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PERTURBED OPTIMIZATION IN BANACH SPACES II: A THEORY BASED ON A STRONG DIRECTIONAL CONSTRAINT QUALIFICATION*

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Abstract. We study the sensitivity of the optimal value and optimal solutions of perturbed optimization problems in two cases. The first one is when multipliers exist but only the weak (and not the strong) second-order sufficient optimality condition is satisfied. The second case is when no Lagrange multipliers exist. To deal with these pathological cases, we are led to introduce a directional constraint qualification stronger than in part I of this paper, which reduces to the latter in the important case of equality-inequality constrained problems. We give sharp upper estimates of the cost based on paths varying as the square root of the perturbation parameter and, under a *no-gap* condition, obtain the first term of the expansion for the cost. When multipliers exist we study the expansion of approximate solutions as well. We show in the appendix that the strong directional constraint qualification is satisfied for a large class of problems, including regular problems in the sense of Robinson.

Key words. sensitivity analysis, marginal function, square root expansion, approximate solutions, directional constraint qualification

AMS subject classifications. 46N10, 47H19, 49K27, 49K40, 58C15, 90C31

1. Introduction.

This paper is the second in a trilogy (see [4, 5]) devoted to the analysis of parametric optimization problems of the form

$$(P_u) \qquad \min_x\{f(x,u) : G(x,u) \in K\}$$

with X and Y Banach spaces, K a closed convex subset of Y, and $f(x,u)$, $G(x,u)$ mappings of class $\mathcal{C}^2$ from $X \times \mathbb{R}$ into $\mathbb{R}$ and Y, respectively. We denote the feasible set, value function, and set of solutions of (P_u) as

$$F(u) := \{x \in X \ : \ G(x,u) \in K\},$$
$$v(u) := \inf\{f(x,u) \ : \ x \in F(u)\},$$
$$S(u) := \{x \in F(u) \ : \ f(x,u) = v(u)\},$$

respectively. Similarly $v(P)$, $F(P)$, $S(P)$ will respectively denote the optimal value, feasible set, and solution set of an optimization problem (P).

Our aim is to study the expansion of $v(u)$ and possibly $S(u)$ in the vicinity of a local solution x_0 of (P_0). Such sensitivity analysis usually relies (among other assumptions) upon stability properties of the feasible set $F(u)$ that follow from so-called *constraint qualification conditions*. In part I of this work (see [4]) our study was based on the following generalization of Gollan's constraint qualification (see [1, 10]):

$$(DCQ) \qquad 0 \in \operatorname{int}\left[G(x_0,0) + G'(x_0,0)X \times (0,\infty) - K\right],$$

which is a *directional* version of Robinson's condition [14]

$$(CQ) \qquad 0 \in \operatorname{int}\left[G(x_0,0) + G'_x(x_0,0)X - K\right].$$

Under *(DCQ)* we obtained the following upper estimate of the optimal value:

$$(1.1) \qquad v'_+(0) \le v(L),$$

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where $v'_+(0)$ and $v'_-(0)$ denote the upper and lower Dini derivatives of the value function:

$$v'_+(0) := \limsup_{u \downarrow 0} \frac{v(u) - v(0)}{u},$$
$$v'_-(0) := \liminf_{u \downarrow 0} \frac{v(u) - v(0)}{u},$$

and (L) is the problem with linearized data:

$$(L) \qquad \min_d \{ f'(x_0, 0)(d, 1) : \; G'(x_0, 0)(d, 1) \in T_K(G(x_0, 0)) \}.$$

Using duality theory we proved that $v(D) = v(L) < \infty$, where (D) is the problem

$$(D) \qquad \max\{ \mathcal{L}'_u(x_0, \lambda, 0) : \lambda \in \Lambda_0 \},$$

with $\mathcal{L}$ the Lagrangian and Λ_0 the set of multipliers associated with x_0, that is to say, denoting by $N_K(y)$ the cone of outward normals at a point $y \in K$,

$$\mathcal{L}(x, \lambda, u) := f(x, u) + \langle \lambda, G(x, u) \rangle,$$
$$\Lambda_0 := \{ \lambda \in Y^* : \lambda \in N_K(G(x_0, 0)); \; \mathcal{L}'_x(x_0, \lambda, 0) = 0 \}.$$

Define a *path* as a mapping $u \to x_u$ from $\mathbb{R}_+$ to X, with $x_u \to x_0$ when $u \downarrow 0$. The path is said to be feasible if $G(x_u, u) \in K$ for u small enough. Under a strong second-order condition on the Lagrangian it can be shown [4] that any $o(u^2)$-optimal path x_u, i.e., a feasible path x_u such that $f(x_u, u) \leq v(u) + o(u^2)$, satisfies $x_u = x_0 + O(u)$. In this case $v'(0)$ exists, being equal to $v(L)$, and some estimates for the second-order variation of $v(u)$ can be obtained. In fact, under suitable conditions we proved that

$$(1.2) \qquad v(u) = v(0) + u \, v(L) + \frac{1}{2} u^2 \, v(\tilde{Q}) + o(u^2),$$

where $(\tilde{Q})$ is a subproblem involving the expansion of orders 1 and 2 of the data at $(x_0, 0)$. A remarkable property in this case is that every weak limit of $(x_u - x_0)/u$, with x_u an $o(u^2)$-optimal path, belongs to $S(\tilde{Q})$.

The available perturbation theory for nonlinear programming shows that this is not the end of the story. Under the directional qualification hypothesis of Gollan [10] and the weak second-order sufficient condition, it appears (see [9] by Gauvin and Janin) that $v'(0)$ exists but may be strictly less than $v(L)$. In that case, a path of $O(u)$-optimal solutions satisfies only $x_u = x_0 + O(\sqrt{u})$. One can still formulate (see Bonnans, Ioffe, and Shapiro [6]) a subproblem (M) such that $v'(0) = v(M)$ and $S(M)$ coincides with the limit points of $(x_u - x_0)/\sqrt{u}$ where x_u ranges over the set of all possible $o(u)$-optimal paths. For this it is necessary to assume the existence of at least one multiplier. A similar theory for the case when no multiplier exists was developed in [3] by Bonnans; here the variation of the cost as well that of the solutions is of order $O(\sqrt{u})$.

The aim of this paper is to extend these two theories to the Banach space setting. Our main results are Theorem 3.9 and Theorem 4.6.

The first one, valid under the weak second-order sufficient condition, provides a first-order expansion of the form

$$v(u) = v(0) + u v(\tilde{D}) + o(u),$$

where $(\tilde{D})$ is a problem involving the expansion of orders 1 and 2 of the data. Moreover, it shows that every weak limit point of $(x_u - x_0)/\sqrt{u}$, with x_u an $o(u)$-optimal path, solves $(\tilde{D})$.

The second one is concerned with problems where no Lagrange multipliers exist. In this case we obtain a square root expansion of the form

$$v(u) = v(0) + \sqrt{u}v(\hat{D}) + o(\sqrt{u}),$$

where $(\hat{D})$ is another linear-quadratic approximating problem.

To prove these results we need a constraint qualification that is still directional but, apparently, stronger than (DCQ). Specifically, in addition to (DCQ) we need a restorability property that, roughly speaking, asserts that to certain almost feasible *square root* paths (i.e., paths satisfying $x_u = x_0 + O(\sqrt{u})$), one can associate a sufficiently close feasible path. In the case of nonlinear programming, that stronger hypothesis $(SDCQ)$ reduces to the condition of Gollan (see [1, 10]) used in [9, 3, 6], so we recover the main results of these three references. Let us mention that square root paths have already been used for sensitivity analysis in a Banach space setting (see [2, 11, 12]). However, our qualification condition is weaker than those in these references.

As in part I of this work, in our extension to the Banach space setting, an additional difficulty related to the possible curvature of the convex K appears. To be more precise, let us recall the definition of first- and second-order tangent sets:

$$T_K(y) := \{h \in Y : \text{ there exists } o(t) \text{ such that } y + th + o(t) \in K\},$$

$$T_K^2(y, h) := \left\{k \in Y : \text{ there exists } o(t^2) \text{ such that } y + th + \frac{1}{2}t^2k + o(t^2) \in K\right\}.$$

The fact that in general 0 does not belong to the set $T_K^2(y, h)$ may cause a *gap* between the upper and lower estimates for the cost. Some cases when the curvature makes no contribution to the second-order variation of the cost were analyzed in part I, yielding the expansion (1.2) under a condition of generalized polyhedricity. The results in this paper are obtained under similar assumptions.

The paper is organized as follows. In §2 we describe the strong directional constraint qualification $(SDCQ)$. Then in §3 we develop a perturbation theory assuming the set of multipliers Λ_0 to be nonempty, whereas §4 deals with the case when Λ_0 is empty. In both cases we obtain sharp upper estimates as well as some lower estimates of the cost and, under a *no-gap* condition, obtain the first term in the expansion of the cost. Finally in the appendix we discuss sufficient conditions for the strong directional constraint qualification $(SDCQ)$.

2. The strong directional qualification condition. Our upper estimates are based on paths that vary as the square root of the perturbation parameter. Specifically, we consider paths satisfying, for given d, w in X, the two conditions

$$x_u = x_0 + \sqrt{u}d + uw + o(u), \tag{2.3}$$

$$\operatorname{dist}(G(x_u, u), K) = o(u). \tag{2.4}$$

Note that we can express (2.4) using the concept of a second-order tangent set. Namely, if x_u satisfies (2.3), then the expansion

$$G(x_u, u) = G(x_0, 0) + \sqrt{u}G_x'(x_0, 0)d + u\left[G'(x_0, 0)(w, 1) + \frac{1}{2}G_x''(x_0, 0)dd\right] + o(u)$$

shows that (2.4) is equivalent to

$$\Psi_G(w, d) \in T_2^K(d), \tag{2.5}$$

where we have set

$$T_2^K(d) := \frac{1}{2} T_K^2(G(x_0, 0), G'_x(x_0, 0)d),$$

$$\Psi_G(w, d) := G'(x_0, 0)(w, 1) + \frac{1}{2} G''_x(x_0, 0)dd,$$

$$\Psi_f(w, d) := f'(x_0, 0)(w, 1) + \frac{1}{2} f''_x(x_0, 0)dd.$$

Remark. The set $T_2^K(d)$ should not be confused with the set

$$T_K^2(d) := T_K^2(G(x_0, 0), G'(x_0, 0)(d, 1))$$

defined in part I of this paper and which will not be used here.

DEFINITION 1. *We say that x_0 is* restorable *(with respect to G and K) if, given a path x_u satisfying* (2.3) *and* (2.4), *for $\gamma < 1$ close to 1 one can find $w_\gamma \in X$ with $w_\gamma \to w$ and feasible paths of the form*

$$x_u^\gamma = x_0 + \gamma \sqrt{u} d + u w_\gamma + o(u). \tag{2.6}$$

We say that the strong directional constraint qualification (*SDCQ*) *holds at x_0 if x_0 is restorable and the weak directional constraint qualification* (*DCQ*) *holds.*

We discuss some sufficient conditions for (*SDCQ*) in the appendix at the end of this paper. We show in particular that for equality-inequality constrained problems (i.e., when $K = \{0\} \times K_2$ with K_2 a closed convex cone with nonempty interior), property (*SDCQ*) is equivalent to (*DCQ*). In fact, it may be that the restorability property is always a consequence of (*DCQ*), but we do not have a proof nor a counterexample for this.

Before proceeding with the sensitivity analysis we summarize in the next lemma four general properties that will be of constant use throughout the paper. Here $\sigma(\lambda, T_2^K(d)) := \sup\{\langle \lambda, k\rangle : k \in T_2^K(d)\}$ denotes the support function of $T_2^K(d)$.

LEMMA 2.1. *For every $d \in X$ we have the following.*

($P1$) $T_2^K(d) + T_K(G(x_0, 0)) - \mathbb{R}_+ G'_x(x_0, 0)d \subset T_2^K(d)$.

($P2$) *If* (*DCQ*) *holds, then* $0 \in \operatorname{int}[T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1\}]$.

($P3$) $T_2^K(\gamma d) = \gamma^2 T_2^K(d)$ *for all* $\gamma > 0$.

($P4$) *If* $T_2^K(d) \neq \phi$, *then the following are equivalent:*

(a) $\sigma(\lambda, T_2^K(d)) \leq 0$.

(b) $\sigma(\lambda, T_2^K(d))$ *is finite.*

(c) $\lambda \in N_K(G(x_0, 0))$ *and* $\langle \lambda, G'_x(x_0, 0)d\rangle = 0$.

Proof. Properties ($P1$) and ($P2$) are straightforward consequences of [8, Prop. 3.1] and [4, Lem. B.3], respectively, while ($P3$) is an easy exercise.

Let us prove ($P4$). Since $T_2^K(d) \neq \phi$, the implication (a) $\Rightarrow$ (b) is straightforward. Also, the nonemptyness of $T_2^K(d)$ implies $G'_x(x_0, 0)d \in T_K(G(x_0, 0))$ and then (b) $\Rightarrow$ (c) follows from property ($P1$). To prove (c) $\Rightarrow$ (a) let us pick $y \in T_2^K(d)$ and choose $y_t \to y$ with $z_t := G(x_0, 0) + tG'_x(x_0, 0)d + t^2 y_t \in K$. Using (c) we deduce

$$0 \geq \langle \lambda, z_t - G(x_0, 0)\rangle = \langle \lambda, tG'_x(x_0, 0)d + t^2 y_t\rangle = t^2 \langle \lambda, y_t\rangle,$$

so that $\langle \lambda, y\rangle = \lim \langle \lambda, y_t\rangle \leq 0$, proving (a). □

3. Perturbation analysis assuming the existence of multipliers. In this section we study the case when $\Lambda_0 \neq \phi$. First we give an upper estimate of $v'_+(0)$, which we can express as a supremum of a certain function over Λ_0. We then rely on second-order conditions to obtain lower estimates for $v'_-(0)$ and to investigate the coincidence of both estimates.

3.1. Sharp first-order upper estimates of the cost. Let C_0 denote the cone of critical directions at x_0, i.e.,

$$C_0 := \{d \in X : \ f'_x(x_0, 0)d \le 0; \ G'_x(x_0, 0)d \in T_K(G(x_0, 0))\}.$$

When $\Lambda_0 \neq \phi$ one has in fact $f'_x(x_0, 0)d = 0$ for all $d \in C_0$. To a path satisfying (2.3) and (2.4) is associated the constraint (2.5), whereas $\Psi_f(w, d)$ is the first term of the expansion of the cost. This leads to the problem

$$(L^d) \qquad \inf_{w \in X} \{\Psi_f(w, d) : \Psi_G(w, d) \in T_2^K(d)\},$$

which admits the dual

$$(D^d) \qquad \sup_{\lambda \in \Lambda_0} \left\{ \mathcal{L}'_u(x_0, \lambda, 0) + \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd - \sigma(\lambda, T_2^K(d)) \right\}.$$

We also consider the problem

$$(\tilde{L}) \qquad \inf_d \{v(L^d) : d \in C_0\},$$

which plays a role in the following upper estimate of the cost.

THEOREM 3.1. *Assume Λ_0 to be nonempty and (SDCQ). Then*

$$v'_+(0) \le v(\tilde{L}) = \inf_{d \in C_0} v(D^d) \le v(L) < \infty.$$

In particular, if $v(\tilde{L})$ is finite, then

$$v(u) \le v(0) + uv(\tilde{L}) + o(u).$$

The theorem is an immediate consequence of the next two lemmas. The first one gives the primal upper estimate of $v'_+(0)$.

LEMMA 3.2. *Assuming (SDCQ) we have*

$$v'_+(0) \le v(\tilde{L}) \le v(L) < \infty.$$

Proof. Let $d \in C_0$ and take a feasible $w \in F(L^d)$. Using the restorability property we may find $w_\gamma \to w$ and feasible paths of the form

$$x_u^\gamma = x_0 + \gamma\sqrt{u}d + uw_\gamma + o(u).$$

Expanding $f(x_u^\gamma, u)$ and using the fact that d is critical, it follows that

$$v(u) \le f(x_u^\gamma, u) \le f(x_0, 0) + u\Psi_f(w_\gamma, \gamma d) + o(u)$$

so that $v'_+(0) \le \Psi_f(w_\gamma, \gamma d)$. Passing to the limit when $\gamma \uparrow 1$ we deduce that $v'_+(0) \le \Psi_f(w, d)$, and taking the infimum over $w \in F(L^d)$ and $d \in C_0$ we get

$$v'_+(0) \le v(\tilde{L}).$$

We conclude by noting that for $d = 0$ problem (L^d) reduces to problem (L) and that $v(L) < \infty$ by [4, Prop. 2.2]. □

Let us prove next the dual expression for $v(\tilde{L})$.

LEMMA 3.3. *Assume Λ_0 to be nonempty and (SDCQ). For each $d \in C_0$ we have the following.*

(i) $v(D^d) \leq v(L^d)$.
(ii) *If* (L^d) *is feasible, then for all* $\gamma \in (0, 1)$, $v(D^{\gamma d}) = v(L^{\gamma d}) \in \mathbb{R}$ *and* $S(D^{\gamma d})$ *is nonempty and bounded.*
(iii) *If* (L^d) *is infeasible, then* $v(D^{\gamma d}) = \infty$ *for all* $\gamma > 1$.
(iv) $\limsup_{\gamma \uparrow 1} v(D^{\gamma d}) \leq v(D^d)$.

As a consequence we obtain

$$v(\tilde{L}) = \inf_{d \in C_0} v(D^d). \tag{3.7}$$

Proof. Let us begin by showing that (3.7) is a consequence of (i)–(iv). The inequality $v(\tilde{L}) \geq \inf_{d \in C_0} v(D^d)$ is obvious from (i). To show the converse inequality it suffices to check that $v(D^d) \geq v(\tilde{L})$ for those $d \in C_0$ such that $v(D^d) < \infty$. By (iii) this implies $(L^{\gamma d})$ is feasible for each $\gamma \in (0, 1)$, and then (ii) gives $v(D^{\gamma d}) = v(L^{\gamma d}) \geq v(\tilde{L})$ for all $\gamma \in (0, 1)$. We conclude by letting $\gamma \uparrow 1$ and using (iv).

We now prove properties (i)–(iv).

(i) It suffices to show that if w and λ are feasible for (L^d) and (D^d), respectively, then the dual cost is not greater than the primal one. From the primal constraint it follows that

$$\sigma(\lambda, T_2^K(d)) \geq \langle \lambda, \Psi_G(w, d) \rangle,$$

which implies

$$\begin{aligned}\Psi_f(w, d) &\geq \Psi_f(w, d) + \langle \lambda, \Psi_G(w, d) \rangle - \sigma(\lambda, T_2^K(d)) \\ &= \mathcal{L}'_u(x_0, \lambda, 0) + \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd - \sigma(\lambda, T_2^K(d)),\end{aligned}$$

as was to be proved.

(ii) We first claim that $v(L^d)$ and $v(D^d)$ are finite and equal with $S(D^d)$ nonempty and bounded, whenever

$$Y = \mathbb{R}_+ \left[T_2^K(d) - G'(x_0, 0)X \times \{1\} - \frac{1}{2}G''_x(x_0, 0)dd \right]. \tag{3.8}$$

To motivate this relation, let us consider the family of problems obtained by perturbing additively the constraint of (L^d), that is, $\min_{w \in X} \varphi(w, y)$ with

$$\varphi(w, y) := \begin{cases} \Psi_f(w, d) & \text{if } \Psi_G(w, d) + y \in T_2^K(d), \\ \infty & \text{otherwise.} \end{cases}$$

Property (3.8) amounts to $Y = \mathbb{R}_+ \cup_w \operatorname{dom} \varphi(w, \cdot)$, so we may apply the convex duality theorem of part I [4, Thm. A.2] to deduce

$$v(L^d) = \inf_{w \in X} \varphi(w, 0) = - \min_{\lambda \in Y^*} \varphi^*(0, \lambda) \tag{3.9}$$

as well as the boundedness and nonemptiness of the set of dual solutions. Now we compute

$$\begin{aligned}\varphi^*(0, \lambda) &= \sup_{w \in X, y \in Y} \{\langle \lambda, y \rangle - \Psi_f(w, d) : \Psi_G(w, d) + y \in T_2^K(d)\} \\ &= \sup_{w \in X} \left\{ \sigma(\lambda, T_2^K(d)) - \mathcal{L}'(x_0, \lambda, 0)(w, 1) - \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd \right\}.\end{aligned}$$

Maximizing over w we deduce that $\varphi^*(0, \lambda) = \infty$ if $\mathcal{L}'_x(x_0, \lambda, 0) \neq 0$, and then using $(P4)$ we get

$$\varphi^*(0, \lambda) = \begin{cases} \sigma(\lambda, T_2^K(d)) - \mathcal{L}'_u(x_0, \lambda, 0) - \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd & \text{if } \lambda \in \Lambda_0, \\ \infty & \text{otherwise.} \end{cases}$$

This and (3.9) imply the equality $v(L^d) = v(D^d)$. Moreover, since the dual is attained, property $(P4)$ shows that this common value is finite. This proves our claim.

In view of the previous discussion, to prove (ii) it suffices to check that for each $\gamma \in (0, 1)$ property (3.8) holds with d replaced by $d_\gamma := \gamma d$. To see this let us choose a feasible $w \in F(L^d)$, that is,

$$G'(x_0, 0)(w, 1) + \frac{1}{2} G''_x(x_0, 0)dd \in T_2^K(d).$$

Multiplying by γ^2 and using $(P3)$ we deduce that

$$G'(x_0, 0)(\gamma^2 w, \gamma^2) + \frac{1}{2} G''_x(x_0, 0)d_\gamma d_\gamma \in T_2^K(d_\gamma).$$

From this and $(P1)$ we get

$$T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1 - \gamma^2\} \subset T_2^K(d_\gamma) - G'(x_0, 0)X \times \{1\} - \frac{1}{2} G''_x(x_0, 0)d_\gamma d_\gamma,$$

which multiplied by $\mathbb{R}_+$ and using $(P2)$ yields (3.8) for d_γ as required.

(iii) Let $\gamma > 1$ and set $d_\gamma := \gamma d$ as before. If $T_2^K(d)$ is empty, by $(P3)$ so is $T_2^K(d_\gamma)$ and then $\sigma(\lambda, T_2^K(d_\gamma)) = -\infty$, hence $v(D^{\gamma d}) = \infty$.

Let us then assume $T_2^K(d)$ to be nonempty. Since (L^d) is infeasible, the convex set $T_2^K(d) - G'(x_0, 0)X \times \{1\}$ does not contain $\frac{1}{2} G''_x(x_0, 0)dd$. But $(P1)$ and $(P2)$ show that this convex set has a nonempty interior, so that the Hahn–Banach theorem gives a nonzero $\mu \in Y^*$ that separates the set and the point, that is,

$$\left\langle \mu, G'(x_0, 0)(w, 1) + \frac{1}{2} G''_x(x_0, 0)dd \right\rangle \geq \sigma(\mu, T_2^K(d)) \quad \text{for all } w \in X. \tag{3.10}$$

This inequality and property $(P4)$ imply $\mu \in N_K(G(x_0, 0))$. Also, taking the infimum over $w \in X$ we deduce $\mu \circ G'_x(x_0, 0) = 0$ (that is to say, μ is a *singular multiplier*, as defined in the next section) so that for each $\lambda \in \Lambda_0$ and $t > 0$ we have $\lambda + t\mu \in \Lambda_0$. Since $S(D)$ is bounded (see [4, Prop. 3.1]), it follows that

$$\langle \mu, G'_u(x_0, 0) \rangle < 0.$$

With these observations property (3.10) reduces to

$$\Xi(\mu, d) := \left\langle \mu, G'_u(x_0, 0) + \frac{1}{2} G''_x(x_0, 0)dd \right\rangle - \sigma(\mu, T_2^K(d)) \geq 0,$$

which multiplied by γ^2 and using $(P3)$ gives

$$\Xi(\mu, d_\gamma) \geq (1 - \gamma^2)\langle \mu, G'_u(x_0, 0) \rangle > 0. \tag{3.11}$$

Let us fix $\lambda \in \Lambda_0$. Since $\Xi(\cdot, d_\gamma)$ is positively homogeneous and concave, and since $\lambda + t\mu \in \Lambda_0$, it follows that

$$\begin{aligned} v(D^{\gamma d}) &\geq f'_u(x_0, 0) + \frac{1}{2} f''_x(x_0, 0)d_\gamma d_\gamma + \Xi(\lambda + t\mu, d_\gamma) \\ &\geq f'_u(x_0, 0) + \frac{1}{2} f''_x(x_0, 0)d_\gamma d_\gamma + \Xi(\lambda, d_\gamma) + t\,\Xi(\mu, d_\gamma). \end{aligned}$$

To conclude we observe that $(P4)$ implies the finiteness of $\Xi(\lambda, d_\gamma)$, so that letting $t \uparrow \infty$ and using (3.11) we get $v(D^{\gamma d}) = \infty$.

(iv) Using $(P3)$ we obtain

$$\begin{aligned} v(D^{\gamma d}) &= \sup_{\lambda\in\Lambda_0} \left\{ \mathcal{L}'_u(x_0, \lambda, 0) + \frac{\gamma^2}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd - \gamma^2\sigma(\lambda, T_2^K(d)) \right\} \\ &\le \sup_{\lambda\in\Lambda_0} \{(1-\gamma^2)\mathcal{L}'_u(x_0, \lambda, 0) + \gamma^2 v(D^d)\} \\ &= (1-\gamma^2)v(L) + \gamma^2 v(D^d). \end{aligned}$$

As $v(L) < \infty$, passing to the limit with $\gamma \uparrow 1$ we get the desired inequality. □

When (CQ) holds, for every $d \in C_0$ problem (L^d) is feasible and then $v(D^d) = v(L^d)$. Otherwise the previous lemma shows that $v(D^{\gamma d}) = v(L^{\gamma d})$ except for at most an exceptional value γ_0. The optimal values are finite for $\gamma < \gamma_0$ and equal to $+\infty$ for $\gamma > \gamma_0$. The following lemma shows that $\gamma_0 = 0$ iff $T_2^K(d)$ is empty. It will be useful in §4 as well.

LEMMA 3.4. *Assume (DCQ) and suppose $T_2^K(d)$ is not empty. Then letting $d_\gamma := \gamma d$ we have $F(L^{d_\gamma}) \ne \phi$ for all $\gamma > 0$ sufficiently small.*

Proof. Taking $k \in T_2^K(d)$ and using $(P2)$ we get

$$\frac{\gamma^2}{2}G''_x(x_0, 0)dd - \gamma^2 k \in T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1\}$$

for all $\gamma > 0$ sufficiently small. Then, using $(P1)$ and $(P3)$ we deduce

$$\frac{1}{2}G''_x(x_0, 0)d_\gamma d_\gamma \in T_2^K(d_\gamma) - G'(x_0, 0)X \times \{1\},$$

so we may find $w \in X$ with $\Psi_G(w, d_\gamma) \in T_2^K(d_\gamma)$. □

We end this section by giving a condition under which the upper estimate of Theorem 3.1 coincides with $v(L)$. Using $(P4)$, it is easy to see that this condition is satisfied in particular if (P_0) is *convex* in the sense that for all $y \in K$ and $\lambda \in N_K(y)$, the mapping $\mathcal{L}(\cdot, \lambda, 0)$ is convex. In that case the right-derivative $v'(0)$ is actually equal to $v(L)$ (see [4, Prop. 3.2]).

PROPOSITION 3.5. *Assume $(SDCQ)$. Then $v(\tilde{L}) = v(L)$ whenever*

$$\inf_{d\in C_0} \sup_{\lambda\in S(D)} \left\{ \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd - \sigma(\lambda, T_2^K(d)) \right\} \ge 0.$$

Proof. By Lemma 3.3 and using the equality $v(L) = v(D)$ we get

$$\begin{aligned} v(\tilde{L}) &= \inf_{d\in C_0} v(D^d) \\ &\ge \inf_{d\in C_0} \sup_{\lambda\in S(D)} \left\{ \mathcal{L}'_u(x_0, \lambda, 0) + \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd - \sigma(\lambda, T_2^K(d)) \right\} \\ &\ge v(L) + \inf_{d\in C_0} \sup_{\lambda\in S(D)} \left\{ \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd - \sigma(\lambda, T_2^K(d)) \right\} \\ &\ge v(L), \end{aligned}$$

and we conclude with Lemma 3.2. □

3.2. Lower estimates and expansion of solutions. We derive next some lower estimates for $v'_{-}(0)$. As $v'_{-}(0) \le v'_{+}(0) \le v(\tilde{L})$ whenever $(SDCQ)$ holds, this is only of interest if $v(\tilde{L}) > -\infty$. We give conditions that imply $v'_{-}(0) > -\infty$, based on a result of part I (see [4, Prop. 6.1]) that we recall for the convenience of the reader.

For each set $\Omega \subset \Lambda_0$ we consider the second-order condition

$$SOC(\Omega) \qquad \text{There exist } \alpha, \epsilon > 0 \text{ s.t. } \max_{\lambda\in\Omega} \mathcal{L}''_x(x_0, \lambda, 0)dd \ge \alpha\|d\|^2 \quad \forall\, d \in C_\epsilon,$$

where

$$C_\epsilon := \{d \in X : f'_x(x_0, 0)d \le \epsilon\|d\|,\ G'_x(x_0, 0)d \in T_K(G(x_0, 0)) + \epsilon\|d\|B_Y\}.$$

Note that for $\epsilon = 0$ the *extended critical cone* C_ϵ reduces to the critical cone C_0.

PROPOSITION 3.6. *Assume (DCQ) and suppose $SOC(\Omega)$ holds for some bounded $\Omega \subset \Lambda_0$. Then, for each $O(u)$-optimal path x_u, we have $x_u = x_0 + O(\sqrt{u})$.*

Now consider the function

$$\Pi(d) := \sup_{\lambda\in\Lambda_0} \left\{ \mathcal{L}'_u(x_0, \lambda, 0) + \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)dd \right\}$$

and the problems

$$(\tilde{D}) \qquad \min\{\Pi(d) : d \in C_0\},$$

$$(\tilde{D}_\epsilon) \qquad \min\{\Pi(d) : f'_x(x_0, 0)d \le \epsilon,\ G'_x(x_0, 0)d \in T_K(G(x_0, 0))\}.$$

Note that $v(\tilde{D}_\epsilon)$ is a nonincreasing function of ϵ; in particular, $\lim_{\epsilon\downarrow 0} v(\tilde{D}_\epsilon) \le v(\tilde{D})$. Moreover, from $(P4)$ we get $\Pi(d) \le v(D^d)$, so under the conditions of Theorem 3.1 we deduce that

$$\lim_{\epsilon\downarrow 0} v(\tilde{D}_\epsilon) \le v(\tilde{D}) \le v(\tilde{L}). \tag{3.12}$$

PROPOSITION 3.7. *Assume (DCQ), the existence of an $o(u)$-optimal path, and $SOC(\Omega)$ for some bounded $\Omega \subset \Lambda_0$. Then $v'_{-}(0) > -\infty$ and*

(i) *if (CQ) holds, then for each $\varepsilon > 0$ we have*

$$v'_{-}(0) \ge v(\tilde{D}_\varepsilon); \tag{3.13}$$

(ii) *if any of the following conditions hold:*

(a) *the path may be expanded as $x_u = x_0 + \sqrt{u}d_0 + o(\sqrt{u})$,*

(b) *X is reflexive and $d \to \mathcal{L}''_x(x_0, \lambda, 0)dd$ is weakly lower semicontinuous at each $d \in C_0$,*

then the previous lower bound may be strengthened to

$$v'_{-}(0) \ge v(\tilde{D}). \tag{3.14}$$

Proof. Let x_u be an $o(u)$-optimal path. By Proposition 3.6 $d_u := (x_u - x_0)/\sqrt{u}$ stays bounded as $u \downarrow 0$, and then for each $\lambda \in \Lambda_0$ we have

$$\begin{aligned} v(u) &= f(x_u, u) + o(u) \\ &\ge v(0) + \mathcal{L}(x_u, \lambda, u) - \mathcal{L}(x_0, \lambda, 0) + o(u) \\ &\ge v(0) + u\left[\mathcal{L}'_u(x_0, \lambda, 0) + \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)d_u d_u\right] + o_\lambda(u), \end{aligned} \tag{3.15}$$

with $\|o_\lambda(u)\|/u \to 0$ uniformly when λ varies over bounded sets. From this and the boundedness of d_u, it follows that $v'_-(0) > -\infty$.

To prove (i) we apply Robinson's theorem [14] to the mapping $\tilde{G}(x) := G(x_0, 0) + G'_x(x_0, 0)(x - x_0)$ in order to find $\tilde{x}_u = x_u + o(\sqrt{u})$ such that $\tilde{G}(\tilde{x}_u) \in K$. Then, by suitably modifying the small term $o_\lambda(u)$, in (3.15) we can replace d_u by $\tilde{d}_u := (\tilde{x}_u - x_0)/\sqrt{u}$. Moreover, under (CQ) we know that Λ_0 is bounded so that taking the supremum over λ we get

$$v(u) \geq v(0) + u\Pi(\tilde{d}_u) + o(u),$$

from which (3.13) follows.

To show (ii), let us choose $u_k \downarrow 0$ realizing the lower limit $v'_-(0)$. When (a) holds we have $d_{u_k} \to d_0$, while in case (b) we may assume that $d_{u_k} \rightharpoonup d_0$. In both cases, $d_0 \in C_0$, and using (3.15) we get

$$v'_-(0) \geq \mathcal{L}'_u(x_0, \lambda, 0) + \frac{1}{2}\mathcal{L}''_x(x_0, \lambda, 0)d_0d_0,$$

where in case (b) we use the weak lower semicontinuity of $\mathcal{L}''_x(x_0, \lambda, 0)dd$. Taking the supremum over $\lambda \in \Lambda_0$ we conclude (3.14). □

We now analyze under which conditions the gap between the estimate of Theorem 3.1 and (3.14) is null. We start with sufficient conditions for the equality between the optimal values of the subproblems giving the upper and lower estimates. We define *extended polyhedricity of the second kind* (for problem (P_0) at point x_0) as

$$0 \in T_2^K(d) \text{ for all } d \text{ in a dense subset of } C_0.$$

We note that in the definition of *extended polyhedricity* given in part I, the set $S(L)$ was considered instead of C_0. If the constraints are unperturbed, then $S(L) = C_0$ and both definitions coincide.

PROPOSITION 3.8. *Assume Λ_0 nonempty and $(SDCQ)$. If one of the two following conditions hold:*

(a) *$0 \in T_2^K(d)$ for all d in C_0,*

(b) *(CQ) and extended polyhedricity of the second kind hold,*

then $v(\tilde{L}) = v(\tilde{D})$ and $S(\tilde{L}) \subset S(\tilde{D})$.

Proof. From $(P4)$ it follows that when $0 \in T_2^K(d)$ we have $\sigma(\lambda, T_2^K(d)) = 0$ for all $\lambda \in \Lambda_0$, and then $\Pi(d) = v(D^d)$. Consider now a minimizing sequence $\{d^k\}$ for $(\tilde{D})$ satisfying $\sigma(\lambda, T_2^K(d^k)) = 0$. The existence of such a sequence is obvious in case (a); while in case (b) it is a consequence of the fact that, due to (CQ), $\Pi(d)$ is continuous. Along this sequence we have, by Theorem 3.1, $\Pi(d^k) = v(D^{d^k}) \geq v(\tilde{L})$. It follows that $v(\tilde{L}) \leq v(\tilde{D})$. Reminding (3.12), we get $v(\tilde{L}) = v(\tilde{D})$. The inclusion $S(\tilde{L}) \subset S(\tilde{D})$ follows easily from this. □

The following is our main result in this section. It provides a formula for the derivative of the marginal value function $v'(0)$ and analyzes the behavior of paths of approximate solutions, for problems with existence of multipliers and satisfying the weak (but not the strong) second-order sufficient optimality condition.

THEOREM 3.9. *Assume X reflexive, the existence of an $o(u)$-optimal path, $\mathcal{L}''_x(x_0, \lambda, 0)dd$ weakly lower semicontinuous and one of the two hypotheses below.*

(i) *(CQ), $SOC(\Lambda_0)$, and extended polyhedricity of the second kind;*

(ii) *$(SDCQ)$, $SOC(\Omega)$ for some bounded $\Omega \subset \Lambda_0$, and $0 \in T_2^K(d)$ for all d in C_0.*

Then:

(a) *There exists $v'(0) = v(\tilde{L}) = v(\tilde{D})$, and $S(\tilde{L}) \subset S(\tilde{D})$.*

(b) *For every $o(u)$-optimal path x_u, the weak accumulation points of $(x_u - x_0)/\sqrt{u}$ belong to $S(\tilde{D})$.*

(c) *If* $d_0 \in S(\tilde{L})$ *and* $w_0 \in S(L^{d_0})$, *then there exists an* $o(u)$*-optimal path of the form* $x_u = x_0 + \sqrt{u}d_0 + o(\sqrt{u})$.

Proof. (a) This follows by combining Theorem 3.1 and Propositions 3.7 and 3.8.

(b) Let d_0 be a weak limit point of $(x_u - x_0)/\sqrt{u}$. Expanding the Lagrangian as in (3.15) we get $v(\tilde{D}) = v'(0) \geq \Pi(d_0)$. As d_0 is feasible for $v(\tilde{D})$, d_0 is a solution of $v(\tilde{D})$.

(c) Using (*SDCQ*) let us select $w_\gamma \to w_0$ and feasible paths of the form $x_u^\gamma = x_0 + \gamma\sqrt{u}d_0 + uw_\gamma + o_\gamma(u)$, with (for each γ) $\|o_\gamma(u)\|/u \to 0$ when $u \to 0$. Take $\gamma_k \uparrow 1$ and choose a strictly decreasing sequence $u_k \downarrow 0$ such that

$$\|o_{\gamma_k}(u)\| \leq \frac{u}{k} \qquad \forall u \in [0, u_k]$$

from which we construct the feasible path

$$x_u = x_u^{\gamma_k} \qquad \forall u \in [u_{k+1}, u_k).$$

Then we have

$$\|x_u - x_0 - \sqrt{u}d_0\| \leq \sqrt{u}(1-\gamma_k)\|d_0\| + u\|w_{\gamma_k}\| + \frac{u}{k} \qquad \forall u \in [u_{k+1}, u_k)$$

from which we get $x_u = x_0 + \sqrt{u}d_0 + o(\sqrt{u})$. Also, a second-order expansion implies that for $u \in [u_{k+1}, u_k)$ we have

$$f(x_u, u) = f(x_0, 0) + u\left[f'(x_0, 0)(w_{\gamma_k}, 1) + \frac{1}{2}f_x''(x_0, 0)d_0d_0\right] + o(u)$$

so that

$$\begin{aligned} f(x_u, u) &= f(x_0, 0) + u\Psi_f(w_0, d_0) + o(u) \\ &= v(0) + uv(\tilde{L}) + o(u) = v(u) + o(u). \end{aligned}$$

The conclusion follows. □

4. Perturbation analysis assuming nonexistence of multipliers.

4.1. Preliminaries. In this section we analyze the situation when the set of multipliers Λ_0 is empty, extending the theory of perturbed singular nonlinear programs of [3]. The qualitative behavior is radically different from the case studied in §3, so we are led to introduce some new objects. Indeed, if Λ_0 is empty we have $v(L) = -\infty$ and by part I it follows that $v'(0) = -\infty$.

We will check that, under suitable second-order assumptions, the variation of the cost is of order $O(\sqrt{u})$. This leads us to define, analogously to the Dini derivatives, the following quantities:

$$v^\#(0) := \limsup_{u\downarrow 0} \frac{v(u) - v(0)}{\sqrt{u}},$$
$$v_\#(0) := \liminf_{u\downarrow 0} \frac{v(u) - v(0)}{\sqrt{u}}.$$

We define the singular Lagrangian, the set of singular multipliers (at x_0, for problem (P_0)), and the set of *normalized* singular multipliers as

$$\begin{aligned} &\hat{\mathcal{L}}(x, \lambda, u) := \langle \lambda, G(x, u)\rangle, \\ &\quad \Lambda^s := \{\lambda \in Y^* \setminus \{0\} : \lambda \in N_K(G(x_0, 0)),\ \hat{\mathcal{L}}_x'(x_0, \lambda, 0) = 0\}, \\ &\quad \Lambda_N^s := \{\lambda \in \Lambda^s : \|\lambda\| \leq 1\}. \end{aligned}$$

The next proposition shows that Λ_0 and Λ^s are both empty only in some very special situations.

PROPOSITION 4.1. *If both* Λ_0 *and* Λ^s *are empty, then the set*

$$\mathcal{A} := \mathbb{R}_+[K - G(x_0, 0)] - G'_x(x_0, 0)X$$

is dense in Y *but not equal to* Y.

Proof. If $\mathcal{A} = Y$ we know that $\Lambda_0 \neq \phi$ [13, 14]. Suppose next that $\mathcal{A}$ is not dense in Y and select $y \in Y$ not belonging to the closure of $\mathcal{A}$. By the Hahn–Banach theorem there exists $\lambda \in Y^* \setminus \{0\}$ such that

$$\langle \lambda, y \rangle > \langle \lambda, t[k - G(x_0, 0)] - G'_x(x_0, 0)w \rangle \quad \text{for all } w \in X,\ k \in K,\ t > 0.$$

Taking the supremum over $w \in X$, we get $\lambda \circ G'_x(x_0, 0) = 0$, and letting $t \uparrow \infty$ we deduce $\langle \lambda, k - G(x_0, 0) \rangle \leq 0$ for all $k \in K$, so $\lambda \in N_K(G(x_0, 0))$ and then $\Lambda^s \neq \phi$. □

4.2. Upper estimate of the cost. To obtain upper estimates for $v^\#(0)$ we consider the following optimization problems:

$$(\hat{L}) \qquad \min_{d \in C_0} \left\{ f'_x(x_0, 0)d : \frac{1}{2} G''_x(x_0, 0)dd \in T_2^K(d) - G'(x_0, 0)X \times \{1\} \right\}$$

and

$$(\hat{D}) \qquad \min_{d \in C_0} \left\{ f'_x(x_0, 0)d : \frac{1}{2} G''_x(x_0, 0)dd \in \overline{T_2^K(d) - G'(x_0, 0)X \times \{1\}} \right\}.$$

Problem $(\hat{L})$ will give an upper estimate of the value function whereas $(\hat{D})$, which has the same optimal value as $(\hat{L})$, will provide a comparison with the lower estimate of $v_\#(0)$. We remark that, when Λ^s is not empty, problem $(\hat{D})$ is equivalent to

$(\hat{D}')$

$$\min_{d \in C_0} \left\{ f'_x(x_0, 0)d : \hat{\mathcal{L}}'_u(x_0, \lambda, 0) + \frac{1}{2} \hat{\mathcal{L}}''_x(x_0, \lambda, 0)dd \leq \sigma(\lambda, T_2^K(d)),\ \text{for all } \lambda \in \Lambda^s \right\}.$$

To prove this equivalence it suffices to check that the constraints in $(\hat{D})$ and $(\hat{D}')$ coincide, which follows from the next result applied with $y = G'_u(x_0, 0) + \frac{1}{2} G''_x(x_0, 0)dd$.

PROPOSITION 4.2. *If* $\Lambda^s \neq \phi$, *then the following are equivalent.*

(a) $y \in \overline{T_2^K(d) - G'_x(x_0, 0)X}$.

(b) $\langle \lambda, y \rangle \leq \sigma(\lambda, T_2^K(d))$ *for all* $\lambda \in \Lambda^s$.

Proof. Both (a) and (b) are false if $T_2^K(d)$ is empty, so we may assume the contrary. The implication (a) $\Rightarrow$ (b) is straightforward and the converse follows by a separation argument. Indeed, if (a) fails we may find a *strictly* separating hyperplane, that is, $\lambda \in Y^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \lambda, y \rangle > \alpha \geq \langle \lambda, k - G'_x(x_0, 0)w \rangle$$

for all $k \in T_2^K(d)$, $w \in X$. Taking the supremum over $w \in X$ it follows that $\lambda \circ G'_x(x_0, 0) = 0$, and then taking the supremum over k we deduce that

$$\langle \lambda, y \rangle > \alpha \geq \sigma(\lambda, T_2^K(d)). \tag{4.16}$$

Using this and $(P4)$ we get $\lambda \in N_K(G(x_0, 0))$, so $\lambda \in \Lambda^s$ and (4.16) contradicts (b). □

We now state the upper estimate.

THEOREM 4.3. *If (SDCQ) holds, then*

$$v^\#(0) \leq v(\hat{L}) = v(\hat{D}) \leq 0,$$

so when $v(\hat{L})$ is finite, we have

$$v(u) \leq v(0) + \sqrt{u}v(\hat{L}) + o(\sqrt{u}).$$

In addition, $v(\hat{L}) < 0$ iff there exists a direction d such that $f'_x(x_0, 0)d < 0$ and $T_2^K(d) \neq \phi$.

Proof. We begin by showing $v^{\#}(0) \leq v(\hat{L}) \leq 0$. Let $d \in F(\hat{L})$ and select $w \in X$ such that $G'(x_0, 0)(w, 1) + \frac{1}{2}G''_x(x_0, 0)dd \in T_2^K(d)$. Using the restorability property we may find feasible paths of the form $x_u^\gamma = x_0 + \gamma\sqrt{u}d + uw_\gamma + o(u)$ with $w_\gamma \to w$ as $\gamma \uparrow 1$. Expanding f it follows that

$$v(u) \leq f(x_u^\gamma, u) = f(x_0, 0) + \gamma\sqrt{u} f'_x(x_0, 0)d + o(\sqrt{u}),$$

from which we deduce

$$v^{\#}(0) \leq \gamma f'_x(x_0, 0)d.$$

Letting $\gamma \uparrow 1$ and then taking the infimum over $d \in F(\hat{L})$ we get $v^{\#}(0) \leq v(\hat{L})$. Moreover, $(P2)$ implies $0 \in F(\hat{L})$, so $v(\hat{L}) \leq 0$.

We prove next $v(\hat{L}) = v(\hat{D})$. Since clearly $v(\hat{D}) \leq v(\hat{L})$, it suffices to show that $v(\hat{L}) \leq f'_x(x_0, 0)d$ for each $d \in F(\hat{D})$. Let $d \in F(\hat{D})$ and select sequences $k_n \in T_2^K(d)$, $w_n \in X$ such that $\frac{1}{2}G''_x(x_0, 0)dd = \lim_n[k_n - G'(x_0, 0)(w_n, 1)]$. Using $(P2)$ we find that given any $t > 0$ we will have for all n large enough

$$\frac{1}{2}tG''_x(x_0, 0)dd - tk_n + tG'(x_0, 0)(w_n, 1) \in T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1\},$$

which rearranged gives

$$\frac{1}{2}\frac{t}{1+t}G''_x(x_0, 0)dd \in \frac{t}{1+t}k_n + T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1\}. \tag{4.17}$$

Letting $d_t := \sqrt{t/(1+t)}d$ and using $(P1)$ and $(P3)$ we deduce that

$$\frac{1}{2}G''_x(x_0, 0)d_t d_t \in T_2^K(d_t) - G'(x_0, 0)X \times \{1\}.$$

Hence $d_t \in F(\hat{L})$ and then

$$v(\hat{L}) \leq f'_x(x_0, 0)d_t.$$

Letting t tend to $+\infty$ we conclude that $v(\hat{L}) \leq f'_x(x_0, 0)d$, as required.

We conclude by proving the sufficient condition for $v(\hat{L}) < 0$ (the necessity is evident). If $d \in X$ is such that $f'_x(x_0, 0)d < 0$ and $T_2^K(d) \neq \phi$, from Lemma 3.4 we get $\alpha d \in F(\hat{L})$ for all $\alpha > 0$ sufficiently small, so that $v(\hat{L}) \leq \alpha f'_x(x_0, 0)d < 0$. □

Remark. From the estimate (1.1) we already know that $v_{\#}(0) \leq 0$. Henceforth Theorem 4.3 improves the upper estimate of the cost only if $v(\hat{L}) < 0$.

4.3. Lower estimates and expansion of solutions. As in the case when $\Lambda_0 \neq \phi$, we will give a lower estimate of the cost that is sharp when the contribution of the curvature of K happens to be null.

We consider the *singular* second-order conditions

$$(SSOC) \qquad \text{there exist } \alpha, \epsilon > 0 \text{ s.t. } \sup_{\lambda \in \Lambda_N^s} \hat{\mathcal{L}}''_x(x_0, \lambda, 0)dd \geq \alpha\|d\|^2 \quad \forall\, d \in C_\epsilon.$$

PROPOSITION 4.4. *If (SSOC) holds, then for each $O(\sqrt{u})$-optimal path x_u we have $x_u = x_0 + O(\sqrt{u})$.*

Proof. Let x_u be an $O(\sqrt{u})$-optimal path and let $\beta_u := \|x_u - x_0\|$, $d_u := (x_u - x_0)/\beta_u$. For each $\lambda \in \Lambda_N^s$ we have

$$\begin{aligned} 0 &\geq \hat{\mathcal{L}}(x_u, \lambda, u) - \hat{\mathcal{L}}(x_0, \lambda, 0) \\ &= u\hat{\mathcal{L}}'_u(x_0, \lambda, 0) + \frac{\beta_u^2}{2}\hat{\mathcal{L}}''_x(x_0, \lambda, 0)d_u d_u + o(u) + o(\beta_u^2). \end{aligned}$$

The small terms $o(u)$ and $o(\beta_u^2)$ may be chosen independent of $\lambda \in \Lambda_N^s$, so we may take the supremum to deduce that

$$\beta_u^2 \max_{\lambda \in \Lambda_N^s} \mathcal{L}''_x(x_0, \lambda, 0)d_u d_u \leq O(u) + o(\beta_u^2). \tag{4.18}$$

If for some sequence $u_n \downarrow 0$ one has $\beta_{u_n}^2/u_n \uparrow \infty$, then for n large enough d_{u_n} is in C_ϵ. With $(SSOC)$ and (4.18), we obtain a contradiction. □

To obtain the desired lower estimate for $v_\#(0)$ we consider a *relaxed* version of problem $(\hat{D})$, namely,

$$(\hat{R}) \qquad \min_{d \in C_0} \left\{ f'_x(x_0, 0)d : \frac{1}{2}G''_x(x_0, 0)dd \in \overline{T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1\}} \right\}.$$

As for problem $(\hat{D})$, when Λ^s is not empty one may use Proposition 4.2 (with $d = 0$) to derive the following equivalent formulation for $(\hat{R})$:

$$(\hat{R}') \qquad \min_{d \in C_0} \left\{ f'_x(x_0, 0)d : \hat{\mathcal{L}}'_u(x_0, \lambda, 0) + \frac{1}{2}\hat{\mathcal{L}}''_x(x_0, \lambda, 0)dd \leq 0 \text{ for all } \lambda \in \Lambda^s \right\}.$$

Comparing with $(\hat{D}')$ and using $(P4)$, we see that $F(\hat{D}') \subset F(\hat{R}')$. As these two problems have the same cost, it follows that

$$v(\hat{R}) = v(\hat{R}') \leq v(\hat{D}') = v(\hat{D}). \tag{4.19}$$

PROPOSITION 4.5. *Assume there exists an $o(\sqrt{u})$-optimal path x_u. If $(SSOC)$ is satisfied, then $v_\#(0) > -\infty$. Moreover, if any of the two following properties hold:*

(a) *the path may be expanded as $x_u = x_0 + \sqrt{u}d_0 + o(\sqrt{u})$,*

(b) *X is reflexive and for each $\lambda \in \Lambda^s$ the mapping $d \to \hat{\mathcal{L}}''_x(x_0, \lambda, 0)dd$ is weakly lower semicontinuous at every $d_0 \in C_0$,*

then

$$v_\#(0) \geq v(\hat{R}). \tag{4.20}$$

Proof. By Proposition 4.4 we have $x_u = x_0 + O(\sqrt{u})$ and then

$$v(u) = f(x_u, u) + O(\sqrt{u}) = f(x_0, 0) + O(\sqrt{u}),$$

so $v_\#(0) > -\infty$.

Now let us choose $u_n \downarrow 0$ realizing the lower limit $v_\#(0)$, and let $d_n := (x_{u_n} - x_0)/\sqrt{u_n}$. When (a) holds we have $d_n \to d_0$, while in case (b) we may assume that $d_n \rightharpoonup d_0$ for some $d_0 \in X$. In both cases, $d_0 \in C_0$ and we have

$$v_\#(0) = f'_x(x_0, 0)d_0.$$

On the other hand for all $\lambda \in \Lambda^s$

$$0 \geq \hat{\mathcal{L}}(x_u, \lambda, u) - \hat{\mathcal{L}}(x_0, \lambda, 0)$$
$$= u\hat{\mathcal{L}}'_u(x_0, \lambda, 0) + \frac{u}{2}\hat{\mathcal{L}}''_x(x_0, \lambda, 0)d_u d_u + o(u),$$

so, using in case (b) the lower semicontinuity of $\hat{\mathcal{L}}''_x(x_0, \lambda, 0)dd$ we get

$$0 \geq \hat{\mathcal{L}}'_u(x_0, \lambda, 0) + \frac{1}{2}\hat{\mathcal{L}}''_x(x_0, \lambda, 0)d_0 d_0.$$

It follows that $d_0 \in F(\hat{R}')$. Combining with (4.19) we get

$$v(\hat{R}) = v(\hat{R}') \leq f'_x(x_0, 0)d_0 = v_{\#}(0),$$

as was to be proved. □

Let us put together the different bounds obtained so far. If (*SDCQ*) and the assumptions of Proposition 4.5 hold, then

$$v(\hat{R}) = v(\hat{R}') \leq v_{\#}(0) \leq v^{\#}(0) \leq v(\hat{D}') = v(\hat{D}) = v(\hat{L}) \leq 0.$$

In our next statement, which is our main result for problems with nonexistence of multipliers, we give a condition for all these optimal values to be equal. This gives the first term of the expansion of the optimal value $v(u)$.

THEOREM 4.6. *Assume the existence of an $O(\sqrt{u})$-optimal path x_u, (SSOC), X reflexive, the lower semicontinuity of $d \to \hat{\mathcal{L}}''_x(x_0, \lambda, 0)dd$ for each $\lambda \in \Lambda^s$, (SDCQ), and finally*

$$0 \in T_2^K(d) \quad \text{for all } d \in C_0.$$

Then $v(\hat{R}) = v(\hat{D})$, $S(\hat{R}) = S(\hat{D})$, and

$$v(u) = v(0) + \sqrt{u}\, v(\hat{D}) + o(\sqrt{u}). \tag{4.21}$$

Proof. The equivalence between $(\hat{R})$ and $(\hat{D})$ follows by noting that when $0 \in T_2^K(d)$ (see [8, Prop. 3.1])

$$T_2^K(d) = \overline{T_K(G(x_0, 0)) - \mathbb{R}_+ G'_x(x_0, 0)d},$$

from which we deduce

$$\overline{T_2^K(d) - G'(x_0, 0)X \times \{1\}} = \overline{T_K(G(x_0, 0)) - G'(x_0, 0)X \times \{1\}}.$$

The expansion of $v(u)$ then follows from Theorem 4.3 and Proposition 4.5. □

5. Appendix: Checking the strong directional constraint qualification. We give some sufficient conditions for checking (*SDCQ*) in the case of *decomposed* constraints of the form: $Y := Y_1 \times Y_2$ with Y_1 and Y_2 Banach spaces and $K := K_1 \times K_2$ with K_1 and K_2 closed convex subsets of Y_1 and Y_2. We denote by $G = (G_1, G_2)$ the components of G, and we consider the decomposed directional constraint qualification:

$$(DDCQ) \qquad \begin{cases} \text{(i)} & 0 \in \text{int}[G_1(x_0, 0) + G'_1(x_0, 0)X \times \{0\} - K_1], \\ \text{(ii)} & \text{there exists } \bar{w} \in X \text{ such that } G'_1(x_0, 0)(\bar{w}, 1) \in \text{Rec}(K_1) \text{ and} \\ & G_2(x_0, 0) + \alpha G'_2(x_0, 0)(\bar{w}, 1) \in \text{int } K_2 \text{ for some } \alpha > 0, \end{cases}$$

where $\text{Rec}(K_1)$ denotes the recession cone of K_1, that is,

$$\text{Rec}(K_1) := \limsup_{t\to\infty} \frac{K_1}{t}.$$

To illustrate this condition, let us mention two particular cases. The first one is when $K_2 = Y_2$ so that the constraint is only with K_1. Then (*DDCQ*) reduces to Robinson's condition [14]. The second case is when $K_1 = \{0\}$. Then (*DDCQ*) (i) amounts to the surjectivity of $G'_{1x}(x_0, 0)$ and (*DDCQ*) appears as a natural generalization of Gollan's condition [10] used in the aforementioned literature devoted to nonlinear programming.

THEOREM 5.1. *(DDCQ) implies (SDCQ).*

Proof. We first prove that x_0 is restorable. Let x_u be a path satisfying (2.3) and (2.4). Choose $w_\gamma := \gamma^2 w + (1-\gamma^2)\bar{w}$ and consider

$$y_u := x_0 + \gamma\sqrt{u}d + uw_\gamma. \tag{5.22}$$

Expanding in series we get

$$\begin{aligned} G(y_u, u) &= G(x_0, 0) + \gamma\sqrt{u}G'_x(x_0, 0)d + u\Psi_G(w_\gamma, \gamma d) + o(u) \\ &= G(x_0, 0) + \gamma\sqrt{u}G'_x(x_0, 0)d + \gamma^2 u\Psi_G(w, d) \\ &\quad + (1-\gamma^2)uG'(x_0, 0)(\bar{w}, 1) + o(u) \\ &= G(x(\gamma^2 u), \gamma^2 u) + (1-\gamma^2)uG'(x_0, 0)(\bar{w}, 1) + o(u). \end{aligned}$$

Using (*DDCQ*) (ii) and (2.4) we deduce $d(G_1(y_u, u), K_1) = o(u)$. Then (*DDCQ*) (i) allows us to use Robinson's theorem to find a small correction x_u^γ of y_u,

$$x_u^\gamma = x_0 + \gamma\sqrt{u}d + uw_\gamma + o(u), \tag{5.23}$$

such that $G_1(x_u^\gamma, u) \in K_1$.

Expanding $G_2(x_u^\gamma, u)$ as above, we get

$$G_2(x_u^\gamma, u) = G_2(x(\gamma^2 u), \gamma^2 u) + (1-\gamma^2)uG'_2(x_0, 0)(\bar{w}, 1) + o(u), \tag{5.24}$$

so that letting $z := G'_2(x_0, 0)(\bar{w}, 1)$ and using (2.4) we have

$$G_2(x_u^\gamma, u) = t_u + (1-\gamma^2)uz + o(u)$$

for some $t_u \in K_2, t_u \to G_2(x_0, 0)$. Moreover, letting $\alpha_u := (1-\gamma^2)u/\alpha$ we may write $G_2(x_u^\gamma, u) = (1-\alpha_u)t_u + \alpha_u r_u$ with

$$r_u = t_u + \alpha z + \alpha o(u)/(1-\gamma^2)u = t_u + \alpha z + o(1).$$

By (*DDCQ*) (i) we have $r_u \in K_2$ for u small; since also $t_u \in K_2$ and $\alpha_u \in (0, 1)$, it follows that $G_2(x_u^\gamma, u) \in K_2$. Hence x_u^γ is a feasible path and x_0 is restorable.

We now check that (*DCQ*) is satisfied. By (*DDCQ*) (i) (see [14]) there exist $\epsilon > 0$ and $\beta > 0$ such that, whenever $y_1 \in Y_1$ satisfies $\|y_1\| < \epsilon$, there exist $\hat{d} \in X$ and $k_1 \in K_1$ such that $\|\hat{d}\| < \beta\|y_1\|$ and

$$G_1(x_0, 0) + G'_1(x_0, 0)(\hat{d}, 0) - k_1 = y_1.$$

Now take d of the form $d = \hat{d} + \alpha\bar{w}$. Then

$$G_1(x_0, 0) + G'_1(x_0, 0)(d, \alpha) - [k_1 + \alpha G'(x_0, 0)(\bar{w}, 1)] = y_1$$

and

$$G_2(x_0, 0) + G_2'(x_0, 0)(d, \alpha) - y_2 = G_2(x_0, 0) + \alpha G_2'(x_0, 0)(\bar{w}, 1) + G_2'(x_0, 0)(\hat{d}, 0) - y_2.$$

We may choose ϵ small so that for all $\|y_1\| < \epsilon$, $\|y_2\| < \epsilon$ we have $\|G_2'(x_0, 0)(\hat{d}, 0) - y_2\|$ small enough to deduce, using (*DDCQ*) (ii), that the left-hand side above is in K_2. From this (*DCQ*) follows easily. □

Remark. We do not know (even for nonlinear programming problems) if the property $d(G(x_0 + \sqrt{u}d + uw, u), K) = o(u)$ together with (*DCQ*) suffices or not to construct a feasible path of the form $x_u = x_0 + \sqrt{u}d_0 + uw + o(u)$ (without γ and w_γ).

Our final result shows that, for the important class of equality-inequality constrained problems, the restorability property is a consequence of the directional constraint qualification condition (*DCQ*). So in this case the strong qualification (*SDCQ*) is equivalent to (*DCQ*).

PROPOSITION 5.2. *If* $K := \{0\} \times K_2$ *with* $\operatorname{int}(K_2)$ *nonempty, then* (*DCQ*), (*SDCQ*), *and* (*DDCQ*) *are equivalent and are satisfied iff the condition* (*EDCQ*) *holds.*

$$(EDCQ) \qquad \begin{cases} \text{(i)} & G_1'(x_0, 0)X \times \{0\} = Y_1, \\ \text{(ii)} & \text{there exists } \bar{w} \in X \text{ such that } G_1'(x_0, 0)(\bar{w}, 1) = 0 \text{ and} \\ & G_2(x_0, 0) + \alpha G_2'(x_0, 0)(\bar{w}, 1) \in \operatorname{int} K_2 \text{ for some } \alpha > 0. \end{cases}$$

Proof. Obviously each of the conditions (*DCQ*), (*SDCQ*), (*DDCQ*), and (*EDCQ*) is a consequence of the one that follows. Therefore it suffices to prove that (*DCQ*) implies (*EDCQ*). From (*DCQ*), $G_1'(x_0, 0)X \times (0, \infty)$ contains a neighborhood of 0. Being a cone, this set is equal to Y_1. In particular there exist $d_0 \in X, \alpha_0 > 0$ such that $G_1'(x_0, 0)(d_0, \alpha_0) = 0$, i.e., $G_u'(x_0, 0) \in G_1'(x_0, 0)X \times \{0\}$. We deduce that

$$Y_1 = G_1'(x_0, 0)X \times (0, \infty) = G_1'(x_0, 0)X \times \{0\},$$

i.e., (*EDCQ*) (i) holds. Now pick $a \in \operatorname{int}(K_2)$ close enough to $G_2(x_0, 0)$ so that there exist $d \in X$ and $\tilde{\alpha} > 0$ such that $(0, a - G_2(x_0, 0)) \in G(x_0, 0) + G'(x_0, 0)(d, \tilde{\alpha}) - K$. It is easily checked that (*EDCQ*) (ii) is satisfied with $\bar{w} := d/\tilde{\alpha}, \alpha := \tilde{\alpha}/2$. □

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