e, u_e, w_e)\} \) is bounded in \( L^\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega) \); hence \( \{y_e\} \) is bounded in \( Y \) by (3.6) and (2.6). This implies that for \( \varepsilon \in D, D \) being a subset of \( ]0, \infty[ \) having zero as limit point, we have for some \((y, u, w)\) in \( Y \times L^2(\Omega) \times L^2(\Omega) \) when \( \varepsilon \to 0:\n
\[
y_e \to y \quad \text{in } Y \text{ weak}, \quad C_0(\Omega) \text{ strong},
\]

\[
u_e \to u \quad \text{in } L^2(\Omega) \text{ weak},
\]

\[
 w_e \to w \quad \text{in } L^2(\Omega) \text{ weak},
\]

with \((y, u, w)\) satisfying (3.6). As \( K \) and \( B \) are closed and convex in \( L^2(\Omega) \) and \( Z \) we have \( u \in K \) and \( Ly \in B \). The form of \( J \) implies that \( \| w_e + \phi(y_e) \|_{L^2(\Omega)} \to 0 \) and \( \| Ty_e - a \| \to 0 \); hence \( w + \phi(y) = 0 \). With (3.6) this implies that \((y, u)\) satisfies (2.1). As \( J \) is lower semicontinuous, we have that

\[
J(y, u, w) \geq \limsup_{\varepsilon \to 0} J_e(y_e, u_e, w_e)
\]

\[
\geq \limsup \{ J(y_e, u_e) + \frac{1}{2}\| u_e - \bar{u} \|_{L^2(\Omega)}^2 + \frac{1}{2}\| w_e + \phi(\bar{y}) \|_{L^2(\Omega)}^2 \}
\]

\[
\geq J(y, u) + \frac{1}{2}\| u - \bar{u} \|_{L^2(\Omega)}^2 + \frac{1}{2}\| w + \phi(\bar{y}) \|_{L^2(\Omega)}^2.
\]

As \((y, u)\) is feasible for \( (P) \), this implies that \( u = \bar{u} \) and \( w + \phi(\bar{y}) = 0 \); hence \( \phi(y) = \phi(\bar{y}) \). With (2.1) this implies that \( y = \bar{y} \). But the inequality above also implies \( \| u_e - \bar{u} \|_{L^2(\Omega)} \to 0 \) and \( \| w_e + \phi(\bar{y}) \|_{L^2(\Omega)} \to 0 \); when we use (2.6), the result follows. □

We now are in position to prove Theorem 2, by passing to the limit in the optimality system of \((P_e)\).

**Proof of Theorem 2.** Let \((y_e, u_e, w_e)\) denote a solution of \((P_e)\) and \((P_e, \mu_e, \lambda_e)\) be given by Theorem 3. If \( \{(p_e, \mu_e, \lambda_e)\} \) is bounded we obtain the result with \( \alpha = 1 \) by passing to the limit in the optimality system of \((P_e)\) with the help of Lemma 3. Now suppose that \( \alpha_e = 1/\| p_e \|_{L^2(\Omega)} + \| \mu_e \|_Z + \| \lambda_e \| \) converges toward zero. Multiplying by \( \alpha_e \) the optimality system given by Theorem 3 and defining

\[
\bar{p}_e = \alpha_e p_e, \quad \bar{\mu}_e = \alpha_e \mu_e, \quad \bar{\lambda}_e = \alpha_e \lambda_e,
\]

we obtain, eliminating \( (1/\varepsilon)(w_e + \phi(y_e)) \) from the last equality of Theorem 3,

\[
A^* \bar{p}_e + \phi'(y_e) \bar{p}_e = \alpha_e \| y_e - y_d \|_{L^2(\Omega)}^2 + T^* \bar{\lambda}_e + L^* \bar{\mu}_e
\]

\[
- \alpha_e \phi'(y_e)(w_e + \phi(y)) \quad \text{in } \Omega,
\]

(3.7)

\[
\bar{p}_e = 0 \quad \text{on } \Gamma,
\]

\[
\langle \bar{\mu}_e, z - Ly_e \rangle \equiv 0 \quad \forall z \in B,
\]

\[
\int_\Omega [ \bar{p}_e + \alpha_e (Nu_e + u_e - \bar{u})](v - u_e) \equiv 0 \quad \forall v \in K.
\]

As \( \| \bar{p}_e \|_{L^2(\Omega)} + \| \bar{\mu}_e \|_Z + \| \bar{\lambda}_e \| \) is bounded, we may pass to the limit in the systems above by using Lemma 3; then we obtain (3.3)-(3.5), with \( \bar{\alpha} = 0 \) here. It remains to prove that \( \bar{p} \neq 0 \). If \( \bar{p} = 0 \), then \( T^* \bar{\lambda} + L^* \bar{\mu} = 0 \) by (3.3). However, (3.1) and the injectivity of \( T^* \) and \( L^* \) (by (2.5)) then imply that \( \bar{\mu} = 0 \) and \( \bar{\lambda} = 0 \). Since \( \{\bar{\lambda}_e\} \) is in \( \mathbb{R}^m \) and
because of Lemma 1, we infer that \( \lim \inf \| \tilde{\mu} \|_{Z'} = 0 \) and \( \| \tilde{\lambda} \| \to 0; \) hence \( \| \tilde{\nu} \|_{L^2(\Omega)} \to 1. \) From (3.7) and Lemma 3 we deduce that \( A^* \tilde{\nu} \) is bounded in \( M(\Omega); \) hence \( \{ \tilde{\nu} \} \) is bounded in \( W^{1,s}_0(\Omega) \) for all \( s < n/(n-1). \) The compact injection from \( W^{1,s}_0(\Omega) \) into \( L^2(\Omega) \) (for \( n \geq 3 \) and \( s \) close to \( n/(n-1) \)) implies that \( \| \tilde{\nu} \|_{L^2(\Omega)} \to \| \tilde{\nu} \|_{L^2(\Omega)} = 0, \) which gives a contradiction.

4. Applications. In this section we consider the three examples stated in § 2, and we derive the optimality system for each of them.

Example 1.

Theorem 4. Let \((\tilde{y}, \tilde{u}) \in Y \times K \) be a solution of \((P1). \) Then there exist a real number \( \tilde{\alpha} \geq 0 \) and elements \( \tilde{\lambda} \in R^m \) and \( \tilde{\nu} \in W^{1,s}_0(\Omega) \) for all \( s < n/(n-1) \) satisfying

\[
\begin{align*}
\tilde{\alpha} + \| \tilde{\nu} \|_{W^{1,s}_0(\Omega)} &> 0, \\
A\tilde{y} + \phi(\tilde{y}) &= \tilde{u} \quad \text{in } \Omega, \\
\tilde{y} &= 0 \quad \text{on } \Gamma, \\
A^* \tilde{\nu} + \phi'(\tilde{y})\tilde{\nu} &= \tilde{\alpha} |\tilde{y} - y_d|^{r-2}(\tilde{y} - y_d) + \sum_{i=1}^{m} \tilde{\lambda}_i \delta_{\{x_i\}} \quad \text{in } \Omega, \\
\tilde{\nu} &= 0 \quad \text{on } \Gamma, \\
\int_{\Omega} (\tilde{\nu} + \tilde{\alpha} N\tilde{u})(v - \tilde{u}) \, dx &\geq 0 \quad \forall \, v \in K.
\end{align*}
\]

Proof. Hypothesis (3.1) is trivially satisfied as \( B = C_0(\Omega). \) Hence we may apply Theorem 2, which gives the result.

In some cases it is possible to prove that the previous theorem is true with \( \tilde{\alpha} = 1. \) We are going to study two situations where this is so.

Theorem 5. Let \( a_{ij} \in C^2(\Omega), 1 \leq i \leq j \leq n. \) Then the results of Theorem 2 are obtained with \( \tilde{\alpha} = 1 \) if \( \Omega \) is connected and one of the two following hypotheses holds:

(i) There exists an open subset \( \Omega_0 \) of \( \Omega \) such that \( K = K + L^2(\Omega_0)(L^2(\Omega_0)) \) is the extension by zero from \( L^2(\Omega_0) \) to \( L^2(\Omega). \)

(ii) \( K = \{ v \in L^2(\Omega); v(x) \geq 0 \text{ a.e. } x \in \Omega \}, \) and \( u = 0 \) is not optimal for \((P1). \)

Proof. (i) If \( \tilde{\alpha} = 0, \) it follows from (4.3) that

\[
\begin{align*}
A^* \tilde{\nu} + \phi'(\tilde{y})\tilde{\nu} &= \sum_{i=1}^{m} \tilde{\lambda}_i \delta_{\{x_i\}} \quad \text{in } \Omega, \\
\tilde{\nu} &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

Now from (4.4) and the property of \( K, \) we get that \( \tilde{\nu} = 0 \) in \( \Omega_0. \) Taking \( \Omega_1 = \{ x \in \Omega \}_{i=1}^{m}, \) we have

\[
\begin{align*}
A^* \tilde{\nu} + \phi'(\tilde{y})\tilde{\nu} &= 0 \quad \text{in } \Omega_1, \\
\tilde{\nu} &= 0 \quad \text{in } \Omega \setminus \{ x_i \}_{i=1}^{m}.
\end{align*}
\]

Then we can use the Prolongation Unicity Theorem (Saut and Scheurer [20]) and we deduce that \( \tilde{\nu} = 0 \) in \( \Omega_1, \) hence in \( \Omega, \) which contradicts (4.1).

(ii) If \( \tilde{\alpha} = 0, \) we deduce from (4.4) that \( \tilde{\nu} \geq 0 \) in \( \Omega. \) If \( \tilde{\nu} \) is null on an open subset \( \Omega_0 \) of \( \Omega, \) we can do as in (i) and obtain a contradiction. Otherwise, for each open subset \( \Omega_0 \) with \( \Omega_0 \) included in \( \Omega_1 \) we have

\[
\max_{x \in \Omega_0} \tilde{\nu}(x) > 0.
\]
We remark that $\bar{p}$ satisfies

$$A^*\bar{p} + \max(0, \phi'(\bar{y}))\bar{p} \geq 0 \quad \text{in } \Omega_1,$$

$$\bar{p} = 0 \quad \text{on } \Gamma.$$ 

Applying the Harnack inequality to $A^* + \max(0, \phi'(\bar{y}))$ (Stampacchia [21]) as in [5], we deduce that $\bar{p}(x) > 0$ everywhere in $\Omega_1$, which with (4.4) implies that $\bar{u} = 0$ almost everywhere. \(\square\)

**Example 2.**

**Theorem 6.** If $(\bar{y}, \bar{u}) \in Y \times K$ is solution of (P2), then there exists a real number $\bar{\alpha} \geq 0$ and elements $\bar{\mu} \in L^\infty(\Omega)$ and $\bar{p} \in W^{1,1}_0(\Omega)$ such that for all $s < n/(n-1)$:

$$\bar{\alpha} + \| \bar{p} \|_{W^{1,1}_0(\Omega)} > 0,$$

$$A\bar{y} + \phi(\bar{y}) = \bar{u} \quad \text{in } \Omega,$$

$$\bar{p} = 0 \quad \text{on } \Gamma,$$

$$\int_\Omega \bar{\mu}(z - \bar{y}) \, dz \leq 0 \quad \forall z \in B,$$

$$\int_\Omega (\bar{p} + \bar{\alpha}N\bar{u})(v - \bar{u}) \, dx \geq 0 \quad \forall v \in K.$$ 

**Proof.** Here again, (3.1) is satisfied because $T = 0$. Hence we may apply Theorem 2 and remark that $Z' = L^\infty(\Omega)$ and $L^*$ is the canonical injection into $M(\Omega)$. Moreover, the regularity of $\bar{p}$ follows from (2.6), (4.10), and the fact that $\bar{\alpha}|\bar{y} - y_d|^{n-2}(\bar{y} - y_d) + \bar{\mu} - \phi'(\bar{y})\bar{p}$ belongs to $L^1(\Omega)$. \(\square\)

**Example 3.**

**Theorem 7.** If $(\bar{y}, \bar{u}) \in Y \times K$ is solution of (P3), then there exist a real number $\lambda \geq 0$ and elements $\bar{p} \in W^{1,1}_0(\Omega)$ for all $s < n/(n-1)$, $\lambda \in \mathbb{R}$, and $\bar{\mu} \in M(\Omega)$ such that

$$\bar{\alpha} + \| \bar{p} \|_{W^{1,1}_0(\Omega)} > 0,$$

$$A\bar{y} + \phi(\bar{y}) = \bar{u} \quad \text{in } \Omega,$$

$$\bar{p} = 0 \quad \text{on } \Gamma,$$

$$\int_\Omega (z - \bar{y}) \, d\bar{\mu} \leq 0 \quad \forall z \in B,$$

$$\int_\Omega (\bar{p} + \bar{\alpha}N\bar{u})(v - \bar{u}) \, dx \geq 0 \quad \forall v \in K.$$ 

**Proof.** We have to verify that (3.1) is satisfied. Remember that in this case $L$ is the identity in $C_0(\Omega)$ and $T \in C_0(\Omega)'$. Take $\mu \in \partial I_B(\bar{y})$ and $\lambda \in \mathbb{R}$ such that

$$\langle \mu, z \rangle = \langle T^*\lambda, z \rangle = \lambda \int_\Omega z \, dx \quad \forall z \in C_0(\Omega);$$

this implies that $\mu = \lambda m$, where $m$ is the Lebesgue measure. If $\lambda \neq 0$, this implies that $\bar{y}(x) = \pm \delta$ almost everywhere, which contradicts the boundary condition. \(\square\)
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