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PERTURBED OPTIMIZATION IN BANACH SPACES I: A GENERAL THEORY BASED ON A WEAK DIRECTIONAL CONSTRAINT QUALIFICATION*

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Abstract. Using a directional form of constraint qualification weaker than Robinson's, we derive an implicit function theorem for inclusions and use it for first- and second-order sensitivity analyses of the value function in perturbed constrained optimization. We obtain Hölder and Lipschitz properties and, under a *no-gap* condition, first-order expansions for exact and approximate solutions. As an application, differentiability properties of metric projections in Hilbert spaces are obtained, using a condition generalizing polyhedricity. We also present in the appendix a short proof of a generalization of the convex duality theorem in Banach spaces.

Key words. sensitivity analysis, marginal function, approximate solutions, directional constraint qualification, regularity and implicit function theorems, convex duality

AMS subject classifications. 46A20, 46N10, 47H19, 49K27, 49K40, 58C15, 90C31

1. Introduction. This paper is the first of a trilogy devoted to sensitivity analysis of parametrized optimization problems of the form

$$(P_u) \qquad \min_x \{f(x, u) : G(x, u) \in K\}$$

where f and G are C^2 mappings from $X \times \mathbb{R}_+$ to $\mathbb{R}$ and Y, respectively, X and Y are Banach spaces, and K is a closed convex subset of Y.

While the theory is fairly complete in the case of finite-dimensional mathematical programming, that is, optimization problems with finitely many equality and inequality constraints, the sensitivity of perturbed optimization problems in Banach spaces is still being developed. Just to mention a couple of recent works related to this topic, see for instance [3, 8, 9, 11, 19, 21, 26] as well as the monographs [10, 13, 18] and references therein.

Loosely speaking, the assumptions that support a complete sensitivity analysis of the value function and optimal solutions are uniqueness of the optimal solution for the unperturbed problem, constraint qualification, existence of Lagrange multipliers, and second-order sufficient optimality conditions.

Concerning constraint qualification, the standard assumption is Robinson's generalization [23] of the Mangasarian–Fromovitz condition [20]. Following the lines of previous works in mathematical programming [2, 5, 7, 12, 14], in this paper we show that sensitivity analysis is still possible under a weak directional form of constraint qualification that takes into account the nature of perturbations. This condition is used to derive a generalization of Robinson's implicit function theorem for systems of inequalities that, in conjunction with a strong second-order sufficient condition, allows us to obtain first- and second-order upper and under estimates of the marginal function. When these two estimates coincide (we give some sufficient conditions for this) the first-order sensitivity of approximate optimal solutions of (P_u) is obtained.

Our second-order expansion includes a term that takes into account the possible curvature of the boundary of K and does not appear in the classical setting of mathematical programming where K is a polyhedral set. This curvature term, studied in [9, 17] in the context of second-order necessary conditions (see also the previous work [4]), leads to a generalization of the

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notion of polyhedric set and to new results on differentiability of metric projections onto convex sets in Hilbert spaces.

We observe that in the case of the trivial perturbation $f(x, u) = f(x, 0)$ and $G(x, u) = G(x, 0)$ for all u, the directional constraint qualification reduces to Robinson's condition and our upper estimates to the necessary optimality conditions obtained in [9]. Similarly, from our under estimates one can easily derive (new) sufficient conditions for local optimality.

When the strong second-order condition fails, and particularly when the set of Lagrange multipliers is empty, we know that directional differentiability of solutions and of the marginal function may fail [5]. It seems that the directional constraint qualification considered in this paper is too weak to obtain a satisfactory sensitivity analysis in such cases. This motivates a strenghtened form of directional qualification, which is the subject of part II of this work.

Finally, in part III we study the application of both theories to semi-infinite programming, that is to say, optimization problems with X finite dimensional and infinitely many inequality constraints. In that case there is a gap between the upper and lower estimates, so we will fill this gap by computing sharper lower estimates.

We denote the feasible set, optimal value, and solution set of (P_u) as

$$\begin{aligned} F(u) &:= \{x \in X : G(x, u) \in K\}, \\ v(u) &:= \inf\{f(x, u) : x \in F(u)\}, \\ S(u) &:= \{x \in F(u) : f(x, u) = v(u)\}. \end{aligned}$$

Similarly, given an optimization problem (P) we denote by $F(P)$, $v(P)$, and $S(P)$ its feasible set, optimal value, and optimal solution set, respectively.

The set of Lagrange multipliers associated with an optimal solution $x \in S(u)$ is

$$\Lambda_u(x) := \{\lambda \in Y^* : \lambda \in N_K(G(x, u)), \mathcal{L}'_x(x, \lambda, u) = 0\}$$

with Y^* denoting the dual space of Y, $N_K(y)$ the normal cone to K at y, and $\mathcal{L}$ the Lagrangian function

$$\mathcal{L}(x, \lambda, u) := f(x, u) + \langle \lambda, G(x, u) \rangle.$$

For the rest of this paper we assume $v(0)$ finite and $S(0)$ nonempty. We also consider a fixed optimal solution $x_0 \in S(0)$ and denote by $\Lambda_0 := \Lambda_0(x_0)$ the corresponding set of multipliers.

Finally, we recall the definition of the first- and second-order tangent sets:

$$\begin{aligned} T_K(y) &:= \{h \in Y : \text{ there exists } o(u) \text{ such that } y + uh + o(u) \in K\}, \\ T_K^2(y, h) &:= \left\{k \in Y : \text{ there exists } o(u^2) \text{ such that } y + uh + \frac{1}{2}u^2 k + o(u^2) \in K\right\}. \end{aligned}$$

Throughout this paper $o(u)$ and $o(u^2)$ will be used freely to denote any terms that are negligible compared to u and u^2. Similary, $O(u)$ and $O(u^2)$ denote terms of orders u and u^2.

2. Upper estimates of the value function. We are interested in sensitivity analysis of (P_u), that is to say, the study of differentiability properties of the optimal value function v and the optimal (set-valued) map S. To this end we consider the *linear* and *quadratic* approximating problems:

$$(L) \qquad \min_d \{f'(x_0, 0)(d, 1) : \ G'(x_0, 0)(d, 1) \in T_K(G(x_0, 0))\},$$

$$(Q) \qquad \min\{v(L_d) : \ d \in S(L)\},$$

$$(L_d) \qquad \min_w \{f'_x(x_0, 0)w + \Phi_f(d) : \ G'_x(x_0, 0)w + \Phi_G(d) \in T_K^2(d)\},$$

where we have set

$$\Phi_f(d) := f''(x_0, 0)(d, 1)(d, 1),$$
$$\Phi_G(d) := G''(x_0, 0)(d, 1)(d, 1),$$
$$T_K^2(d) := T_K^2(G(x_0, 0), G'(x_0, 0)(d, 1)).$$

The motivation for these approximating problems is the following.

We say that $u \to x_u$ is a *feasible path* if $x_u \in F(u)$ for $u > 0$ small enough and x_u tends to x_0 when $u \downarrow 0$. Suppose that we have a feasible path of the form $x_u = x_0 + ud + o(u)$. A first-order expansion gives $G(x_u, u) = G(x_0, 0) + uG'(x_0, 0)(d, 1) + o(u) \in K$, so d is feasible for (L) and also

$$v(u) \leq f(x_u, u) = v(0) + uf'(x_0, 0)(d, 1) + o(u), \tag{1}$$

suggesting that $v(u) \leq v(0) + u\ v(L) + o(u)$.

Similarly, if $d \in S(L)$ and $x_u = x_0 + ud + \frac{1}{2}u^2 w + o(u^2)$ is a feasible path, a second-order Taylor expansion of $G(x_u, u)$ shows that $w \in F(L_d)$, and

$$v(u) \leq f(x_u, u) = v(0) + u\ v(L) + \frac{1}{2}u^2[f_x'(x_0, 0)w + \Phi_f(d)] + o(u^2), \tag{2}$$

so we may expect $v(u) \leq v(0) + u\ v(L) + \frac{1}{2}u^2\ v(Q) + o(u^2)$.

To prove these upper estimates it suffices to show that each $d \in F(L)$ admits an $o(u)$ correction such that $x_0 + ud + o(u) \in F(u)$ and similarly that each $w \in F(L_d)$ admits an $o(u^2)$ correction such that $x_0 + ud + \frac{1}{2}u^2 w + o(u^2) \in F(u)$. The existence of such corrections may be established by using Robinson's regularity theorem [23, Thm. 1], which is based on the constraint qualification condition

$$0 \in \operatorname{int}\,[G(x_0, 0) + G_x'(x_0, 0)X - K]. \tag{CQ}$$

However, this condition does not take into account the specific form of perturbations, so, loosely speaking, it will work uniformly no matter what type of perturbations are being considered. We shall rather use the following refinement of Robinson's regularity theorem proved in Appendix B, which allows us to discriminate those perturbations for which sensitivity analysis can be carried out.

THEOREM B.5. *Let us assume the* directional constraint qualification

$$0 \in \operatorname{int}\,[G(x_0, 0) + G'(x_0, 0)X \times (0, \infty) - K]. \tag{DCQ}$$

Then for each trajectory $x_u = x_0 + O(u)$ there exist constants $c \geq 0$, $u_0 > 0$ and a second trajectory y_u such that $G(y_u, u) \in K$ and

$$\|y_u - x_u\| \leq c\ d(G(x_u, u), K)$$

for all $u \in [0, u_0]$.

It may not be apparent that (CQ) implies (DCQ). To see this we remark (see Appendix B) that the latter is equivalent to

$$0 \in \operatorname{int}\,[G(x_0, 0) + G'(x_0, 0)X \times [0, \infty) - K]. \tag{DCQ)$'$}$$

PROPOSITION 2.1. *Suppose* (DCQ) *holds. Then* $\limsup_{u \downarrow 0}[v(u) - v(0)]/u \leq v(L)$ *and when $v(L) > -\infty$ we have the first-order upper estimate*

$$v(u) \leq v(0) + u\ v(L) + o(u). \tag{3}$$

Also, $\limsup_{u\downarrow 0} 2[v(u) - v(0) - u\ v(L)]/u^2 \leq v(Q)$ *and when* $v(Q) > -\infty$ *the following second-order upper estimate holds:*

$$v(u) \leq v(0) + u\ v(L) + \frac{1}{2}u^2\ v(Q) + o(u^2). \tag{4}$$

Proof. Let d be feasible for (L). Applying Theorem B.5 with $x_u = x_0 + ud$ we find a feasible trajectory y_u such that $\|y_u - x_u\| \leq c\ d(G(x_u, u), K) = o(u)$. Then $y_u = x_0 + ud + o(u)$ and the first-order estimate follows from (1).

To prove the second-order estimate, let $d \in S(L)$ and $w \in F(L_d)$. Applying Theorem B.5 with $x_u = x_0 + ud + \frac{1}{2}u^2 w$ we get a feasible trajectory y_u with $\|y_u - x_u\| \leq c\ d(G(x_u, u), K) = o(u^2)$. Then $y_u = x_0 + ud + \frac{1}{2}u^2 w + o(u^2)$ and the conclusion follows from (2). □

The above upper estimates are only meaningful if $v(L) < +\infty$ and $v(Q) < +\infty$. Let us then prove the following result.

PROPOSITION 2.2. *Assuming (DCQ) we have* $v(L) < +\infty$. *Moreover, in this case* $v(Q) < +\infty$ *if and only if there exists* $d \in S(L)$ *such that* $T_K^2(d) \neq \phi$.

Proof. Using (DCQ) we may find $t > 0$ and $d \in X$ with $G'(x_0, 0)(d, t) \in K - G(x_0, 0)$. Then d/t is feasible for (L) and consequently $v(L) < +\infty$.

Clearly $v(Q) < +\infty$ requires $T_K^2(d) \neq \phi$ for some $d \in S(L)$.

To prove the converse we fix $k \in T_K^2(d)$ so that, according to [9, Prop. 3.1],

$$k + \mathbb{R}_+[T_K(G(x_0, 0)) - G'(x_0, 0)(d, 1)] \subset T_K^2(d). \tag{5}$$

Using (DCQ) we find $\mu > 0$ with $\mu[k - \Phi_G(d)] \in G(x_0, 0) + G'(x_0, 0)X \times (0, \infty) - K$, and then for some $z \in X$ and $t > 0$ we get

$$\begin{aligned}\Phi_G(d) &\in k + \frac{1}{\mu}[K - G(x_0, 0) - G'(x_0, 0)(z, t)]\\ &\in k - G'_x(x_0, 0)\frac{z - td}{\mu} + \frac{1}{\mu}[T_K(G(x_0, 0)) - tG'(x_0, 0)(d, 1)].\end{aligned}$$

Letting $w := (z - td)/\mu$ and using (5) we deduce that

$$G'_x(x_0, 0)w + \Phi_G(d) \in k + \frac{t}{\mu}[T_K(G(x_0, 0)) - G'(x_0, 0)(d, 1)] \subset T_K^2(d),$$

proving that (L_d) is feasible and then $v(Q) \leq v(L_d) < +\infty$. □

3. Differentiability of the value function and suboptimal trajectories. To find lower estimates of the cost and sufficient conditions for the existence of the right derivative $v'(0)$, we use convex duality theory to get the following characterization for $v(L)$.

PROPOSITION 3.1. *Assume (DCQ). Then* $v(L) = v(D)$ *and* $S(D) \neq \phi$, *where*

$$\max\{\mathcal{L}'_u(x_0, \lambda, 0) : \lambda \in \Lambda_0\}. \tag{D}$$

Moreover, $v(L) > -\infty$ *if and only if* $\Lambda_0 \neq \phi$, *in which case* $S(D)$ *is a nonempty weak* compact subset of* Λ_0.

Proof. This is a consequence of the convex duality theorem of Appendix A, Theorem A.2, applied to problem (L) with the perturbation function

$$\varphi(d, y) := \begin{cases} f'(x_0, 0)(d, 1) & \text{if } G'(x_0, 0)(d, 1) + y \in T_K(G(x_0, 0)), \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, from (DCQ) we get

$$\begin{aligned} Y &= T_K(G(x_0,0)) - G'(x_0,0)X \times (0,\infty) \\ &= \mathbb{R}_+ \bigcup_{d\in X} [T_K(G(x_0,0)) - G'(x_0,0)(d,1)], \end{aligned}$$

so $\mathbb{R}_+ \cup_d \operatorname{dom} \varphi(d,\cdot) = Y$ and Theorem A.2 can be used to deduce

$$v(L) = -\min_{\lambda} \varphi^*(0,\lambda). \tag{6}$$

A straightforward computation shows that

$$\varphi^*(x^*,\lambda) = \begin{cases} -\mathcal{L}'_u(x_0,\lambda,0) & \text{if } \lambda \in N_K(G(x_0,0)),\ \mathcal{L}'_x(x_0,\lambda,0) = x^*, \\ +\infty & \text{otherwise,} \end{cases}$$

which combined with (6) yields the desired conclusions. □

We state our next results using suboptimal paths. We say that x_u is an *$o(u)$-optimal trajectory* if it is a feasible path and $v(u) = f(x_u,u) + o(u)$.

Existence of $o(u)$- and $o(u^2)$-optimal paths requires finiteness of $v(u)$. Conversely, when the latter holds, one may always find $o(u)$ or $o(u^2)$ approximate solutions of (P_u). The fact that these paths do converge to x_0 as u tends to 0 can be proved in a number of particular situations (see for instance [6, 12]).

In addition, we shall either assume Hölder and Lipschitz stability of these suboptimal paths (these assumptions will be discussed in §6) or we shall suppose that problem (P_0) is convex in the sense that for all $y \in K$ and $\lambda \in N_K(y)$ the mapping $\mathcal{L}(\cdot,\lambda,0)$ is convex. The next result, under the convexity assumption, extends that given by Gol'stein [15].

PROPOSITION 3.2. *Suppose that (DCQ) holds, there exists an $o(u)$-optimal trajectory x_u, and either (P_0) is convex or $x_u = x_0 + o(\sqrt{u})$. Then v is right differentiable at* 0 *with $v'(0) = v(L)$. Moreover, when $\Lambda_0 \neq \phi$ we have*

$$v(u) = v(0) + u\, v(L) + o(u).$$

Proof. If $\Lambda_0 = \phi$ we have $v(L) = -\infty$ and the result follows immediately from Proposition 2.1. Otherwise, by Proposition 3.1 we may take $\lambda \in S(D) \subset \Lambda_0$ so that

$$\begin{aligned} v(u) - v(0) &= f(x_u,u) - f(x_0,0) + o(u) \\ &\geq \mathcal{L}(x_u,\lambda,u) - \mathcal{L}(x_0,\lambda,0) + o(u). \end{aligned}$$

Since $\mathcal{L}'_x(x_0,\lambda,0) = 0$, when (P_0) is convex we get $\mathcal{L}(x_0,\lambda,0) \leq \mathcal{L}(x_u,\lambda,0)$ and when $x_u = x_0 + o(\sqrt{u})$ a second-order expansion gives $\mathcal{L}(x_0,\lambda,0) = \mathcal{L}(x_u,\lambda,0) + o(u)$. In both cases we obtain

$$v(u) - v(0) \geq \mathcal{L}(x_u,\lambda,u) - \mathcal{L}(x_u,\lambda,0) + o(u)$$

and, since x_u tends to x_0, deduce that

$$\liminf_{u\downarrow 0} \frac{v(u) - v(0)}{u} \geq \mathcal{L}'_u(x_0,\lambda,0) = v(D) = v(L),$$

which combined with Proposition 2.1 yields the desired conclusions. □

As a further consequence we establish a relation between the solution set $S(L)$ and the right derivatives of suboptimal trajectories.

PROPOSITION 3.3. *With the assumptions of Proposition* 3.2 *we have:*

(a) *$S(L)$ is the set of all weak accumulation points of $(x_u - x_0)/u$, where x_u ranges over all possible $o(u)$-optimal trajectories.*

(b) *If $S(L) \neq \phi$, then there exists an $o(u)$-optimal trajectory such that $x_u = x_0 + O(u)$. The converse holds if X is reflexive.*

(c) *If x_u is chosen as in* (b), *then $\Lambda_u(x_u)$ is uniformly bounded for u small. Moreover, if $\lambda_u \in \Lambda_u(x_u)$, then every weak* accumulation point of λ_u belongs to $S(D)$.*

Proof. (a) Let x_u be an $o(u)$-optimal trajectory and $u_k \downarrow 0$ be such that $(x_{u_k} - x_0)/u_k \rightharpoonup d$. Then we have $[G(x_{u_k}, u_k) - G(x_0, 0)]/u_k \rightharpoonup G'(x_0, 0)(d, 1)$ and, since $T_K(G(x_0, 0))$ is weakly closed, deduce that $G'(x_0, 0)(d, 1) \in T_K(G(x_0, 0))$, proving that $d \in F(L)$. Similarly, $[f(x_{u_k}, u_k) - f(x_0, 0)]/u_k \to f'(x_0, 0)(d, 1)$ and then

$$v(u_k) = f(x_{u_k}, u_k) + o(u_k) = v(0) + u_k f'(x_0, 0)(d, 1) + o(u_k),$$

so Proposition 2.1 implies $f'(x_0, 0)(d, 1) \leq v(L)$, which shows $d \in S(L)$.

Conversely, let $d \in S(L)$ and apply Theorem B.5 to the trajectory $x_u = x_0 + ud$ to find $y_u = x_0 + ud + o(u) \in F(u)$. Proposition 3.2 then implies

$$f(y_u, u) = f(x_0, 0) + uf'(x_0, 0)(d, 1) + o(u) = v(0) + u\, v(L) + o(u) = v(u) + o(u),$$

proving that y_u is an $o(u)$-optimal trajectory with $(y_u - x_0)/u \rightharpoonup d$ (notice that the limit can be taken in the strong sense as well).

(b) The argument developed in (a) shows that $S(L) \neq \phi$ implies the existence of $o(u)$-optimal trajectories with $x_u = x_0 + O(u)$. Conversely, if such a trajectory exists, then by reflexivity we may find a sequence $u_k \downarrow 0$ such that $(x_{u_k} - x_0)/u_k$ converges weakly. From (a) the limit belongs to $S(L)$, which is then nonempty.

(c) Let $\lambda_u \in \Lambda_u(x_u)$ and select $r_u \in B_Y$ with $\|\lambda_u\|/2 \leq \langle r_u, -\lambda_u\rangle$. From relation (17) in Lemma B.4, for all u small enough there exist $d_u \in B_X$ and $k_u \in K$ such that

$$u\varepsilon r_u = G(x_u, u) + umG'_x(x_0, 0)d_u - k_u$$

where $\varepsilon > 0$ and $m > 0$ are given constants. Taking the product with $-\lambda_u$ we get

$$\begin{aligned}\frac{\varepsilon}{2}\|\lambda_u\| &\leq m\langle\lambda_u, G'_x(x_0, 0)d_u\rangle \\ &\leq m\|G'_x(x_0, 0) - G'_x(x_u, u)\|\,\|\lambda_u\| + m\langle\lambda_u, G'_x(x_u, u)d_u\rangle \\ &\leq \frac{\varepsilon}{4}\|\lambda_u\| - mf'_x(x_u, u)d_u \\ &\leq \frac{\varepsilon}{4}\|\lambda_u\| + m(\|f'_x(x_0, 0)\| + 1)\end{aligned}$$

for u small, and the desired uniform bound on $\Lambda_u(x_u)$ follows.

Now let $\lambda := \lim_k \lambda_{u_k}$ be a weak* accumulation point of λ_u where $u_k \downarrow 0$. Then

$$\begin{aligned}&\forall y \in K \qquad \langle\lambda, y - G(x_0, 0)\rangle = \lim_k \langle\lambda_{u_k}, y - G(x_{u_k}, u_k)\rangle \leq 0, \\ &\forall d \in X \qquad \mathcal{L}'_x(x_0, \lambda, 0)d = \lim_k \mathcal{L}'_x(x_{u_k}, \lambda_{u_k}, u_k)d = 0,\end{aligned}$$

proving that $\lambda \in \Lambda_0 = F(D)$. To show λ is also optimal for (D) we observe that

$$\begin{aligned}v(u) &\leq f(x_u, u) \\ &\leq f(x_u, u) - \langle\lambda_u, G(x_0, 0) - G(x_u, u)\rangle \\ &= v(0) + \mathcal{L}(x_u, \lambda_u, u) - \mathcal{L}(x_0, \lambda_u, 0) \\ &= v(0) + u\mathcal{L}'_u(x_0, \lambda_u, 0) + o(u + \|x_u - x_0\|).\end{aligned}$$

Dividing by u and passing to the limit in the subsequence u_k we get $\mathcal{L}'_u(x_0, \lambda, 0) \geq v'(0) = v(D)$, so $\lambda \in S(D)$. $\square$

Remark. In part (a) above we also showed that $S(L)$ is the set of all *strong* limits of differential quotients of the type $(x_{u_k} - x_0)/u_k$ with $u_k \downarrow 0$ and even the set of *continuous* strong limits

$$d := \lim_{u \downarrow 0} \frac{x_u - x_0}{u},$$

where now x_u ranges over all $o(u)$-optimal trajectories for which this limit exists.

4. Second-order expansion of the value function. In this section we supplement Proposition 2.1 by deriving second-order lower estimates for the value function. The next simple result shows that (4) is a sharp bound.

PROPOSITION 4.1. *Suppose (DCQ) holds and assume there exists an $o(u^2)$-optimal path x_u that admits an expansion of the form $x_u = x_0 + ud_0 + \frac{1}{2}u^2 w_0 + o(u^2)$. Then $d_0 \in S(Q)$, $w_0 \in S(L_{d_0})$, and we have*

$$v(u) = v(0) + u\, v(L) + \frac{1}{2}u^2\, v(Q) + o(u^2).$$

Proof. Propositions 3.2 and 3.3(a) imply $v'(0) = v(L)$ and $d_0 \in S(L)$. On the other hand, a second-order expansion of $G(x_u, u)$ shows that $w_0 \in F(L_{d_0})$ and also

$$\begin{aligned} v(u) &= f(x_u, u) + o(u^2) \\ &= f(x_0, 0) + uf'(x_0, 0)(d_0, 1) + \frac{1}{2}u^2[f'_x(x_0, 0)w_0 + \Phi_f(d_0)] + o(u^2) \\ &= v(0) + u\, v(L) + \frac{1}{2}u^2[f'_x(x_0, 0)w_0 + \Phi_f(d_0)] + o(u^2), \end{aligned}$$

which combined with Proposition 2.1 gives the desired conclusions. $\square$

Unfortunately this result is of more theoretical than practical interest since we must ensure a priori the existence of a second-order expansion of x_u. While it is possible to find conditions giving a first-order expansion (see §6), we dispose of no analogue for the second-order case. To overcome this difficulty we tackle the second-order lower estimates using duality theory as was done in the previous section for the first order. Let us then dualize problem (L_d).

PROPOSITION 4.2. *Suppose (DCQ) holds. Then $v(L_d) = v(D_d)$ where*

$$(D_d) \qquad \max\{\mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) - \sigma(\lambda, T_K^2(d)) : \lambda \in S(D)\},$$

and $\sigma(\lambda, T_K^2(d)) := \sup\{\langle \lambda, k \rangle : k \in T_K^2(d)\}$ is the support function of $T_K^2(d)$. Moreover the solution set $S(D_d)$ is nonempty.

Proof. The case $T_K^2(d) = \phi$ being trivial, we shall assume $T_K^2(d) \neq \phi$ (notice that in this case $d \in F(L)$). Let us consider problem (L_d) with the perturbation function

$$\varphi(w, y) := \begin{cases} f'_x(x_0, 0)w + \Phi_f(d) & \text{if } G'_x(x_0, 0)w + \Phi_G(d) + y \in T_K^2(d), \\ +\infty & \text{otherwise.} \end{cases}$$

To apply Theorem A.2 we must check that $\mathbb{R}_+ \cup_w \operatorname{dom} \varphi(w, \cdot) = Y$. To this end we fix $k \in T_K^2(d)$ and use property (5) to get

$$\begin{aligned} \bigcup_w \operatorname{dom} \varphi(w, \cdot) &= T_K^2(d) - G'_x(x_0, 0)X - \Phi_G(d) \\ &\supset k + T_K(G(x_0, 0)) - G'(x_0, 0)X \times (0, \infty) - \Phi_G(d) \\ &= Y, \end{aligned}$$

the last equality since (DCQ) implies $T_K(G(x_0, 0)) - G'(x_0, 0)X \times (0, \infty) = Y$.

We may then use the convex duality theorem to deduce

$$v(L_d) = -\min_{\lambda} \varphi^*(0, \lambda)$$

and a straightforward computation to obtain

$$\varphi^*(0, \lambda) = \begin{cases} \sigma(\lambda, T_K^2(d)) - \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) & \text{if } \mathcal{L}'_x(x_0, \lambda, 0) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

To complete the proof we note that if λ satisfies $\mathcal{L}'_x(x_0, \lambda, 0) = 0$, we may have $\sigma(\lambda, T_K^2(d)) < +\infty$ only if $\lambda \in S(D)$ (and $d \in S(L)$). Indeed, if $\sigma(\lambda, T_K^2(d)) < +\infty$, property (5) shows that

$$\langle \lambda, h - G'(x_0, 0)(d, 1) \rangle \leq 0 \quad \text{for all } h \in T_K(G(x_0, 0)).$$

This implies $\lambda \in N_K(G(x_0, 0))$; hence $\lambda \in \Lambda_0$, and also $\langle \lambda, G'(x_0, 0)(d, 1) \rangle \geq 0$ so that

$$f'(x_0, 0)(d, 1) \leq \mathcal{L}'(x_0, \lambda, 0)(d, 1) = \mathcal{L}'_u(x_0, \lambda, 0).$$

Since $\lambda \in F(D)$ and $d \in F(L)$, this inequality proves that $\lambda \in S(D)$ and $d \in S(L)$. □

With this result we have the following min-max characterization of $v(Q)$:

$$v(Q) = \min_{d \in S(L)} \max_{\lambda \in S(D)} \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) - \sigma(\lambda, T_K^2(d)).$$

The term $\sigma(\lambda, T_K^2(d))$ above will be referred to as the "σ-term" for short and is related, loosely speaking, to the curvature of the set K (see also [9, 17]). Neglecting this σ-term we obtain second-order lower estimates that, however, may not be sharp. To be precise, let us consider the function

$$\Gamma(d) := \max_{\lambda \in S(D)} \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1)$$

and the optimization problems

$$(\tilde{Q}) \qquad \min\{\Gamma(d) : d \in S(L)\},$$

$$(\tilde{Q}_\varepsilon) \qquad \min\{\Gamma(d) : d \in S_\varepsilon(L)\},$$

where $S_\varepsilon(L)$ is the set of approximate solutions of (L)

$$S_\varepsilon(L) := \{d \in F(L) : f'(x_0, 0)(d, 1) \leq v(L) + \varepsilon\}.$$

To obtain meaningful second-order lower bounds we must assume that $v(L) > -\infty$. By Proposition 3.3 this amounts to $\Lambda_0 \neq \phi$, in which case $S(D)$ is a weak* compact subset of Λ_0.

PROPOSITION 4.3. *Suppose (DCQ) holds, $\Lambda_0 \neq \phi$, and assume there exists an $o(u^2)$-optimal path x_u such that $x_u = x_0 + O(u)$. Then, for each $\varepsilon > 0$ we have*

$$v(u) \geq v(0) + u\, v(L) + \frac{1}{2} u^2\, v(\tilde{Q}_\varepsilon) + o(u^2). \tag{7}$$

Moreover, if any of the following conditions hold:

(a) *the path may be expanded as $x_u = x_0 + u d_0 + o(u)$,*

(b) *X is reflexive and Γ is weakly l.s.c. at each $d_0 \in S(L)$,* *then the previous lower bound may be strengthened to*

$$v(u) \geq v(0) + u\ v(L) + \frac{1}{2}u^2\ v(\tilde{Q}) + o(u^2). \tag{8}$$

Proof. For each $\lambda \in S(D)$ we have

$$\begin{aligned} v(u) &= f(x_u, u) + o(u^2) \\ &\geq f(x_u, u) + \langle \lambda, G(x_u, u) - G(x_0, 0) \rangle + o(u^2) \\ &= v(0) + \mathcal{L}(x_u, \lambda, u) - \mathcal{L}(x_0, \lambda, 0) + o(u^2) \\ &= v(0) + u\ v(L) + \frac{1}{2}\mathcal{L}''(x_0, \lambda, 0)(x_u - x_0, u)(x_u - x_0, u) + o(u^2) \end{aligned} \tag{9}$$

and the small term $o(u^2)$ may be chosen uniform in λ since $S(D)$ is bounded.

Applying Theorem B.4 to the mapping $\tilde{G}(x, u) := G(x_0, 0) + G'(x_0, 0)(x - x_0, u)$ we find a path y_u with $\tilde{G}(y_u, u) \in K$ and

$$\|y_u - x_u\| \leq c\ d(\tilde{G}(x_u, u), K) \leq c\ \|\tilde{G}(x_u, u) - G(x_u, u)\| = o(u). \tag{10}$$

Replacing in (9) we find

$$v(u) \geq v(0) + u\ v(L) + \frac{1}{2}\mathcal{L}''(x_0, \lambda, 0)(y_u - x_0, u)(y_u - x_0, u) + o(u^2)$$

with $o(u^2)$ still independent of λ. Thus, letting $d_u := (y_u - x_0)/u$ and taking the supremum in λ over the bounded set $S(D)$, we get

$$v(u) \geq v(0) + u\ v(L) + \frac{1}{2}u^2\Gamma(d_u) + o(u^2). \tag{11}$$

But $\tilde{G}(y_u, u) \in K$ implies $d_u \in F(L)$, and the equality $v(u) = f(x_u, u) + o(u^2) = f(y_u, u) + o(u)$ implies that for each $\varepsilon > 0$ the vector d_u belongs to $S_\varepsilon(L)$ for u small, so (7) follows immediately from (11).

Let us next choose $u_k \downarrow 0$, realizing the lower limit $\liminf_u 2[v(u) - v(0) - uv(L)]/u^2$. When (a) holds we have $d_{u_k} \to d_0$, while in case (b) we may assume (by eventually passing to a subsequence) that d_{u_k} converges weakly to some d_0. In both cases Proposition 3.3 implies $d_0 \in S(L)$ and using (11) (and the l.s.c. of Γ) we get

$$v(u_k) \geq v(0) + u_k\ v(L) + \frac{1}{2}u_k^2\Gamma(d_0) + o(u_k^2), \tag{12}$$

from which (8) follows. □

5. Asymptotic expansions of suboptimal solutions. In this section we prove the analogue of Proposition 3.3 for the second-order problem (Q). Roughly speaking, the solution set $S(Q)$ is the set of right derivatives of $o(u^2)$-optimal paths.

This result is obtained under a strong assumption, namely, that there exists no gap between the upper and lower estimates (4) and (8). This *no-gap* condition is not true in general—we will see in part III that semi-infinite programming does not satisfy this property—but is still valid for a large class of applications, one of which will be considered in §7.

The next result gives sufficient conditions for having no gap.

PROPOSITION 5.1. (a) *For $\lambda \in S(D)$ and $d \in S(L)$ one has $\sigma(\lambda, T_K^2(d)) \leq 0$.*

(b) *If $d \in S(L)$ and $0 \in T_K^2(d)$, then $\sigma(\lambda, T_K^2(d)) = 0$ for all $\lambda \in S(D)$.*

(c) *If $0 \in T_K^2(d)$ for all d in a (strongly) dense subset of $S(L)$, then $v(Q) = v(\tilde{Q})$.*

Proof. For all $\lambda \in S(D)$ and $d \in S(L)$ we have $\langle \lambda, G'(x_0,0)(d,1)\rangle = 0$. Moreover, since $\lambda \in N_K(G(x_0,0))$, for each $k \in T_K^2(d)$ we get

$$\langle \lambda, G(x_0,0) + uG'(x_0,0)(d,1) + \frac{1}{2}u^2 k + o(u^2) - G(x_0,0)\rangle \leq 0,$$

from which $\langle \lambda, k\rangle \leq 0$ and (a) follows.

Property (b) is obvious from (a). To prove (c) we notice that (a) implies $v(Q) \geq v(\tilde{Q})$, so we must only show the converse inequality. To this end it suffices to assume $S(L) \neq \phi$, in which case $S(D)$ is weak* compact and then Γ is strongly continuous. The required inequality follows using (b). □

Note that $0 \in T_K^2(y,h)$ when K is polyhedral in the sense that $T_K(y) = \mathbb{R}_+(K-y)$. This is the case for optimization problems with equality constraints and finitely many inequality constraints, where $K = \{0\} \times \mathbb{R}_-^p$. Thus, the condition "$0 \in T_K^2(d)$ for all d in a dense subset of $S(L)$" may be interpreted as a generalization of polyhedrality which, in a certain sense, rules out any curvature of K. We shall refer to this condition as *extended polyhedricity* (see also the discussion at the end of §7).

COROLLARY 5.2. *Let the hypothesis of Proposition* 4.3(b) *be satisfied, and suppose that the extended polyhedricity condition holds. Then* $v(Q) = v(\tilde{Q})$ *and we have*

$$v(u) = v(0) + u\ v(L) + \frac{1}{2}u^2\ v(Q) + o(u^2).$$

The previous results raise the question whether a second-order expansion compatible with curvature may hold. In this sense, we mention that the *sharp* lower estimate

$$v(u) \geq v(0) + u\ v(L) + \frac{1}{2}u^2\ v(Q) + o(u^2) \tag{13}$$

holds under assumption (a) of Proposition 4.3 and the additional hypothesis:

(H) For all sequences $u_n \downarrow 0$ and $y_n = y + u_n h + o(u_n) \in K$, there exists $k_n \in T_K^2(y,h)$ with $y_n = y + u_n h + \frac{1}{2}u_n^2 k_n + o(u_n^2)$.

The proof is similar to that of Proposition 4.3 and is left to the reader. In the case of assumption (b) in Proposition 4.3, (H) must be suitably modified in terms of weakly convergent sequences.

While (H) is not always satisfied, we observe that it holds whenever $0 \in T_K^2(y,h)$. To see that (H) is in fact more general than the latter one may consider the set $K = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}$ that satisfies (H) but $0 \notin T_K^2(y,h)$. Unfortunately, we do not know an easy way to check (H) in the general case. Nevertheless, in part III of this work we obtain sufficient conditions for obtaining the sharp lower estimate (13) in semi-infinite programming problems.

The next result links $S(Q)$ with the asymptotic behavior of suboptimal paths. Part (b) is a converse of Proposition 4.1.

PROPOSITION 5.3. *Suppose (DCQ) holds,* $\Lambda_0 \neq \phi$, *there exists an* $o(u^2)$*-optimal path* x_u *such that* $x_u = x_0 + O(u)$, *and suppose in addition that* $v(Q) = v(\tilde{Q})$ *and* Γ *is weakly l.s.c. at every* $d \in S(L)$. *Then:*

(a) $S(Q) \subset S(\tilde{Q})$ *and for every* $o(u^2)$*-optimal path* z_u, *the weak accumulation points of* $(z_u - x_0)/u$ *belong to* $S(\tilde{Q})$.

(b) *If* X *is reflexive,* $d_0 \in S(Q)$, *and* $w_0 \in S(L_{d_0})$, *then there exists an* $o(u^2)$*-optimal path of the form* $z_u = x_0 + ud_0 + \frac{1}{2}u^2 w_0 + o(u^2)$.

Proof. (a) Since $v(Q) = v(\tilde{Q})$ and the cost of (Q) dominates the cost of $(\tilde{Q})$, we deduce $S(Q) \subset S(\tilde{Q})$. If d_0 is the weak limit of $(z_{u_k} - x_0)/u$, reasoning as in the proof of (9) and using (4) we obtain $v(Q) \geq \Gamma(d_0)$. But Proposition 3.3 implies $d_0 \in S(L)$, so $\Gamma(d_0) \geq v(\tilde{Q}) = v(Q)$ and then $d_0 \in S(\tilde{Q})$.

(b) Using Theorem B.4 we may find a feasible path $z_u = x_0 + ud_0 + \frac{1}{2}u^2 w_0 + o(u^2)$. Expanding $f(z_u, u)$ we get

$$\begin{aligned} f(z_u, u) &= f(x_0, 0) + uf'(x_0, 0)(d_0, 1) + \frac{1}{2}u^2[f'_x(x_0, 0)w_0 + \Phi_f(d_0)] + o(u^2) \\ &= v(0) + uv(L) + \frac{1}{2}u^2 v(Q) + o(u^2) \\ &= v(0) + uv(L) + \frac{1}{2}u^2 v(\tilde{Q}) + o(u^2) \\ &\leq v(u) + o(u^2), \end{aligned}$$

where the last inequality follows from Proposition 4.3. This shows that z_u is $o(u^2)$-optimal and the proof is complete. □

Remark. In the next section we check that, under some reasonable hypothesis, every $o(u^2)$-optimal path satisfies $x_u = x_0 + O(u)$. When X is reflexive this implies the existence of weak accumulation points of $(x_u - x_0)/u$, so that $S(\tilde{Q})$ is nonempty. We also observe that when $0 \in T_K^2(d)$ for all $d \in S(L)$, the cost function in (Q) and $(\tilde{Q})$ coincide so that $S(Q) = S(\tilde{Q})$.

6. Hölder and Lipschitz properties of suboptimal paths. We discuss next the Hölder and Lipschitz stability properties of suboptimal paths assumed in the previous sections. The results we present are simple variants of known results (e.g., [8, 12, 14, 26]). The essential difference lies in the use of the weaker directional regularity condition (*DCQ*) and the extension to the infinite-dimensional setting.

Typically, the stability properties follow from different second-order sufficient optimality conditions. More precisely, for each set $\Omega \subset \Lambda_0$ we consider the second-order condition

$$SOC(\Omega) \qquad \text{There exist } \alpha, \eta > 0 \text{ s.t. } \max_{\lambda\in\Omega} \mathcal{L}''_x(x_0, \lambda, 0)dd \geq \alpha \quad \forall\, d \in C_\eta,$$

where

$$C_\eta = \{d \in X : \|d\| = 1,\ f'_x(x_0, 0)d \leq \eta,\ G'_x(x_0, 0)d \in T_K(G(x_0, 0)) + \eta B_Y\}.$$

When the space X is finite dimensional, or more generally when C_η is strongly compact for some $\eta > 0$, this condition is equivalent to the positive definiteness requirement:

$$SOC'(\Omega) \qquad \text{For each } d \in C_0 \text{ we have } \max_{\lambda\in\Omega} \mathcal{L}''_x(x_0, \lambda, 0)dd > 0,$$

where only the *critical cone* C_0 needs to be considered. Also, when (*CQ*) holds, one can replace C_η by a smaller set (see [8]).

PROPOSITION 6.1. *Assume (DCQ), $\Lambda_0 \neq \phi$, and suppose $SOC(\Omega)$ holds for some bounded $\Omega \subset \Lambda_0$. Then for each $O(u)$-optimal path x_u we have $x_u = x_0 + O(\sqrt{u})$.*

Proof. By contradiction suppose there exists $u_k \downarrow 0$ such that $\lim_k \tau_k^2/u_k = +\infty$, where $\tau_k := \|x_{u_k} - x_0\|$.

Then $\lim_k u_k/\tau_k = 0$ and letting $d_k := (x_{u_k} - x_0)/\tau_k$ we have $G(x_{u_k}, u_k) = G(x_0, 0) + \tau_k G'_x(x_0, 0)d_k + o(\tau_k)$ so that $G'_x(x_0, 0)d_k \in T_K(G(x_0, 0)) + \eta B_Y$ for k large. On the other hand, since x_u is an $O(u)$-optimal path and using Proposition 2.1, we may find a constant M such that for u small

$$f(x_u, u) \leq v(0) + Mu, \tag{14}$$

and since $f(x_{u_k}, u_k) = f(x_0, 0) + \tau_k f'_x(x_0, 0)d_k + o(\tau_k)$, we deduce $f'_x(x_0, 0)d_k \leq \eta$ for all k large enough. The previous argument shows that $d_k \in C_\eta$ for large k.

Now, using (14), for each $\lambda \in \Omega$ we have

$$\mathcal{L}(x_u, \lambda, u) - \mathcal{L}(x_0, \lambda, 0) \leq f(x_u, u) - f(x_0, 0) \leq Mu,$$

and since $\mathcal{L}'_x(x_0, \lambda, 0) = 0$, a second-order expansion of f and G leads to

$$\frac{1}{2}\mathcal{L}''(x_0, \lambda, 0)(x_u - x_0, u)(x_u - x_0, u) \leq [M - \mathcal{L}'_u(x_0, \lambda, 0)]u + (1 + \|\lambda\|)o(\|x_u - x_0\|^2 + u^2)$$

with the small term $o(\|x_u - x_0\|^2 + u^2)$ not depending on λ. Since Ω is bounded, we deduce that

$$\max_{\lambda \in \Omega} \mathcal{L}''(x_0, \lambda, 0)(d_k, u_k/\tau_k)(d_k, u_k/\tau_k) \leq M' \frac{u_k}{\tau_k^2} + M'' \frac{o(\tau_k^2 + u_k^2)}{\tau_k^2}$$

for some constants M' and M'', from which we get

$$\limsup_{k \to \infty} \max_{\lambda \in \Omega} \mathcal{L}''_x(x_0, \lambda, 0)d_k d_k \leq 0,$$

contradicting $SOC(\Omega)$. □

COROLLARY 6.2. *Assume $\Lambda_0 \neq \phi$ and any of the two following conditions:*

(a) *(CQ) and $SOC(\Lambda_0)$,*

(b) *(DCQ), C_η is strongly compact for some $\eta > 0$, and $SOC'(\Lambda_0)$.*

Then for each $O(u)$-optimal path x_u we have $x_u = x_0 + O(\sqrt{u})$.

Proof. In case (a) the set $\Omega := \Lambda_0$ is bounded and the result follows at once from the previous proposition.

In case (b) the set C_0 is compact and then, letting $\Lambda_0^k := \Lambda_0 \cap B(0, k)$, we get

$$\lim_{k \uparrow \infty} \min_{d \in C_0} [\max_{\lambda \in \Lambda_0^k} \mathcal{L}''_x(x_0, \lambda, 0)dd] = \min_{d \in C_0} [\max_{\lambda \in \Lambda_0} \mathcal{L}''_x(x_0, \lambda, 0)dd] > 0.$$

Hence, for k large $SOC'(\Omega)$ holds with $\Omega := \Lambda_0^k$, and we may conclude again using the previous proposition. □

The preceding results are not as strong as to ensure the property $x_u = x_0 + o(\sqrt{u})$ needed in Proposition 3.2. Let us then prove a Lipschitz stability result, valid for general Banach spaces, that can be used to check the hypothesis of both Proposition 4.3 and Proposition 3.2.

PROPOSITION 6.3. *Suppose (DCQ), $\Lambda_0 \neq \phi$, and assume $SOC(\Omega)$ holds for $\Omega := S(D)$. Suppose also that $v(Q) < +\infty$. Then, for each $O(u^2)$-optimal path x_u we have $x_u = x_0 + O(u)$.*

Proof. The proof is similar to that of Proposition 6.1. We proceed by contradiction assuming $\lim_k \tau_k/u_k = +\infty$ for a given sequence $u_k \downarrow 0$ and $\tau_k := \|x_{u_k} - x_0\|$, so that $d_k := (x_{u_k} - x_0)/\tau_k$ belongs to C_η for k large.

Since x_u is an $O(u^2)$-optimal path, using Proposition 2.1 we may find a constant M such that for u small

$$f(x_u, u) \leq v(0) + u\, v(L) + Mu^2,$$

and then for each $\lambda \in S(D)$ we have

$$\mathcal{L}(x_u, \lambda, u) - \mathcal{L}(x_0, \lambda, 0) \leq f(x_u, u) - f(x_0, 0) \leq u\, \mathcal{L}'_u(x_0, \lambda, 0) + Mu^2.$$

Expanding f and G we get

$$\mathcal{L}''(x_0, \lambda, 0)(d_k, u_k/\tau_k)(d_k, u_k/\tau_k) \leq 2M \left(\frac{u_k}{\tau_k}\right)^2 + \frac{o(\tau_k^2 + u_k^2)}{\tau_k^2}$$

with the small term $o(\tau_k^2 + u_k^2)$ not depending on λ (here we use the boundedness of $S(D)$). The conclusion follows as in Proposition 6.1. □

7. Directional differentiability of metric projections. In this section we use the preceding results to compute the directional derivatives of projections onto convex sets in Hilbert spaces. More precisely, the problem is to study the right differentiability of the unique optimal solution of

$$(P_u) \qquad \min\left\{\frac{1}{2}\|x - y_u\|^2 : x \in K\right\},$$

where K is a closed convex subset of a Hilbert space H and $u \to y_u$ is a smooth mapping from $\mathbb{R}_+$ to H. Let us consider the slightly more general format

$$(P'_u) \qquad \min\left\{\frac{1}{2}\|x - y_u\|^2 : G(x, u) \in K\right\},$$

assuming that $G(\cdot, 0)$ is a linear mapping $G(x, 0) = Ax$ and that (DCQ) and $\Lambda_0 \neq \phi$ hold. Notice that these properties are satisfied when we have (CQ), which is obviously the case if A is surjective and particularly if $G(x, 0) = x$ as in (P_u).

Since $G(x, 0)$ is linear, we have $\mathcal{L}''_x(x_0, \lambda, 0) = I$, so $SOC(\Omega)$ is automatically satisfied for $\Omega = S(D)$ and problem $(\tilde{Q})$ is strongly convex. In particular, $S(\tilde{Q})$ is reduced to a singleton.

PROPOSITION 7.1. *Suppose (DCQ), $\Lambda_0 \neq \phi$, and the extended polyhedricity condition. Then the unique solution x_u of (P'_u) may be expanded as*

$$x_u = x_0 + ud_0 + o(u)$$

where d_0 is the unique solution of $(\tilde{Q})$.

Proof. Propositions 6.3 and 5.3(a) imply that $d_u := (x_u - x_0)/u$ converges weakly to d_0, the unique solution of $(\tilde{Q})$. Now, using the second-order bound (4), the equality $v(Q) = v(\tilde{Q}) = \Gamma(d_0)$, and inequality (9), we deduce that

$$\limsup_{u \downarrow 0} \Gamma(d_u) \leq \Gamma(d_0).$$

Since Γ is strongly convex, we conclude that d_u converges strongly to d_0, completing the proof. □

In the special case $G(x, u) = x$ and $y_u = y_0 + uh_0$; that is, when we study directional differentiability of the projection onto K at y_0 in the direction h_0, the set $S(L)$ is just the critical cone

$$S(L) = C_0 = \{d \in T_K(x_0) : d \perp (y_0 - x_0)\},$$

so the problem $(\tilde{Q})$ reduces to

$$\min\{\|d - h_0\|^2 : d \in C_0\}.$$

Hence we get as an immediate consequence the following result.

COROLLARY 7.2. *Assuming the extended polyhedricity condition, the projection x_u of $y_0 + uh_0$ onto K can be expanded as*

$$x_u = x_0 + ud_0 + o(u),$$

where d_0 is the projection of h_0 onto C_0.

Among the papers studying differentiability properties of metric projections we mention [11, 16, 19, 22, 26, 27]. A common hypothesis in these studies is that K has to be *polyhedric* in this sense that for each $x \in K$ and every $\lambda \in N_K(x)$ one has

$$T_K(x) \cap \lambda^\perp = \overline{\mathbb{R}_+(K - x) \cap \lambda^\perp}.$$

Since $S(L) = T_K(x_0) \cap (y_0 - x_0)^\perp$ and $0 \in T_K^2(x_0, d)$ whenever $d \in \mathbb{R}_+(K - x_0)$, the extended polyhedricity condition is in fact a generalization of polyhedricity. Notice that this hypothesis always holds when $y_0 \in K$ since then $C_0 = T_K(G(x_0, 0))$, which was the case studied in [27]. Another extension of polyhedricity is considered in [3].

8. Conclusion and further problems. We have shown that a satisfactory sensitivity analysis for perturbed problems of the form

$$(P_u) \qquad \min_x \{f(x, u) : G(x, u) \in K\}$$

may be obtained under directional constraint qualification conditions that are weaker than the standard Robinson's condition.

Since the results are scattered throughout the paper, we provide a summarized (though necessarily incomplete) version of the main results obtained in the paper. The precise meaning of the stated assumptions and notation is made clear in the preceding sections of the paper, to which the reader is referred.

THEOREM 8.1. *Let the functions f, G defining (P_u) be of class C^2, and suppose X is a reflexive Banach space. Let x_0 be an optimal solution for (P_0) at which the following assumptions are satisfied:*

(i) *directional constraint qualification (DCQ),*
(ii) *existence of multipliers $\Lambda_0 \neq \emptyset$,*
(iii) *second-order sufficient condition $SOC(\Omega)$ for $\Omega = S(D)$,*
(iv) *existence of an $o(u^2)$-optimal trajectory,*
(v) *extended polyhedricity,*
(vi) $d \to \mathcal{L}''_x(x_0, \lambda, 0)dd$ *is weakly lower semicontinuous for all $\lambda \in S(D)$.*

Then:

(a) *The optimal value function may be expanded as*

$$v(u) = v(0) + uv(L) + \frac{1}{2}u^2 v(\tilde{Q}) + o(u^2),$$

where (L) and $(\tilde{Q})$ are the linear and quadratic approximating optimizing problems.
(b) *The optimal solutions of (L) are the same as the weak accumulation points of the differential quotients $(x_u - x_0)/u$ where x_u ranges over the set of all possible $o(u)$-optimal trajectories.*
(c) *Every $o(u^2)$-optimal path z_u satisfies $z_u = z_0 + O(u)$, and the weak accumulation points of $(z_u - z_0)/u$ are optimal solutions for $(\tilde{Q})$.*

We remark that a key ingredient for attaining these results is the generalization of Robinson's implicit function theorem presented in Appendix B, which is based on the weak directional constraint qualification condition (*DCQ*).

The main results of this paper are limited to problems for which there is existence of multipliers and satisfying the strong second-order sufficient condition stated as (iii) above, which ensure the existence of suboptimal paths of the form $x_u = x_0 + O(u)$.

In the setting of finite-dimensional mathematical programming we know [5] that this type of expansion may fail. For instance, when $\Lambda_0 \neq \phi$ but only the weak second-order condition holds, suboptimal paths may only satisfy $x_u = x_0 + O(\sqrt{u})$ and it may happen that $v'(0) < v(L)$. On the other hand, when $\Lambda_0 = \phi$ we may even have $v(u) = v(0) + O(\sqrt{u})$.

It seems that (DCQ) is too weak to extend these results to the general framework discussed in the present paper. Theorem B.4 may not be used since it requires the a priori bound $x_u = x_0 + O(u)$, and its refinement Theorem B.1 may only handle those paths such that $\|x_u - x_0\| \le \gamma\sqrt{u}$ for a sufficiently small γ.

These remarks lead us to consider a strenghtened form of directional constraint qualification, well suited to the analysis of problems of the form

$$\min\{f(x,u) : G_1(x,u) \in K_1, G_2(x,u) \in K_2\}$$

where K_1 and K_2 are closed convex subsets of some Banach spaces with $\mathrm{int}(K_2) \neq \phi$. This study will be the subject of part II of this work.

Appendix A. The convex duality theorem in Banach spaces. This short appendix presents a short proof of the convex duality theorem of Robinson [24]. This result is a generalization of [25, Thm. 18(c)] (see also [1, Thm. 1.1]). We include it since the version we present is more directly applicable to the dualization of problems (L) and (L_d) in the previous sections and also since the method of proof is very simple. The basic argument is the following lemma due to Robinson [24] (also used in Appendix B) for which we provide a simplified proof too.

Given a subset $C \subset X \times Y$ we denote by C_X and C_Y the projections of C onto X and Y, respectively.

LEMMA A.1. *Let X, Y be two normed spaces with X complete. Let $C \subset X \times Y$ be a closed convex set with C_X bounded. Then*

$$\mathrm{int}\,(C_Y) = \mathrm{int}\,(\overline{C_Y}).$$

Proof. It clearly suffices to show $\mathrm{int}\,(\overline{C_Y}) \subset C_Y$; that is, given $\overline{y} \in \mathrm{int}\,(\overline{C_Y})$ we must find $\overline{x} \in X$ such that $(\overline{x}, \overline{y}) \in C$. To this end let us take $\varepsilon > 0$ with $B(\overline{y}, \varepsilon) \subset \overline{C_Y}$ and choose an arbitrary point $(x_0, y_0) \in C$ from which we generate a sequence $(x_k, y_k) \in C$ using the following "algorithm."

while $(y_k \neq \overline{y})$ *do*

- Let $\alpha_k = \varepsilon/\|y_k - \overline{y}\|$ so that $w := \overline{y} + \alpha_k(\overline{y} - y_k) \in B(\overline{y}, \varepsilon) \subset \overline{C_Y}$.
- Take $(u, v) \in C$ with $\|v - w\| \le \frac{1}{2}\|y_k - \overline{y}\|$ and define

$$(x_{k+1}, y_{k+1}) := \frac{\alpha_k}{1+\alpha_k}(x_k, y_k) + \frac{1}{1+\alpha_k}(u, v) \in C.$$

endwhile.

If the algorithm stops, then we have $y_k = \overline{y}$ and we may take $\overline{x} = x_k$. Otherwise, the generated sequence satisfies

$$\text{(i)} \quad \|x_{k+1} - x_k\| = \frac{\|x_k - u\|}{1+\alpha_k} \le \frac{\mathrm{diam}(C_X)}{\varepsilon}\|y_k - \overline{y}\|,$$

$$\text{(ii)} \quad \|y_{k+1} - \overline{y}\| = \frac{\|v - w\|}{1+\alpha_k} \le \frac{1}{2}\|y_k - \overline{y}\|.$$

From (ii) it follows that $\|y_k - \overline{y}\| \le \|y_0 - \overline{y}\|/2^k$. This implies that $y_k \to \overline{y}$ and also, in combination with (i), that (x_k) is a Cauchy sequence. The completeness of X gives the existence of a limit $\overline{x}$ for (x_k), and the closedness of C implies $(\overline{x}, \overline{y}) \in C$ as required. $\square$

We may now proceed by proving the convex duality theorem.

THEOREM A.2. *Let $\theta(y) := \inf\{\varphi(x, y) : x \in X\}$, where $\varphi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is a closed proper convex function with X, Y Banach spaces and $\mathbb{R}_+ \cup_x \operatorname{dom} \varphi(x, \cdot) = Y$. Then θ is continuous in a neighborhood of 0 and $\theta(0) < +\infty$.*

In particular $\theta(0) = \theta^{**}(0)$, which can be written as

$$\inf_{x \in X} \varphi(x, 0) = - \min_{y^* \in Y^*} \varphi^*(0, y^*), \tag{15}$$

and the solution set of the minimum on the right is $\partial\theta(0)$, which is nonempty and weak-compact when $\theta(0)$ is finite, and the whole space Y^* when $\theta(0) = -\infty$.*

Proof. Since θ is convex, the continuity near 0 is equivalent to θ being bounded above in a certain neighborhood of 0. To show this, let $\alpha \in \mathbb{R}$ and $x_0 \in X$ be such that $\varphi(x_0, 0) < \alpha$ and consider the closed convex set

$$C = \{(x, y) : \varphi(x, y) \leq \alpha;\ \|x\| \leq \|x_0\| + 1\}$$

that is nonempty and has C_X bounded.

Since $\theta(y) \leq \alpha$ for all $y \in C_Y$, it suffices to show that C_Y is a neighborhood of 0. From Lemma A.1 this amounts to $0 \in \operatorname{int}(\overline{C_Y})$, which, by Baire's lemma, is a consequence of the fact that C_Y is absorbing as we show next. For any $y \in Y$ there exist $t > 0$ and $x \in X$ with $\varphi(x, ty) < +\infty$, so for $\varepsilon > 0$ small enough we have

$$\begin{aligned}
&\|(1 - \varepsilon)x_0 + \varepsilon x\| \leq \|x_0\| + 1,\\
&\varphi((1 - \varepsilon)(x_0, 0) + \varepsilon(x, ty)) \leq (1 - \varepsilon)\varphi(x_0, 0) + \varepsilon\varphi(x, ty) \leq \alpha,
\end{aligned}$$

showing that $\varepsilon t y \in C_Y$ for all $\varepsilon > 0$ small.

We observe that $\theta^*(y^*) = \varphi^*(0, y^*)$ so that (15) is just a rewriting of $\theta(0) = \theta^{**}(0)$. From this we also get that $\partial\theta(0)$ is the solution set of $\min \varphi^*(0, y^*)$, and the last claim is a well-known fact in convex analysis (see [25]). $\square$

Appendix B. Regularity theorems under directional constraint qualification conditions. Throughout this section we suppose that $G : X \times \mathbb{R}_+ \to Y$ is a C^2 mapping and the spaces X, Y are Banach. Also $K \subset Y$ is a closed convex set and $x_0 \in X$ is such that $G(x_0, 0) \in K$ and satisfies the constraint qualification

$$0 \in \operatorname{int}\left[G(x_0, 0) + G'(x_0, 0)X \times (0, \infty) - K\right]. \tag{\textit{DCQ}}$$

We begin by stating the equivalence.

PROPOSITION B.1. *Condition (DCQ) is equivalent to*

$$0 \in \operatorname{int}\, [G(x_0, 0) + G'(x_0, 0)X \times [0, \infty) - K]. \tag{\textit{DCQ}$'$}$$

Proof. Clearly (*DCQ*) implies (*DCQ*)$'$. Conversely, suppose (*DCQ*)$'$ holds and choose $\varepsilon > 0$ with

$$\varepsilon B_Y \subset [G(x_0, 0) + G'(x_0, 0)X \times [0, \infty) - K].$$

Let $\delta > 0$ be such that $\delta[B_Y - G'_u(x_0, 0)] \subset \varepsilon B_Y$. Then

$$\delta B_Y \subset [G(x_0, 0) + G'(x_0, 0)X \times [\delta, \infty) - K],$$

from which (*DCQ*) follows. $\square$

THEOREM B.2. *Let x_u be a trajectory such that $\|x_u - x_0\| \le \gamma\sqrt{u}$, and suppose $d(G(x_u, u), K) \le mu$ for some constants γ, m and all $u \ge 0$ close to 0. If γ is small enough, we can find constants $c \ge 0$, $u_0 > 0$ and a trajectory y_u with*

$$\begin{aligned} &G(y_u, u) \in K, \\ &\|y_u - x_u\| \le \frac{c}{u}(u + \|x_u - x_0\|)d(G(x_u, u), K), \end{aligned}$$

for all $u \in (0, u_0]$.

Our proof will be based on the following couple of lemmas.

LEMMA B.3. *Under assumption (DCQ), there exist $\varepsilon > 0$, $\alpha \ge 1$, and $\overline{u} > 0$ such that for all $u \in [0, \overline{u}]$*

$$2u\varepsilon B_Y \subset G(x_0, 0) + uG'_u(x_0, 0) + u\alpha G'_x(x_0, 0)B_X - K.$$

Proof. Letting $A_k := G(x_0, 0) + kG'(x_0, 0)B_X \times [0, 1] - K \cap kB_Y$, condition (*DCQ*) gives

$$0 \in \operatorname{int} \bigcup \{A_k : k \in \mathbb{N}\};$$

thus the completeness of Y implies $0 \in \operatorname{int}(\overline{A_k})$ for some $k \in \mathbb{N}$. But the set A_k can be expressed as the projection over the fourth component of the closed convex set

$$C := \left\{(x, y, t, G(x_0, 0) + G'(x_0, 0)(x, t) - y) : \|x\| \le k, \|y\| \le k, y \in K, t \in [0, k]\right\},$$

and since the projection of C onto its first three components is bounded, Lemma A.1 gives $\operatorname{int}(A_k) = \operatorname{int}(\overline{A_k})$. Therefore we may find $\varepsilon > 0$ such that

$$2\varepsilon k B_Y \subset G(x_0, 0) + kG'_u(x_0, 0) - k[0, 1]G'_u(x_0, 0) + kG'_x(x_0, 0)B_X - K,$$

which multiplied by u/k and rearranged becomes

$$2u\varepsilon B_Y \subset G(x_0, 0) + uG'_u(x_0, 0) + uG'_x(x_0, 0)B_X - S, \tag{16}$$

where

$$S := \left(1 - \frac{u}{k}\right) G(x_0, 0) + [0, 1]uG'_u(x_0, 0) + \frac{u}{k}K.$$

Now, (*DCQ*) implies $G'_u(x_0, 0) = [y - G(x_0, 0) - G'_x(x_0, 0)d]/\delta$ for some $y \in K$, $d \in X$, and $\delta > 0$, so

$$S = \bigcup_{\lambda \in [0,1]} \left[\left(1 - \frac{\lambda u}{\delta} - \frac{u}{k}\right) G(x_0, 0) + \frac{\lambda u}{\delta} y + \frac{u}{k} K\right] - \frac{\lambda u}{\delta} G'_x(x_0, 0)d.$$

Since K is convex, we deduce that $S \subset K - [0, 1]\frac{u}{\delta}G'_x(x_0, 0)d$ for all $u \le \overline{u} := \delta k/(\delta + k)$, which combined with (16) yields the desired conclusion for $\alpha := 1 + \|d\|/\delta$. □

In the next lemma we denote

$$M = \sup\{\|G''(x, u)\| : \|x - x_0\| \le 1, 0 \le u \le \overline{u}\}.$$

LEMMA B.4. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\varphi(u) \le \frac{1}{4}\sqrt{u\varepsilon/M}$ for all u sufficiently small. Then there exists $u_0 > 0$ such that, for each trajectory x_u with*

$$\|x_u - x_0\| \le \varphi(u) \quad \forall u \in [0, u_0],$$

one has for all $u \in [0, u_0]$

$$u\varepsilon B_y \subset G(x_u, u) + (\alpha u + \|x_u - x_0\|)G'_x(x_0, 0)B_X - K. \tag{17}$$

Moreover, we can associate to x_u *another trajectory* y_u *such that for all* $u \in (0, u_0]$

$$\text{(i)} \quad d(G(y_u, u), K) \le \frac{1}{2} d(G(x_u, u), K),$$

$$\text{(ii)} \quad \|y_u - x_u\| \le \frac{2}{u\varepsilon}(\alpha u + \|x_u - x_0\|)d(G(x_u, u), K).$$

Proof. The hypothesis on $\varphi(u)$ ensures the existence of $u_0 \in (0, \overline{u}]$ such that

$$8M[\alpha u + \varphi(u)]^2 \le u\varepsilon \le M \qquad \forall\, u \in [0, u_0]. \tag{18}$$

To show (17) we observe from (18) that $\|x_u - x_0\| \le \varphi(u) \le 1$, and then letting $b := G(x_u, u) - G(x_0, 0) - G'(x_0, 0)(x_u - x_0, u)$ we have

$$\|b\| \le M(u + \|x_u - x_0\|)^2 \le M[\alpha u + \varphi(u)]^2 \le u\varepsilon. \tag{19}$$

Thus, Lemma B.3 gives

$$u\varepsilon B_Y - b \subset 2u\varepsilon B_Y \subset G(x_0, 0) + uG'_u(x_0, 0) + u\alpha G'_x(x_0, 0)B_X - K,$$

and then

$$u\varepsilon B_Y \subset G(x_u, u) + G'_x(x_0, 0)[-(x_u - x_0) + u\alpha B_X] - K,$$

from which (17) follows at once.

Let us construct next the trajectory y_u for $u \in (0, u_0]$.

If $G(x_u, u) \in K$ we just take $y_u = x_u$ so that (i) and (ii) hold trivially.

Otherwise we choose r such that $G(x_u, u) + r \in K$ and

$$\|r\| \le 2d(G(x_u, u), K), \tag{20}$$

and we use (17) to select d with $\|d\| \le \alpha u + \|x_u - x_0\|$ such that

$$u\varepsilon \frac{r}{\|r\|} \in G(x_u, u) + G'_x(x_0, 0)d - K. \tag{21}$$

With these choices we define $y_u = x_u + \beta d$, where $\beta := \|r\|/(u\varepsilon + \|r\|) < 1$.

Property (ii) follows immediately from (20) and the inequality

$$\|y_u - x_u\| = \beta\|d\| \le \frac{\|r\|}{u\varepsilon}(\alpha u + \|x_u - x_0\|).$$

To check property (i) we observe that $\|d\| \le \alpha u + \varphi(u)$ and then, using (18),

$$\|y_u - x_0\| \le \alpha u + 2\varphi(u) \le 1.$$

Then we can apply the mean value theorem to find $\xi \in]x_u, y_u[$ with

$$\begin{aligned} \|G(y_u, u) - G(x_u, u) - \beta G'_x(x_0, 0)d\| &\le \|G'_x(\xi, u) - G'_x(x_0, 0)\|\ \|d\|\ \beta \\ &\le M(u + \|\xi - x_0\|)\|d\| \frac{\|r\|}{u\varepsilon} \\ &\le \frac{2M}{u\varepsilon}[\alpha u + \varphi(u)]^2 \|r\| \\ &\le \frac{1}{2} d(G(x_u, u), K), \end{aligned} \tag{22}$$

where we have used the bound $u + \|\xi - x_0\| \le 2[\alpha u + \varphi(u)]$, (18), and (20).

Now, from (21) we get

$$G(x_u,u)+\beta G'_x(x_0,0)d \in (1-\beta)G(x_u,u)+\beta u\varepsilon\frac{r}{\|r\|}+\beta K,$$

since $1-\beta=\beta u\varepsilon/\|r\|$, we deduce

$$[G(x_u,u)+\beta G'_x(x_0,0)d \in (1-\beta)(G(x_u,u)+r)+\beta K \subset K,$$

which combined with (22) yields (i). $\square$

Proof of Theorem B.2. Let $\varphi(u) := e^{4m/\varepsilon}(\alpha u+\|x_u-x_0\|)$ and suppose that $\gamma < \frac{1}{4}e^{-4m/\varepsilon}\sqrt{\frac{\varepsilon}{M}}$ so that Lemma B.4 can be used to find u_0.

Starting with $y_u^0 := x_u$ we shall construct recursively a sequence y_u^k such that for all $u \in (0,u_0]$ one has

$$\text{(i)}\quad d(G(y_u^k,u),K) \le \frac{1}{2}d(G(y_u^{k-1},u),K),$$

$$\text{(ii)}\quad \|y_u^k-y_u^{k-1}\| \le \frac{2}{u\varepsilon}(\alpha u+\|y_u^{k-1}-x_0\|)d(G(y_u^{k-1},u),K).$$

To prove the existence of such a sequence it suffices to check inductively that

$$\text{(iii)}\quad \|y_u^k-x_0\| \le \varphi(u) \quad \forall\, u \in (0,u_0],$$

so that Lemma B.4 can be used to find the next term y_u^{k+1}. Since (iii) obviously holds for $k=0$, we only need to prove the inductive step. Suppose $y_0, y_1, \dots, y_k$ are such that (i) and (ii) hold; then for every $u \in (0,u_0]$ we have

$$\begin{aligned}\|y_u^k-y_u^{k-1}\| &\le \frac{2}{u\varepsilon}(\alpha u+\|y_u^{k-1}-x_0\|)\frac{d(G(x_u,u),K)}{2^{k-1}}\\ &\le \frac{2m}{\varepsilon 2^{k-1}}(\alpha u+\|y_u^{k-1}-x_0\|),\end{aligned}\tag{23}$$

so that letting $a_k := \alpha u+\|y_u^k-x_0\|$ we get

$$a_k \le a_{k-1}+\|y_u^k-y_u^{k-1}\| \le \left(1+\frac{2m}{\varepsilon 2^{k-1}}\right)a_{k-1}.$$

It follows that

$$\ln a_k \le \ln a_{k-1}+\ln\left(1+\frac{2m}{\varepsilon 2^{k-1}}\right) \le \ln a_{k-1}+\frac{2m}{\varepsilon 2^{k-1}}$$

and then recursively

$$\ln a_k \le \ln a_0+\frac{2m}{\varepsilon}\left(\frac{1}{2^0}+\frac{1}{2^1}+\cdots+\frac{1}{2^{k-1}}\right) \le \ln a_0+\frac{4m}{\varepsilon},$$

from which we obtain the desired conclusion (iii) as

$$\|y_u^k-x_0\| \le a_k \le a_0e^{4m/\varepsilon} = \varphi(u).$$

The existence of the sequence (y_k) being established, we may use the previous bound $a_k \le \varphi(u)$ and (23) to obtain

$$\|y_u^k-y_u^{k-1}\| \le \frac{2\varphi(u)d(G(x_u,u),K)}{u\varepsilon 2^{k-1}},\tag{24}$$

which shows that $(y_u^k)_{k\in\mathbb{N}}$ is a Cauchy sequence for each $u \in (0,u_0]$.

Let $y_u := \lim_{k\uparrow\infty} y_u^k$. From (i) we deduce that $G(y_u, u) \in K$, while (24) implies

$$\|y_u - x_u\| \le \frac{4\varphi(u)}{u\varepsilon} d(G(x_u, u), K),$$

proving the theorem with $c := \frac{4}{\varepsilon} e^{4m/\varepsilon}$. □

A careful analysis of the previous proof shows that the result is still valid if G is supposed of class $\mathcal{C}^1$ (or merely strictly differentiable at $(x_0, 0)$) provided we restrict to the case of trajectories $x_u = x_0 + O(u)$. More precisely we have the following theorem.

THEOREM B.5. *Let $G : X \times \mathbb{R} \to Y$ be strictly differentiable at $(x_0, 0)$ and $K \subset Y$ be a closed convex set. Suppose that $G(x_0, 0) \in K$ and (DCQ) holds. Then for each trajectory $x_u = x_0 + O(u)$ there exist constants $c \ge 0$, $u_0 > 0$ and a second trajectory y_u such that*

$$G(y_u, u) \in K,$$
$$\|y_u - x_u\| \le c\, d(G(x_u, u), K),$$

for all $u \in [0, u_0]$.

Proof. It is clear that the result will follow from Theorem B.2, which is applicable since $x_u = x_0 + O(u)$ implies $x_u = x_0 + o(\sqrt{u})$ and $d(G(x_u, u), K) = O(u)$.

However, we must check that Theorem B.2 remains valid under the weaker $\mathcal{C}^1$ assumption on G and the stronger $x_u = x_0 + O(u)$ condition. To this end all we need is to modify Lemma B.4. More specifically, it suffices to adjust the arguments leading to the bounds (19) and (22), which is easily accomplished by fixing $\ell \in \mathbb{R}$ and $u_0 \in (0, \overline{u}]$ such that $\varphi(u) \le \ell u$ for all $u \in [0, u_0]$ and then reducing u_0 so that

$$\|G(y, v) - G(x, u) - G'(x_0, 0)(y - x, v - u)\| \le \frac{\varepsilon}{4(\alpha + \ell)} (\|y - x\| + |v - u|)$$

for each $u, v \in [0, u_0]$ and every $x, y \in B(x_0, (\alpha + 2\ell)u_0)$. □

As a corollary of the preceding result we obtain the following *directional* version of Robinson–Ursescu's regularity theorem for convex multifunctions.

THEOREM B.6. *Let $M : X \longrightarrow 2^Y$ be a multifunction with closed convex graph. Let $y_0 \in M(x_0)$ and let y_u be a $\mathcal{C}^1$ trajectory with $y(0) = y_0$ and*

$$(RU) \qquad 0 \in \operatorname{int}[M(X) - y(0) - (0, \infty) y'(0)].$$

Then for each trajectory $x_u = x_0 + O(u)$ one has

$$d(x_u, M^{-1}(y_u)) \le c\, d(y_u, M(x_u))$$

for a given constant c and all $u \ge 0$ sufficiently small.

Proof. The result follows as a direct application of Theorem B.5 to the function $G(x, u) = (x, y_u)$ and the closed convex set $K = \operatorname{graph}(M)$. □

APPLICATION. As a particular case of the previous result let us consider $y_0 \in M(x_0)$ and suppose that $d \in Y$ is such that

$$[0 \in \operatorname{int}[M(X) - y_0 - (0, \infty) d].$$

Then, for each trajectory $x_u = x_0 + O(u)$ there exists $\tilde{x}_u$ such that

$$y_0 + ud \in M(\tilde{x}_u),$$
$$\|\tilde{x}_u - x_u\| \le c\, d(y_u, M(x_u)).$$

In particular, letting $x_u \equiv x_0$ we obtain the existence of a trajectory $\tilde{x}_u = x_0 + O(u)$ with $y_0 + ud \in M(\tilde{x}_u)$.

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