

PERTURBED OPTIMIZATION IN BANACH SPACES III: SEMI-INFINITE OPTIMIZATION*

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Abstract. This paper is devoted to the study of perturbed semi-infinite optimization problems, i.e., minimization over \mathbb{R}^n with an infinite number of inequality constraints. We obtain the second-order expansion of the optimal value function and the first-order expansion of approximate optimal solutions in two cases: (i) when the number of binding constraints is finite and (ii) when the inequality constraints are parametrized by a real scalar.

These results are partly obtained by specializing the sensitivity theory for perturbed optimization developed in part I (cf. [*SIAM J. Control Optim.*, 34 (1996), pp. 1151–1171]) and deriving specific sharp lower estimates for the optimal value function which take into account the curvature of the positive cone in the space $C(\Omega)$ of continuous real-valued functions.

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1. Introduction. This paper is the last of a trilogy devoted to the analysis of parametric optimization problems of the form

$$\min_x \{f(x, u) : G(x, u) \in K\},$$

with X and Y Banach spaces, K a closed convex subset of Y , and $f(x, u)$ and $G(x, u)$ mappings of class C^2 from $X \times \mathbb{R}$ into \mathbb{R} and Y , respectively. This third part is devoted to the study of the parametric semi-infinite optimization problem

$$(P_u) \quad \min_x \{f(x, u) : G(x, u)_\omega \geq 0, \forall \omega \in \Omega\},$$

where Ω is a compact metric space; $G(x, u) := \{G(x, u)_\omega\}_{\omega \in \Omega}$ belongs to $C(\Omega)$, the space of continuous functions on Ω endowed with the max norm; and the mapping $(x, u) \rightarrow (f(x, u), G(x, u))$ is of class C^2 from $\mathbb{R}^n \times \mathbb{R}^+$ into $\mathbb{R} \times C(\Omega)$. Since

$$C_+(\Omega) := \{y \in C(\Omega) : y \geq 0\}$$

is a closed convex cone in the Banach space $C(\Omega)$, it follows that (P_u) is a particular case of the above abstract optimization problem.

Semi-infinite optimization problems occur in robust control theory, the design of filters, the design of devices having to respect some specifications in a certain range of pressure and temperature, and optimal control problems when the control has a finite-dimensional parametrization; see [16]. However, the wealth of applications is not the only motivation for studying semi-infinite optimization. In the past few years, a rather complete perturbation theory has been developed for optimization problems with a finite number of constraints, the so-called perturbed nonlinear programming problem; see [2], [6], [8], [19]. The theory of perturbed semi-infinite optimization problems, although it seems much easier than the general perturbation problem in Banach space, includes an essential difficulty related to the curvature of $C_+(\Omega)$. As a consequence, the standard second-order upper and lower estimates for the cost

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do not, in general, coincide. Our main contribution here is to exhibit a sharp lower estimate that, in some cases, is equal to the parabolic upper estimate.

There is already a large body of literature on semi-infinite optimization; see the early references [3] and [11]. The recent review [9] describes in particular the so-called *reduction theory* that reduces (P_u) to an optimization problem with a finite number of constraints (see also [13]). This reduction is possible when the contact set includes a finite number of points and each of them can be expressed locally as a function of the data, typically a local solution of an optimization problem with finitely many constraints. Then the perturbation theory for nonlinear programming can be applied for deriving optimality conditions as well as for conducting a perturbation analysis; see the early reference [18].

In this paper we do not use any reduction device. In this way we may handle some cases where there is a continuum of binding constraints, especially when Ω is a one-dimensional interval. We are also able to treat the case of a finite number of binding constraints in cases where the reduction theory does not apply.

The paper is organized as follows. In §2 we discuss the directional qualification condition introduced in part I. We characterize it and show how to deduce a first-order upper estimate. Section 3 is devoted to the parabolic (second-order) upper estimates. There we combine the technique of parabolic estimates with the directional qualification condition and a characterization of second-order tangent sets to $C_+(\Omega)$, recently obtained in [7]. This upper estimate combined with the strong quadratic growth condition implies the upper Lipschitz property for the set of solutions. In §4 we discuss some sharp lower estimates. We use there specific properties of semi-infinite optimization, among them the fact that an extremal multiplier has a finite support. Then in §5 we recapitulate and state our main result.

2. Directional qualification. We start with some notations. The feasible set, value function, and set of solutions of (P_u) are denoted

$$\begin{aligned} F(u) &:= \{x \in \mathbb{R}^n : G(x, u) \geq 0\}, \\ v(u) &:= \inf\{f(x, u) : x \in F(u)\}, \\ S(u) &:= \{x \in F(u) : f(x, u) = v(u)\}. \end{aligned}$$

Similarly, given any optimization problem (P) , we define $F(P)$, $v(P)$, and $S(P)$ as the feasible set, value function, and set of solutions of (P) .

We recall that the dual space of $C(\Omega)$ is the set $M(\Omega)$ of bounded measures; see, e.g., [22]. If $(\lambda, y) \in M(\Omega) \times C(\Omega)$, then $\langle \lambda, y \rangle = \int_{\Omega} y(\omega) d\lambda(\omega)$. The support of $\lambda \in M(\Omega)$, denoted $\text{supp}(\lambda)$, is defined as the complement of the greatest open subset of Ω over which $|\lambda|$ is null. The negative cone of $M(\Omega)$ is denoted $M_-(\Omega)$.

The Lagrangian function associated with (P_u) is

$$\mathcal{L}(x, \lambda, u) = f(x, u) + \int_{\Omega} G(x, u)_{\omega} d\lambda(\omega).$$

With $x \in F(u)$ we associate the set of Lagrange multipliers

$$\Lambda_u(x) := \{\lambda \in M_-(\Omega) : \text{supp}(\lambda) \subset Z(G(x, u)) \text{ and } \mathcal{L}'_x(x, \lambda, u) = 0\},$$

where, for $y \in C(\Omega)$, the set $Z(y)$ is the contact set defined as

$$Z(y) := \{\omega \in \Omega : y(\omega) = 0\}.$$

Let us fix a particular solution $x_0 \in S(0)$ and denote $\Lambda_0 := \Lambda_0(x_0)$ and $Z_0 := Z(G(x_0, 0))$. The problem with linearized data (L) and its dual (D) are

$$\begin{aligned} \text{(L)} \quad & \min_d \{f'(x_0, 0)(d, 1) : G'(x_0, 0)(d, 1) \geq 0 \text{ on } Z_0\}, \\ \text{(D)} \quad & \max_{\lambda \in \Lambda_0} \mathcal{L}'_u(x_0, \lambda, 0). \end{aligned}$$

We consider the directional qualification hypothesis

$$(DCQ) \quad \exists \hat{d} \in \mathbb{R}^n : G'(x_0, 0)(\hat{d}, 1) > 0 \text{ on } Z_0,$$

which is to be compared to the standard qualification hypothesis (see [12])

$$(CQ) \quad \exists \tilde{d} \in \mathbb{R}^n : G'_x(x_0, 0)\tilde{d} > 0 \text{ on } Z_0.$$

The following is essentially known.

LEMMA 2.1. *Condition (CQ) is equivalent to each of the following two conditions:*

(i) *There exists $\tilde{d} \in \mathbb{R}^n$ such that*

$$G(x_0, 0) + G'_x(x_0, 0)\tilde{d} > 0 \text{ on } \Omega.$$

(ii) *The set Λ_0 is nonempty and bounded.*

Proof. The equivalence between (CQ) and (i) is proven in [20]. That (CQ) implies (ii) follows from [23]. Let us prove that (ii) implies (CQ). If (CQ) does not hold, then the linear semi-infinite optimization problem

$$\min_{d,z} \{z : G'(x_0, 0)_\omega(d, 0) + z \geq 0, \forall \omega \in Z_0\}$$

has value 0. It follows that $(d, z) = (0, 0)$ is a solution of this problem at which the qualification condition is satisfied by the direction $(0, 1)$. By (i) \Rightarrow (ii), there exists at least one multiplier $\hat{\lambda}$. Expressing the optimality conditions, we find that $\hat{\lambda} \in M_-(\Omega) \setminus \{0\}$, $\text{supp}(\hat{\lambda}) \subset Z_0$, and $\hat{\lambda} \circ G'_x(x_0, 0) = 0$. It follows that whenever $\lambda \in \Lambda_0$ and $t \in \mathbb{R}_+$, then $\lambda + t\hat{\lambda} \in \Lambda_0$, in contradiction to (ii). \square

A similar result holds for condition (DCQ).

LEMMA 2.2. *Condition (DCQ) is equivalent to*

(i) *there exist $\varepsilon > 0$ and $\tilde{d} \in \mathbb{R}^n$ such that*

$$G(x_0, 0) + \varepsilon G'(x_0, 0)(\tilde{d}, 1) > 0 \text{ on } \Omega.$$

If in addition $\Lambda_0 \neq \emptyset$, then (DCQ) is equivalent to

(ii) *the set $S(D)$ is nonempty and bounded.*

Proof. Noting that (DCQ) is nothing but the standard qualification condition for the set of constraints $\{G(x, u) \geq 0; u \geq 0\}$, and applying Lemma 2.1, we obtain the equivalence of (DCQ) and (i).

Now assume that $\Lambda_0 \neq \emptyset$. That (DCQ) implies (ii) follows from [4, Prop. 3.1]. Conversely, if (DCQ) does not hold, we have

$$\alpha := \min_{d,z} \{z : G'(x_0, 0)_\omega(d, 1) + z \geq 0, \forall \omega \in Z_0\} \geq 0.$$

Considering the perturbation function

$$\varphi((d, z), h) = \begin{cases} z & \text{if } h(\omega) + G'(x_0, 0)_\omega(d, 1) + z \geq 0, \forall \omega \in Z_0 \\ +\infty & \text{otherwise,} \end{cases}$$

we obtain the dual problem

$$\min_{\lambda} \left\{ - \int_{\Omega} G'_u(x_0, 0)d\lambda : \lambda \in M_-(\Omega), \text{supp}(\lambda) \subset Z_0, \int_{\Omega} d\lambda = -1, \lambda \circ G'_x(x_0, 0) = 0 \right\}.$$

Applying [4, Thm. A2] we get the existence of an optimal solution $\hat{\lambda}$ for this problem, and we have

$$\int_{\Omega} G'_u(x_0, 0)d\hat{\lambda} = \alpha \geq 0.$$

It follows that for each $\lambda \in S(D)$ and every $t > 0$ we have $\lambda + t\hat{\lambda} \in S(D)$, contradicting the boundedness of $S(D)$ stated in (ii). \square

From the above lemma and [5, Prop. 5.2], it follows that (DCQ) is a particular case of the abstract directional constraint qualification of part I.

PROPOSITION 2.3. *If (DCQ) holds, then*

$$\limsup_{u \downarrow 0} \frac{v(u) - v(0)}{u} \leq v(D) = v(L).$$

Proof. This follows from Propositions 2.1 and 3.1 in [4] and Lemma 2.2 above. \square

Remark. The above statements hold when f and G are merely of class C^1 .

3. Second-order upper estimates. Define a *path* as a mapping $u \rightarrow x_u$ from \mathbb{R}_+ to X , with $x_u \rightarrow x_0$ when $u \downarrow 0$. The path is said to be feasible if $G(x_u, u) \in K$ for u small enough. In the study of second-order upper estimates, we analyze feasible paths of the form

$$x_u := x_0 + ud + \frac{u^2}{2}z + o(u^2).$$

Feasibility of x_u implies some relations between the expansion of $G(x_u, u)$ and the geometry of $C_+(\Omega)$. Given a convex subset K of a Banach space Y , we define the first-order tangent set at $y \in K$ as

$$T_K(y) := \{h \in Y : \text{there exists } o(t) \text{ such that } y + th + o(t) \in K\}.$$

Similarly, the second-order tangent set at $y \in K$ in the direction $h \in T(y)$ is

$$T_K^2(y, h) := \left\{ z \in Y : \text{there exists } o(t^2) \text{ such that } y + th + \frac{t^2}{2}z + o(t^2) \in K \right\}.$$

For the sake of simplicity we write $T := T_{C_+(\Omega)}$ and $T^2 := T_{C_+(\Omega)}^2$ and denote the terms of the second-order expansion of $f(x_u, u)$ and $G(x_u, u)$ as

$$\Psi_f(z, d) := f'_x(x_0, 0)z + f''(x_0, 0)(d, 1)(d, 1),$$

$$\Psi_G(z, d) := G'_x(x_0, 0)z + G''(x_0, 0)(d, 1)(d, 1).$$

Expanding $G(x_u, u)$ we obtain that if x_u is a feasible path, then

(1)
$$G'(x_0, 0)(d, 1) \in T(G(x_0, 0)),$$

(2)
$$\Psi_G(z, d) \in T^2(G(x_0, 0), G'(x_0, 0)(d, 1)),$$

and, when $d \in S(L)$, we get

(3)
$$v(u) \leq v(0) + uv(L) + \frac{u^2}{2}\Psi_f(z, d) + o(u^2).$$

In [4], it was shown that an upper estimate of the second-order variation of the cost is obtained by minimizing $\Psi_f(z, d)$ over those z satisfying (2). The purpose of this section is

to make explicit this bound in the case of semi-infinite programming. In the statement of our result, we use some expressions for the tangent sets of $K = C_+(\Omega)$. The first-order tangent cone is well known (see, e.g., [20]):

$$T(y) = \{h \in C(\Omega) : h \geq 0 \text{ on } Z(y)\}.$$

In particular, the tangent cone at $G(x_0, 0)$ is

$$T(G(x_0, 0)) = \{h \in C(\Omega) : h \geq 0 \text{ on } Z_0\}.$$

A formula for the second-order tangent set has been recently obtained in [7]. This formula uses the concept of lower epilimit that we now recall, referring to [1] for a detailed exposition. Let $(A_t)_{t>0}$ be a family of subsets of a Banach space Y . The upper limit of $(A_t)_{t>0}$ at $t = 0$ in the sense of Painlevé–Kuratowski is defined as

$$\limsup_{t \downarrow 0} A_t := \left\{ y \in Y : \liminf_{t \downarrow 0} d(y, A_t) = 0 \right\}.$$

The lower epilimit of a family $(f_t)_{t>0}$ of extended real-valued functions on the topological space K is defined as the function whose epigraph is $\limsup_{t \downarrow 0} \text{epi } f_t$, where

$$\text{epi } f_t := \{(x, r) \in K \times \mathbb{R} : f_t(x) \leq r\}$$

is the epigraph of f_t . An alternative characterization is given by

$$\text{e-lim inf}_{t \downarrow 0} f_t(x) = \sup_{V \in \mathcal{N}(x)} \liminf_{t \downarrow 0} \inf_{y \in V} f_t(y) = \liminf_{(t,y) \rightarrow (0^+,x)} f_t(y),$$

where $\mathcal{N}(x)$ is the set of neighborhoods of x .

PROPOSITION 3.1 (cf. [7]). *Let $y \in C_+(\Omega)$ and $h \in T(y)$. Then*

$$T^2(y, h) = \{h \in C(\Omega) : h + \tau(y, h) \geq 0\},$$

where $\tau(f, v)$ is the lower semicontinuous extended real-valued function defined by

$$\tau(f, v) := \text{e-lim inf}_{t \downarrow 0} \left[\frac{f + tv}{t^2/2} \right].$$

Equivalently, $\tau(f, v)$ is given by the formula

$$(4) \quad \tau(f, v)(\omega) = \begin{cases} 0 & \text{if } \omega \in \text{int } Z(f) \text{ and } v(\omega) = 0, \\ -\theta(\omega) & \text{if } \omega \in \text{bd } Z(f) \text{ and } v(\omega) = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$(5) \quad \theta(\omega) := \limsup_{\substack{y \rightarrow \omega \\ f(y) > 0}} \frac{[-v(y)]_+^2}{2f(y)}.$$

In view of this result, defining

$$\Xi(d, u)_\omega := \frac{G(x_0, 0)_\omega + uG'(x_0, 0)_\omega(d, 1)}{u^2/2},$$

$$\tau_d(\omega) := \text{e-lim inf}_{u \downarrow 0} \Xi(d, u)_\omega,$$

we get the characterization

$$T^2(G(x_0, 0); G'(x_0, 0)(d, 1)) = \{h \in C(\Omega) : h + \tau_d \geq 0\}.$$

Writing $T^2(d)$ for the above set for brevity, we see that $T^2(d) \neq \emptyset$ if and only if $\tau_d > -\infty$. In such a case the support function of $T^2(d)$ can be characterized as

$$\sigma(\lambda, T^2(d)) := \sup \left\{ \int_{\Omega} h(\omega) d\lambda(\omega) : h \in T^2(d) \right\} = - \int_{\Omega} \tau_d(\omega) d\lambda(\omega)$$

for all $\lambda \in S(D)$. (Since $\lambda \leq 0$ and τ_d is lower semicontinuous and nonpositive on $\text{supp}(\lambda)$, the integral on the right-hand side above is well defined.) To make this equality always valid we define $\int_{\Omega} \tau_d(\omega) d\lambda(\omega) := +\infty$ whenever τ_d takes the value $-\infty$.

The function $-\tau_d$ may be interpreted as an *upper curvature function*.

Given $d \in S(L)$, the relations (1)–(3) suggest consideration of the subproblem

$$(L_d) \quad \min_z \{ \Psi_f(z, d) : \Psi_G(z, d) + \tau_d \geq 0 \},$$

with which we associate a dual formulation

$$(D_d) \quad \max_{\lambda \in S(D)} \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) + \int_{\Omega} \tau_d(\omega) d\lambda(\omega).$$

We also introduce the problem

$$(Q) \quad \min \{ v(L_d) : d \in S(L) \}.$$

Whenever $v'(0)$ exists, we define the upper and lower second-order Dini derivatives

$$(6) \quad v''_+(0) := \limsup_{u \downarrow 0} 2[v(u) - v(0) - uv'(0)]/u^2,$$

$$(7) \quad v''_-(0) := \liminf_{u \downarrow 0} 2[v(u) - v(0) - uv'(0)]/u^2.$$

PROPOSITION 3.2. *If (DCQ) holds, then $v(L_d) = v(D_d)$ for all $d \in S(L)$, and we have $v(Q) < +\infty$ iff there exists $d \in S(L)$ such that $\tau_d(\omega) > -\infty$ for all $\omega \in \Omega$. Moreover, if $S(L)$ is nonempty and $v(Q) > -\infty$, we then have*

$$(8) \quad v(u) \leq v(0) + uv(L) + \frac{u^2}{2} v(Q) + o(u^2).$$

In particular, if there exists $v'(0) = v(L)$, we get

$$(9) \quad v''_+(0) \leq \inf_{d \in S(L)} \max_{\lambda \in S(D)} \left\{ \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) + \int_{\Omega} \tau_d(\omega) d\lambda(\omega) \right\}.$$

Proof. This is a consequence of Propositions 2.1, 2.2, and 4.2 in [4]. □

By the above Proposition, $v(Q) < +\infty$ iff $\tau_d > -\infty$ for some $d \in S(L)$. We show that a sufficient condition for this is a quadratic growth condition, recently introduced in [20].

LEMMA 3.3. *Suppose that $G'(x_0, 0)$ is Lipschitz with respect to ω and assume that $G(x_0, 0)$ satisfies the quadratic growth condition*

$$(QGC) \quad \exists c > 0 \text{ such that } G(x_0, 0)_{\omega} \geq c \text{ dist}(\omega, Z_0)^2.$$

Then $\tau_d > -\infty$ for all $d \in S(L)$.

Proof. Using (4) it suffices to show that for all $\omega \in \text{bd } Z_0$ with $G'(x_0, 0)(d, 1)_\omega = 0$ we have $\theta(\omega) < +\infty$. To this end let L be a Lipschitz constant for $\omega \rightarrow G'(x_0, 0)(d, 1)_\omega$. For each $\omega \notin Z_0$ let ω_0 be a projection of ω onto Z_0 . Then ω_0 lies on the boundary of Z_0 so that $G(x_0, 0)_{\omega_0} = 0$, and since $G'(x_0, 0)(d, 1) \in T(G(x_0, 0))$ we deduce $G'(x_0, 0)(d, 1)_{\omega_0} \geq 0$. We obtain

$$-G'(x_0, 0)(d, 1)_\omega \leq -G'(x_0, 0)(d, 1)_{\omega_0} + L \text{dist}(\omega, Z_0).$$

From this and (QGC) we deduce

$$[-G'(x_0, 0)(d, 1)_\omega]_+^2 \leq L^2 \text{dist}(\omega, Z_0)^2 \leq \frac{L^2}{c} G(x_0, 0)_\omega,$$

which implies $\theta(\omega) \leq L^2/(2c) < +\infty$. \square

Remark. Assuming (DCQ), we know by Lemma 2.2 that $S(D)$ is bounded. Let $d \in S(L)$. If (QGC) holds, as a consequence of the bound established for $\theta(\omega)$, the amount $\int_\Omega \tau_d(\omega) d\lambda(\omega)$ is bounded uniformly for $\lambda \in S(D)$, so $v(D_d)$ is finite.

4. Stability of solutions. We state next a sufficient condition for ensuring a Lipschitz behavior of the (sub)optimal paths x_u of the perturbed problems. The result, which is a rather straightforward application of the preceding discussion and [4, Prop. 6.3], is based on the strong second-order sufficient condition

$$(SOC) \quad \max_{\lambda \in S(D)} \mathcal{L}''_x(x_0, \lambda, 0)dd > 0 \text{ for all } d \text{ in } C \setminus \{0\},$$

where C denotes the critical cone

$$C := \{d \in \mathbb{R}^n : f'_x(x_0, 0)d \leq 0 \text{ and } G'_x(x_0, 0)d \geq 0 \text{ on } Z_0\}.$$

PROPOSITION 4.1. *Suppose that*

- (i) (DCQ) holds and $\Lambda_0 \neq \emptyset$.
- (ii) There exists $d \in S(L)$ such that $\tau_d(\omega) > -\infty \forall \omega \in \Omega$.
- (iii) (SOC) holds.

Then every $O(u^2)$ -optimal path x_u satisfies $x_u = x_0 + O(u)$.

Remark. Combining Lemma 3.3 and Proposition 4.1, we obtain a sufficient condition for Lipschitz behavior of solutions, similar to the result of [20].

Remark. Following [10], we may introduce the strong quadratic growth condition

$$(SQG) \quad \exists \alpha > 0, c > 0, F(x, u) \geq v(0) + uv(L) + \alpha \text{dist}(x, S_0)^2 - cu^2,$$

where

$$F(x, u) := \begin{cases} f(x, u) & \text{if } G(x, u) \geq 0, \\ +\infty & \text{if not.} \end{cases}$$

Then we can show that (SOC) \Rightarrow (SQG), and Proposition 4.1 is still valid if we replace assumption (iii) by (SQG).

5. Lower estimates. We recall for reference the following standard lower estimate, which is in fact a particular case of the general lower estimate of part I.

LEMMA 5.1. *Let us assume (DCQ) and suppose that there exists a path of $o(u^2)$ -optimal solutions x_u satisfying $x_u = x_0 + O(u)$. Then $\Lambda_0 \neq \emptyset$, $v'(0)$ exists with $v'(0) = v(L) = v(D)$, $S(L) \neq \emptyset$, and we have*

$$v''_-(0) \geq \inf_{d \in S(L)} \max_{\lambda \in S(D)} \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1).$$

Proof. The existence of $v'(0)$, as well as the equality $v'(0) = v(L) = v(D)$, follows from [4, Prop. 3.2]. The finiteness of $v'(0) = v(D)$ implies that $\Lambda_0 \neq \emptyset$, and [4, Prop. 3.3] gives $S(L) \neq \emptyset$. The lower estimate on $v''_-(0)$ is then obtained by applying [4, Prop. 4.3(b)]. \square

Comparing the previous lower estimate to the upper estimate (9) in §3, we observe a gap due to the curvature of $C_+(\Omega)$ at $G(x_0, 0)$. More precisely, if

$$\tau_d = 0 \text{ on } \text{supp}(\lambda) \quad \forall \lambda \in S(D),$$

then the two estimates coincide, but as one can see from (4) and (5), this may occur only in some very special situations.

We are then led to search for sharper lower estimates. We will obtain a lower estimate on $v''_-(0)$ involving the upper epilimit of $\Xi(d, u)$.

We recall the concept of upper epilimit, for which we refer again to [1]. Let $(A_t)_{t>0}$ be a family of subsets of a Banach space Y . The lower limit of $(A_t)_{t>0}$ at $t = 0$ in the sense of Painlevé–Kuratowski is defined as

$$\liminf_{t \downarrow 0} A_t := \left\{ y \in Y : \limsup_{t \downarrow 0} d(y, A_t) = 0 \right\}.$$

The upper epilimit of a family $(f_t)_{t>0}$ of extended real-valued functions on the topological space K is defined as the function whose epigraph is $\liminf_{t \downarrow 0} \text{epi } f_t$. A useful formula is

$$(10) \quad \text{e-lim sup}_{t \downarrow 0} f_t(x) = \sup_{V \in \mathcal{N}(x)} \limsup_{t \downarrow 0} \inf_{y \in V} f_t(y).$$

When the upper and lower epilimits coincide at a given point, we shall say that the family of functions *epiconverges* at that point, and we shall denote

$$\text{e-lim}_{t \downarrow 0} f_t(x) = \text{e-lim inf}_{t \downarrow 0} f_t(x) = \text{e-lim sup}_{t \downarrow 0} f_t(x).$$

We shall say that the family of functions epiconverges on a subset $K_0 \subset K$ if it epiconverges at each point of K_0 .

The next proposition makes use of the set of extreme points of $S(D)$, which will be denoted $S^*(D)$. The result will be derived under a technical assumption $(H_{\hat{\omega}})$, which will be further clarified afterwards. $B(\hat{\omega}, r)$ denotes the ball of center $\hat{\omega}$ and radius r .

PROPOSITION 5.2. *Assume (DCQ) and suppose that there exists an $o(u^2)$ -optimal path such that $x_u = x_0 + O(u)$. Let $G(x_0, 0)$, $G'(x_0, 0)$, and $G''(x_0, 0)$ be Lipschitz with respect to ω and assume also that for every $d \in S(L)$, each $\lambda \in S^*(D)$, and every $\hat{\omega} \in \text{supp}(\lambda)$, one has*

$$(H_{\hat{\omega}}) \quad \exists V \in \mathcal{N}(\hat{\omega}), \exists r > 0 \text{ s.t. } \inf_{\omega \in V} \Xi(d, u)_\omega = \inf_{\omega \in B(\hat{\omega}, ru)} \Xi(d, u)_\omega + o(1),$$

with $o(1)$ converging to 0 uniformly as $u \downarrow 0$. Then

$$(11) \quad v''_-(0) \geq \inf_{d \in S(L)} \max_{\lambda \in S(D)} \left\{ \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) + \int_{\Omega} \bar{\tau}_d(\omega) d\lambda(\omega) \right\},$$

where

$$\bar{\tau}_d(\omega) := \text{e-lim sup}_{u \downarrow 0} \Xi(d, u)_\omega.$$

For $\int_{\Omega} \bar{\tau}_d(\omega) d\lambda(\omega)$ we adopt the same convention as in §3: its value is the usual integral when $\bar{\tau}_d > -\infty$, and $+\infty$ when $\bar{\tau}_d$ takes the value $-\infty$. With this convention

$$\int_{\Omega} \bar{\tau}_d(\omega) d\lambda(\omega) = -\sigma(\lambda, \{h \in C(\Omega); h + \bar{\tau}_d \geq 0\}),$$

which is an upper semicontinuous function of $\lambda \in S(D)$.

The function $-\bar{\tau}_d$ may be interpreted as a *lower-curvature function*.

We discuss some consequences of this proposition, postponing the proof until the end of the section.

Comparing the bounds obtained for $v''_+(0)$ and $v''_-(0)$ in Propositions 3.2 and 5.2, we see that the only difference is between the terms $e\text{-lim inf}_{u \downarrow 0} \Xi(d, u)$ and $e\text{-lim sup}_{u \downarrow 0} \Xi(d, u)$. The statement below follows.

COROLLARY 5.3. *In addition to the assumptions of Proposition 5.2, let us suppose that $\Xi(d, u)$ epiconverges on $\text{supp}(\lambda)$ for each $\lambda \in S^*(D)$ and $d \in S(L)$. Then there exists*

$$v''(0) = \inf_{d \in S(L)} \max_{\lambda \in S(D)} \left\{ \mathcal{L}''(x_0, \lambda, 0) + \int_{\Omega} e\text{-lim}_{u \downarrow 0} \Xi(d, u)_{\omega} d\lambda(\omega) \right\}.$$

It remains to find sufficient conditions to ensure the technical assumption $(H_{\hat{\omega}})$ in Proposition 5.2. Lemma 5.5 below gives a result in this direction. We first need a technical lemma which describes the structure of extreme points of $S(D)$.

LEMMA 5.4. *Suppose (DCQ) and $\Lambda_0 \neq \emptyset$. Then $S(D)$ is the closed convex hull of $S^*(D)$, and any $\lambda \in S^*(D)$ is of the form*

$$(12) \quad \lambda = \sum_{i=1}^p \lambda_{\omega_i} \delta_{\omega_i}$$

with $p \leq n$, $\lambda_{\omega_i} < 0$, and δ_{ω_i} being the Dirac mass at ω_i . (Recall that n is the dimension of the space to which x belongs.)

Proof. By Lemma 2.2, the set $S(D)$ is nonempty and bounded. Being closed, it is weak* compact. The Krein–Milman theorem implies that $S(D)$ is the closed convex hull of its extreme points (see, e.g., [22]). Now, $S(D)$ is a face of Λ_0 so that the points in $S^*(D)$ are also extreme points of Λ_0 , and the latter are known to be the sum of at most n Dirac masses (see [20]). \square

LEMMA 5.5. *Suppose that $G(x_0, 0)$, $G'(x_0, 0)$, and $G''(x_0, 0)$ are Lipschitz with respect to ω and (QGC) holds. Under each of the following conditions, property $(H_{\hat{\omega}})$ is satisfied $\forall \hat{\omega} \in \text{supp}(\lambda)$, $\forall \lambda \in S^*(D)$, $\forall d \in S(L)$:*

- (i) Z_0 is a finite set.
- (ii) Ω is an interval, and Z_0 is the union of finitely many intervals.

Proof. By the previous lemma, $\lambda \in S^*(D)$ has a finite support included in Z_0 . Then, using $d \in S(L)$, we obtain

$$G(x_0, 0)_{\hat{\omega}} = G'(x_0, 0)(d, 1)_{\hat{\omega}} = 0 \quad \forall \hat{\omega} \in \text{supp}(\lambda).$$

From this we deduce that for all $V \in \mathcal{N}(\hat{\omega})$ we have for all $r > 0$, $u > 0$ small enough

$$(13) \quad \inf_{\omega \in V} \Xi(d, u)_{\omega} \leq \inf_{\omega \in B(\hat{\omega}, ru)} \Xi(d, u)_{\omega} \leq \Xi(d, u)_{\hat{\omega}} = 0.$$

Let V be a closed neighborhood of $\hat{\omega}$ such that $\text{dist}(\omega, Z_0) = d(\omega, \hat{\omega})$ whenever $\omega \in V \setminus Z_0$: this is possible by (i) or (ii). Let us consider a Lipschitz constant L for

$$g(\omega) := G'(x, 0)(d, 1)_{\omega},$$

and let ω_u minimize $\Xi(d, u)_{\omega}$ over $\omega \in V$. If $\omega_u \in Z_0$, then we get $\Xi(d, u)_{\omega_u} \geq 0$, which combined with (13) gives $(H_{\hat{\omega}})$ with $o(1) \equiv 0$ and $r > 0$ arbitrary. Let us then assume that

$\omega_u \notin Z_0$. Then we may use (QGC) and (13) to obtain

$$c \operatorname{dist}(\omega_u, Z_0)^2 + u g(\omega_u) \leq G(x_0, 0)_{\omega_u} + u g(\omega_u) = \frac{u^2}{2} \Xi(d, u)_{\omega_u} \leq \frac{u^2}{2} \Xi(d, u)_{\hat{\omega}} = u g(\hat{\omega}).$$

Since $g(\omega)$ is Lipschitzian, we get

$$c d(\omega_u, \hat{\omega})^2 \leq u(g(\hat{\omega}) - g(\omega_u)) \leq uL d(\omega_u, \hat{\omega}),$$

and then $d(\omega_u, \hat{\omega}) \leq uL/c$. This proves that minimizing $\Xi(d, u)_\omega$ over $\omega \in V$ is equivalent to minimizing it over $B(\hat{\omega}, uL/c)$, proving $(H_{\hat{\omega}})$ with $r = L/c$ and $o(1) \equiv 0$. \square

We now return to the proof of Proposition 5.2, starting with the following lemma, which does not use the specific properties of semi-infinite optimization.

LEMMA 5.6. *Assume (DCQ) and suppose that there exists an $o(u^2)$ -optimal path such that $x_u = x_0 + O(u)$. Then there exists $u_k \downarrow 0$ and $d \in S(L)$ such that for any $\lambda \in S(D)$*

$$(14) \quad v''_-(0) = \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) - \lim_{k \rightarrow \infty} \int_{\Omega} \Psi_G(z^k, d)_\omega d\lambda(\omega),$$

where

$$(15) \quad z^k := \frac{x_{u_k} - x_0 - u_k d}{u_k^2/2}.$$

Proof. Let us take a sequence $u_k \downarrow 0$ such that

$$(16) \quad 2(v(u_k) - v(0) - u_k v'(0))/u_k^2 \rightarrow v''_-(0).$$

Denoting $x^k = x_{u_k}$ and passing to a subsequence we may also assume that

$$(17) \quad (x^k - x_0)/u_k \rightarrow d$$

for some $d \in \mathbb{R}^n$, which, by [4, Prop. 3.3], satisfies $d \in S(L)$.

Since $x_u = x_0 + O(u)$, a second-order expansion of G gives

$$G(x^k, u_k) = G(x_0, 0) + u_k G'(x_0, 0)(d, 1) + \frac{u_k^2}{2} \Psi_G(z^k, d) + o(u_k^2).$$

Since $d \in S(L)$ and $\lambda \in S(D)$ we have

$$\langle \lambda, G(x_0, 0) \rangle = \langle \lambda, G'(x_0, 0)(d, 1) \rangle = 0,$$

from which we get

$$\langle \lambda, G(x^k, u_k) \rangle = \frac{u_k^2}{2} \langle \lambda, \Psi_G(z^k, d) \rangle + o(u_k^2).$$

Taking into account that $\mathcal{L}(x_0, \lambda, 0) = v(0)$, $\mathcal{L}'_x(x_0, \lambda, 0) = 0$, and $\mathcal{L}'_u(x_0, \lambda, 0) = v'(0)$, we deduce

$$v(u_k) = f(x^k, u_k) + o(u_k^2) = \mathcal{L}(x^k, \lambda, u_k) - \langle \lambda, G(x^k, u_k) \rangle + o(u_k^2)$$

$$= v(0) + u_k v'(0) + \frac{u_k^2}{2} [\mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) - \langle \lambda, \Psi_G(z^k, d) \rangle] + o(u_k^2),$$

from which the conclusion follows. \square

In order to get the best lower estimate, and recalling that the multiplier is nonpositive, we must minorize $\Psi_G(z^k, d)$ efficiently. Note that by expanding $G(x^k, u_k)$, we get the following relation, which we shall use later:

$$(18) \quad \Psi_G(z^k, d) + \Xi(d, u^k) \geq o(1),$$

where the inequality is to be understood in $C(\Omega)$, i.e., $o(1) \rightarrow 0$ uniformly when $u \downarrow 0$.

We now proceed with the proof of Proposition 5.2.

Proof. Let u_k, d, z^k be chosen as in Lemma 5.6. Consider the problem

$$\max_{\lambda \in S(D)} \left\{ \mathcal{L}''(x_0, \lambda, 0)(d, 1)(d, 1) + \int_{\Omega} \bar{\tau}_d(\omega) d\lambda(\omega) \right\}.$$

As the cost function is affine and upper semicontinuous and $S(D)$ is weak* compact, the maximum is attained at an extreme point $\lambda^* \in S^*(D)$. By Lemma 5.4, $\lambda^* = \sum_{i=1}^p \lambda_{\omega_i} \delta_{\omega_i}$.

Let us take $\hat{\omega} = \omega_i$. From (18) we have

$$\Psi_G(z^k, d)_{\hat{\omega}} + \Xi(d, u_k)_{\omega} \geq \Psi_G(z^k, d)_{\hat{\omega}} - \Psi_G(z^k, d)_{\omega} + o(1),$$

with $o(1)$ uniform with respect to ω . Minimize the right-hand side first and then the left-hand side for $\omega \in B(\hat{\omega}, ru_k)$ to obtain

$$\Psi_G(z^k, d)_{\hat{\omega}} + \inf_{\omega \in B(\hat{\omega}, ru_k)} \Xi(d, u_k)_{\omega} \geq \Psi_G(z^k, d)_{\hat{\omega}} - \sup_{\omega \in B(\hat{\omega}, ru_k)} \Psi_G(z^k, d)_{\omega} + o(1).$$

Now if we fix $r > 0$, because $G'(x_0, 0)$ and $G''(x_0, 0)$ are Lipschitzian, using the fact that $u_k z^k \rightarrow 0$, we get

$$\Psi_G(z^k, d)_{\hat{\omega}} - \sup_{\omega \in B(\hat{\omega}, ru_k)} \Psi_G(z^k, d)_{\omega} \geq O(u_k) \|z^k\| + o(1) = o(1),$$

which combined with the previous estimate gives

$$\Psi_G(z^k, d)_{\hat{\omega}} + \inf_{\omega \in B(\hat{\omega}, ru_k)} \Xi(d, u_k)_{\omega} \geq o(1).$$

Invoking assumption $(H_{\hat{\omega}})$, we may select $V \in \mathcal{N}(\hat{\omega})$, $r > 0$ such that

$$\liminf_{k \rightarrow +\infty} \Psi_G(z^k, d)_{\hat{\omega}} + \limsup_{u \downarrow 0} \inf_{\omega \in V} \Xi(d, u)_{\omega} \geq 0,$$

and therefore, by (10)

$$\liminf_{k \rightarrow +\infty} \Psi_G(z^k, d)_{\hat{\omega}} + \bar{\tau}_d(\hat{\omega}) \geq 0.$$

Combining this estimate with (14) and noting that $\lambda_{\omega_i} < 0$, we obtain (11) as required. □

6. Conclusion. Let us assume that the original problem (P_0) has a unique solution

$$(H1) \quad S(0) = \{x_0\}$$

and that we have the uniform boundedness of solutions:

$$(H2) \quad \exists r > 0, u_0 > 0 \text{ such that for all } u \leq u_0, S(u) \neq \emptyset \text{ and } S(u) \subset B(0, r).$$

THEOREM 6.1 (main result). *Let us assume (H1), (H2), (DCQ), (QGC), (SOC), and $S(L) \neq \emptyset$. Suppose also that $G(x_0, 0)$, $G'(x_0, 0)$, and $G''(x_0, 0)$ are Lipschitz with respect to*

ω . Suppose finally that for each $d \in S(L)$ there exists $e\text{-}\lim_{u \downarrow 0} \Xi(d, u)$ and also that one of the following conditions holds:

- (i) Z_0 is finite.
- (ii) Ω is an interval, and Z_0 is the union of finitely many intervals.

Then we have

- (a) The value function has first- and second-order (right) derivatives given by $v'(0) = v(L)$, $v''(0) = v(Q)$. Moreover, if $v(Q) > -\infty$, we have the expansion

$$v(u) = v(0) + uv(L) + \frac{u^2}{2}v(Q) + o(u^2).$$

- (b) The set of all limit points of $(x_u - x_0)/u$, where x_u ranges over all paths of $o(u^2)$ -optimal solutions, is included in $S(Q)$. In particular, if $S(Q)$ is a singleton, i.e., $S(Q) = \{d\}$, and x_u is as above, then $x_u = x_0 + ud + o(u)$.
- (c) Let $d \in S(Q)$. If there exists $z \in S(L_d)$ (this is always the case when (i) holds), then there exists an $o(u^2)$ -optimal path $x_u = x_0 + ud + o(u)$.
- (d) Let λ_u be a multiplier associated with a solution x_u of (P_u) . Then all weak* limit points of λ_u belong to $S(D)$.

Proof. From (H1) and (H2) there exists $x_u \in S(u)$ which satisfies $x_u \rightarrow x_0$ when $u \downarrow 0$. Then, from Proposition 4.1 we get $x_u = x_0 + O(u)$, and part (a) follows by combining Lemmas 5.1 and 5.5 with Corollary 5.3.

If x_u is a path of $o(u^2)$ -optimal solutions, expanding $f(x_u, u)$ as in the proof of Lemma 5.6, we obtain the first statement in (b). The second statement is an immediate consequence of the first.

We now prove (c). Let $d \in S(Q)$ and $z \in S(L_d)$. Then there exists a feasible path $x_u = x_0 + ud + \frac{u^2}{2}z + o(u^2)$. Expanding $f(x_u, u)$ and $G(x_u, u)$, we obtain $v(Q) = \Psi_f(z, d)$ as well as (2). The conclusion follows.

Assertion (d) is a consequence of [4, Prop. 3.3]. \square

Concluding remarks. Our final result is an extension to semi-infinite optimization of the results of the sequence of papers [8], [19], [2], and [6] in the following sense: if Ω is a finite set, then we exactly recover the above-mentioned results up to the presence of equality constraints. However, there is no difficulty in adding a finite number of equality constraints to our formulation. We avoided it for the sake of clarity of exposition and in order to concentrate on the real difficulty, which is to handle an infinite number of constraints.

Some of our hypotheses, however, may seem unduly strong. First of all, we assume that $S(L)$ is nonempty. While this hypothesis is automatically satisfied when the contact set is finite (due to the standard theory of linear programming), we are not aware of general criteria allowing to check nonemptiness of $S(L)$ for semi-infinite programming. Performing an analysis of the variation of the solutions when $S(L)$ is empty is an open problem. Some of the results of part II might be useful for dealing with this case.

The other hypothesis that seems excessively strong is the alternative (i) or (ii). We need it in order to satisfy the geometrical hypothesis (H_δ) . Still, the most important contribution of this paper is to present a new way of obtaining sharp lower estimates of the cost, and we hope that the technique presented here can be improved in order to deal with more general contact sets.

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