Authorized documents : lecture notes and personal notes of this course.
Arguments should be concise and stated carefully, without neglecting partial answers.

Problem 1 (Ensta and Master students) : **Optimal Hydropower Generation**

The problem is to schedule the energy production of n dams. Each dam equation is

$$\dot{y}_i(t) = b_i(t) - u_i(t), \quad t \in (0,T); \;\; y_i(0) = y_i^0, \tag{1}$$

where for dam $i = 1$ to n : $y_i(t)$ is the amount of water, with given initial value y_i^0, and $b_i(t) \geq 0$ is the (given) water inflow, u_i (control variable) is the outflow, the corresponding energy produced being $\rho_i(y_i)u_i$, where ρ_i is a positive, *nondecreasing* function (the head effect). The (given) energy demand is $D(t) \geq 0$, and we denote the difference between the demand and hydro production by

$$\delta(t) := D(t) - \sum_{i=1}^{n} u_i(t)\rho_i(y_i(t)). \tag{2}$$

The cost for this difference is $P(\delta)$, where $P(\cdot)$ is *nondecreasing, differentiable and convex.* So the optimal control problem say $(\mathcal{P})$ is to minimize the cost function (where the $a_i > 0$ represent the water value at final time)

$$\int_0^T P\left(D(t) - \sum_{i=1}^{n} u_i(t)\rho_i(y_i(t))\right) \mathrm{d}t - \sum_{i=1}^{n} a_i y_i(T), \tag{3}$$

subject to the state equation (1) and to the control constraints

$$u_i(t) \in [0, u_i^M], \;\; i = 1, \ldots, n, \quad t \in (0,T). \tag{4}$$

We write $U_{ad} := \Pi_i[0, u_i^M]$and assume that the $b_i(t)$ and $D(t)$ are bounded and of class C^∞. There is no constraint on the state other than the given initial state.

1/ Give the expression of the pre-Hamiltonian $H(t, u, y, p)$.
ANSWER:

$$H(t, u, y, p) = P\left(D(t) - \sum_{i=1}^{n} u_i\rho_i(y_i)\right) + \sum_{i=1}^{n} p_i(b_i(t) - u_i). \tag{5}$$

2/ Write the costate equation.
ANSWER: *We have that for $i = 1$ to n :*

$$-\dot{\bar{p}}_i(t) = H_{y_i} = -P'(\delta(t))\rho_i'(\bar{y}_i(t))\bar{u}_i(t), \tag{6}$$

with final condition

$$\bar{p}_i(T) = -a_i. \tag{7}$$

3/ Show that $\bar{p}_i(t)$ is nondecreasing, with negative values.

ANSWER: *By (6), $\dot{\bar{p}}_i(t)$ is nonnegative, so that $\bar{p}_i$ is nondecreasing, with final value $\bar{p}_i(T) = -a_i < 0$. The result follows.*

4/ Write the Hamiltonian inequality.

ANSWER: *We may ignore the term of the pre-Hamiltonian not depending on the control, and we get that for a.a. $t \in (0,T)$, for all $u \in U_{ad}$:*

$$P\left(D(t) - \sum_{i=1}^{n} \bar{u}_i(t)\rho_i(\bar{y}_i(t))\right) - \bar{p}(t) \cdot \bar{u}(t) \leq P\left(D(t) - \sum_{i=1}^{n} u_i\rho_i(\bar{y}_i(t))\right) - \bar{p}(t) \cdot u. \quad (8)$$

5/ Set for $i = 1$ to n :

$$\begin{array}{ll} v_i & = \rho_i(\bar{y}_i(t))u_i, \\ \bar{v}_i(t) & := \rho_i(\bar{y}_i(t))\bar{u}_i(t), \\ \alpha_i(t) & := \bar{p}_i(t)/\rho_i(\bar{y}_i(t)). \end{array} \quad (9)$$

Show that $\bar{v}$ is solution of the problem

$$\begin{array}{l} \text{Max}_v \sum_{i=1}^n \alpha_i(t)v_i; \quad v_i \in [0, u_i^M/\rho_i(\bar{y}_i(t))],\ i = 1, \dots, n, \\ \qquad\qquad\qquad\qquad \sum_{i=1}^n v_i = \sum_{i=1}^n \rho_i(\bar{y}_i(t))\bar{u}_i(t). \end{array} \quad (10)$$

ANSWER: *Clearly $\bar{u}$ maximizes $\bar{p}(t) \cdot u$ over U_{ad}, under the constraint of having*

$$\sum_{i=1}^{n} \rho_i(\bar{y}_i(t))u_i = \sum_{i=1}^{n} \rho_i(\bar{y}_i(t))\bar{u}_i(t). \quad (11)$$

Passing to the variables v, $\bar{v}$ we get the result.

6/ Compute the solution of (10) and deduce an expression of $\bar{u}(t)$.

ANSWER: *For such a (continuous knapsack) problem we know that the optimum is that some $\eta(t) < 0$, $\bar{u}_i(t) = 0$ if $\alpha_i < \eta(t)$ and $\bar{u}_i(t) = u_i^M$ if $\alpha_i > \eta(t)$.*

7/ We assume in this question that P is the identity mapping (case of market with constant unit price for buying/selling), i.e., $P(s) = s$ for all s. Show that the Hamiltonian inequality splits into n independent problems, and express the solutions of these problems when possible.

ANSWER: *We get the n independent problems*

$$-(\rho_i(\bar{y}_i(t)) + \bar{p}_i(t))\bar{u}_i(t) \leq -(\rho_i(\bar{y}_i(t)) + \bar{p}_i(t)) \cdot u_i, \quad \textit{for all } u_i \in [0, u_i^M]. \quad (12)$$

The solution satisfies

$$\bar{u}_i(t) = \left\{ \begin{array}{ll} 0 & \textit{if } \rho_i(\bar{y}_i(t)) + \bar{p}_i(t) < 0, \\ u_i^M & \textit{if } \rho_i(\bar{y}_i(t)) + \bar{p}_i(t) > 0. \end{array} \right. \quad (13)$$

If $\rho_i(\bar{y}_i(t)) + \bar{p}_i(t) < 0$ we can only say that $\bar{u}_i(t) \in [0, u_i^M]$. We recover the previous result with $\eta(t) = -1$.

Note that when $P(s)$ is affine, the original problem itself splits in a natural way into n independent problems, one for each dam.

8/ We assume in this question that P is (again) the identity mapping, and also that $b_i(t) > 0$ and $\rho_i'(\bar{y}_i(t)) > 0$ for all t. Discuss the sign of the time derivative of H_{u_i} (partial derivative) and deduce that the optimal policy is to have $\bar{u}_i(t) = 0$ over $[0, t_i]$ and $\bar{u}_i(t) = u_i^M$ over $[t_i, T]$, with $t_i \in [0, T]$.

ANSWER: *We have that $H_{u_i} = -(\rho_i(\bar{y}_i(t)) + \bar{p}_i(t))$ and*

$$\dot{H}_u = -\frac{d}{dt}(\rho_i(\bar{y}_i(t)) + \bar{p}_i(t)) = -\rho_i'(\bar{y}_i(t))b_i(t)) \quad (14)$$

is negative, so that H_u is positive and then negative (or has a constant sign). The result follows.

Problem 2 (Master students) : **Hydropower Generation with state constraints**

We consider the same problem, with additional state constraints

$$y_i(t) \geq 0, \quad i = 1, \dots, n, \quad t \in (0, T). \tag{15}$$

1/ Write the costate equation and the complementarity relation between the state constraint and multiplier.

ANSWER: *We have for* $i = 1$ *to* n *that*

$$-\mathrm{d}\bar{p}_i(t) = -P'(\delta(t))\rho_i'(\bar{y}_i(t))\bar{u}_i(t)\mathrm{d}t - \mathrm{d}\mu(t), \tag{16}$$

with final value $\bar{p}_i(T) = -a_i$. *Also* $\mathrm{d}\mu_i(t) \geq 0$ *and* $\int_0^T \bar{y}_i(t)\mathrm{d}\mu_i(t) = 0$, *meaning that* $\mathrm{d}\mu_i$ *has support on the contact set of the state constraint* $\bar{y}_i \geq 0$.

2/ Show that $\bar{p}_i(t)$ is nondecreasing, with negative values.

ANSWER: *By* (6), $\mathrm{d}\bar{p}_i(t)$ *is nonnegative with negative final value* $-a_i$. *The result follows.*

3/ Can we extend the results of questions 7-8 of the first problem ?

ANSWER: *If* P *is the identity mapping, then* $H_{u_i} = -(\rho_i(\bar{y}_i(t)) + \bar{p}_i(t))$ *(same expression as for the problem without state constraints) and*

$$\mathrm{d}H_{u_i} = -\frac{d}{dt}(\rho_i(\bar{y}_i(t)))\mathrm{d}t - \mathrm{d}\bar{p}_i(t) = -\rho_i'(\bar{y}_i(t))b_i(t))\mathrm{d}t - \mathrm{d}\mu_i(t) \leq -\rho_i'(\bar{y}_i(t))b_i(t))\mathrm{d}t. \tag{17}$$

When $b_i(t) > 0$ *and* $\rho_i'(\bar{y}_i(t)) > 0$, *we still have that* H_u *is positive and then negative (or has a constant sign). So, by the Hamiltonian inequality, the optimal policy is to have* $\bar{u}_i(t) = 0$ *over* $[0, t_i]$ *and* $\bar{u}_i(t) = u_i^M$ *over* $[t_i, T]$, *for some* $t_i \in [0, T]$.