Exam of the Optimal Control course V 05/12/2018 J.F. Bonnans and H. Zidani
Ensta : Third year course SOD311 ; U. Paris-Saclay Optimization Master

Nov. 16, 2018 ; PART I ONLY

Authorized documents : lecture notes and personal notes of this course.
Arguments should be concise and stated carefully, without neglecting partial answers.

Problem 1 (Ensta and Master students) : **Cell growth control**

Consider the state equation

$$\dot{x}(t) = A(x(t)) - au(t)x(t), \quad t \in (0, T); \quad x(0) = x_0 > 0. \tag{1}$$

Here $x(t) \in \mathbb{R}$ is the number of cells, $u(t)$ is the drug concentration, $a > 0$, $x_0 > 0$ and $T > 0$ are given, $A : \mathbb{R} \to \mathbb{R}$ is of class C^∞, *increasing and strictly concave*, and $A(0) = 0$. The control constraint is that, for some $u_M > 0$:

$$u(t) \in [0, u_M], \quad \text{for a.a. } t \in (0, T). \tag{2}$$

We minimize the cost $x(T)$, and admit that $x(t) > 0$ for all $t \in (0, T)$.

1/ Give the expression of the pre-Hamiltonian and of the costate equation, and show that $\beta = 1$.

Answer: $H = p(A(x) - aux)$. *The costate equation is*

$$-\dot{p}(t) = (A'(x) - au)p; \quad p(T) = \beta. \tag{3}$$

Necessarily $\beta = 1$ *since there is no final state constraint.*

2/ Show that $p(t) > 0$ for all $t \in (0, T)$.

Answer:
Since $\beta = 1$, $p(T) = 1$. *If* $p(t_0) = 0$ *for some* t_0, *then* $p(t) = 0$ *for all* t *by* (3), *which is impossible since* $p(T) = 1$. *Since* p *is continuous it may therefore have only positive values.*

3/ Show that the optimal control is $\bar{u}(t) = u_M$, for all $t \in (0, T)$.

Answer: *For all* $t \in (0, T)$, *since* $x(t) > 0$ *and* $p(t) > 0$, $\bar{H}_u(t) = -ax(t)p(t) < 0$. *By the Hamiltonian inequality, the optimal control is almost always equal to its upper bound.*

4/ We introduce the additional constraint $\int_0^T u(t)\mathrm{d}t \le Z_T$ for some given $Z_T > 0$. Show that this is equivalent to introduce an additional state variable ζ and the additional final state constraint as follows :

$$\dot{\zeta}(t) = u(t), \quad t \in (0, T); \quad \zeta(0) = 0, \quad \zeta(T) \le Z_T. \tag{4}$$

Answer: *Obvious.*

5/ Denoting by p, q the costates associated with the states x, ζ, and by Ψ the multiplier associated with the final state constraint. Give the expression of the pre-Hamiltonian and costate equations of this problem.

ANSWER:

$$H(\beta, u, x, \zeta, p, q) = p(A(x) - aux) + qu. \tag{5}$$

$$-\dot{p}(t) = (A'(x) - au)p; \text{ for a.a. } t \in (0,T), \quad p(T) = \beta, \tag{6}$$

$$-\dot{q}(t) = 0, \text{ for a.a. } t \in (0,T), \quad q(T) = \Psi. \tag{7}$$

6/ Show that

$$\beta + \Psi > 0; \quad q(t) = \Psi, \ t \in (0,T); \ \Psi \geq 0; \quad \Psi(\zeta(T) - Z_T) = 0. \tag{8}$$

ANSWER: *Immediate consequence of the PMP.*

7/ Assuming that $\Psi = 0$, give the expression of the optimal control.

ANSWER: *Necessarily $\beta = 1$ and $q(t) = 0$ for all t, so that (by the same arguments as before) $\bar{H}_u(t) = -ax(t)p(t) < 0$, and so, again, $u(t) = u_M$ for a.a. t.*

8/ Show that $\beta = 0$ is impossible.

ANSWER: *We would have for all t, $p(t) = 0$ and $q(t) = \Psi > 0$, so that $\bar{H}_u(t) = \Psi > 0$, so that $u(t) = 0$ for a.a. t. But then $\zeta(T) = 0 < Z_T$, so that $\Psi = 0$, which is a contradiction.*

9/ Deduce from the strict concavity of $A(x)$ that, for all $x > 0$, $A(x) > A'(x)x$.

ANSWER: *By the mean value theorem, since $A(0) = 0$, $A(x) = A'(y)x$ with $y \in (0,x)$. Since A is strictly concave, $A'(y) > A'(x)$. The conclusion follows.*

10/ In the sequel we assume that $\Psi > 0$. Show that $\bar{H}_u(t)$ is monotone, i.e., either nondecreasing or nonincreasing [HINT : COMPUTE ITS TIME DERIVATIVE]. Deduce the expression of the optimal control.

ANSWER: *Since $\beta = 1$, by the same arguments as before, $p(t) > 0$ for all t. Since $\bar{H}_u(t) = q(t) - ax(t)p(t) = \Psi - ax(t)p(t)$, we have that, skipping time arguments :*

$$\frac{1}{a}\dot{\bar{H}}_u = -\frac{d}{dt}(xp) = -p(A(x) - aux) + xp(A'(x) - au) = -p(A(x) - A'(x)x) < 0. \tag{9}$$

So, $\bar{H}_u(t)$ is decreasing. If its sign is constant, since $\Psi > 0$, the final state constraint is active so that $\bar{H}_u(t) < 0$ for all $t \in (0,T)$ and the control has always its maximal value; this can happens only if $Z_T = Tu_M$. Otherwise, $Z_T < Tu_M$ (since the final state constraint is active), and $\bar{H}_u(t)$ has at most one change of sign at some time $\tau \in (0,T)$. Then $\bar{u}(t) = 0$ if $t < \tau$, and $\bar{u}(t) = u_M$ if $t > \tau$. Observe that τ is determined by the relation $(T - \tau)u_M = Z_T$.

Problem 2 (Master students) : Cell growth control with state constraints

The model is the same as before, with an additional state constraint : minimize $x(T)$ subject to

$$\dot{x}(t) = A(x(t)) - au(t)x(t), \ t \in (0,T); \ x(0) = x_0 > 0, \tag{10}$$

$$\dot{\zeta}(t) = u(t) \in [0, u_M], \ t \in (0,T); \ \zeta(0) = 0, \tag{11}$$

with the running and final constraint

$$x(t) \leq x_M; \ t \in (0,T); \ \zeta(T) \leq Z_M. \tag{12}$$

We assume that $x_0 < x_M$ and $Z_M > 0$.

1/ Give the expression of the pre-Hamiltonian and costate equation.

ANSWER: *As before,* $H = p(A(x) - aux) + qu$. The costate equation is

$$-\mathrm{d}p(t) = H_x \mathrm{d}t + \mathrm{d}\mu(t) = (A'(x(t)) - au(t))p(t)\mathrm{d}t + \mathrm{d}\mu(t); \quad p(T) = \beta, \tag{13}$$

$$-\dot{q}(t) = H_\zeta = 0; \quad q(T) = \Psi. \tag{14}$$

2/ Check that $p(t) \geq 0$, for all $t \in [0, T)$, with strict inequality if $\beta = 1$. [HINT : CHECK THAT $\mathrm{d}p(t) \leq c|p(t)|\mathrm{d}t$, FOR SOME $c > 0$].

ANSWER: *Observe that*

$$\mathrm{d}p(t) = -(A'(x(t)) - au(t))p(t)\mathrm{d}t - \mathrm{d}\mu(t) \leq -(A'(x(t)) - au(t))p(t)\mathrm{d}t \leq c|p(t)|\mathrm{d}t, \tag{15}$$

for some $c > 0$*. Now assume that* $p(t_0) < 0$*, with* $t_0 \in [0, T)$*. Let*

$$t_1 := \sup\{t' \geq t_0;\ p(t) \leq 0 \text{ over } (t_0, t')\}. \tag{16}$$

Since the jumps of p *are nonpositive,* $p(t_1^+) \leq 0$*, and as long as* $p(t) \leq 0$*, we have that* $\mathrm{d}p(t) \leq -cp(t)\mathrm{d}t$*, so that necessarily* $t_1 = T$ *and* $p(T) \leq e^{c(t_0 - t_1)}p(t_0) < 0$*. This contradicts the final condition on* p*. If* $\beta > 0$ *we obtain a contradiction by the same arguments when assuming that* $p(t_0) \leq 0$*, with* $t_0 \in [0, T)$*.*

3/ Compute the optimal control in the case when $\Psi = 0$.

ANSWER: *By the above results, for a.a.* t*,* $\bar{H}_u(t) < 0$ *and so,* $u(t) = u_M$*. This is feasible, and optimal, iff* $Z_M \geq T u_M$*.*

4/ In the sequel we assume that the state constraint becomes active for the first time at time $t_1 \in (0, T)$, that $\beta = 1$, and that

$$A(x_M) - au_M x_M < 0, \tag{17}$$

meaning that the state constraint cannot remain active when the control is maximal. Show that, over $(0, t_1)$, $\bar{H}_u(t)$ is monotonic, with positive values, and that the control is a.e. equal to 0.

ANSWER: *By the same arguments as for problem 1 (without state constraints),* $\bar{H}_u(t)$ *is decreasing over* $(0, t_1)$*, and if it has negative values over a non negligible subset of* $(0, t_1)$*, then the control is maximal over* $(t_1 - \varepsilon, t_1)$ *for some* $\varepsilon > 0$*. Then, by* (17)*, the state constraint cannot be active near* t_1*, which gives a contradiction.*

5/ Let t_2 be the minimal time for which $x(t) < x_M$, for all $t \in (t_2, T]$. Show that $u(t) = u_M$ over (t_2, T).

ANSWER: *Same type of argument : over* (t_2, T)*,* $\bar{H}_u(t)$ *is decreasing since the state constraint is not active. If it had positive values over* $(t_2, t_2 + \varepsilon)$ *for some* $\varepsilon > 0$*, then over* $(t_2, t_2 + \varepsilon)$ *the control would have zero value, and therefore the state constraint would be violated. So, over* (t_2, T)*,* $\bar{H}_u(t)$ *has negative values and the conclusion follows with the Hamiltonian inequality.*

6/ Show that the state constraint is active over $[t_1, t_2]$. [HINT : ASSUME THE CONTRARY AND DEDUCE A CONTRADICTION BASED ON THE ANALYSIS OF PROBLEM 1.]

ANSWER: *If the state constraint is not active at* $t_0 \in (t_1, t_2)$*, it is not active over a maximal interval* (t', t'') *such that* $t_1 \leq t' < t_0 < t'' \leq t_2$*. By the analysis of problem 1, over* (t', t'')*,* *the control is first minimal then maximal with possibly one of these two phases empty. But since necessarily the state constraint is active at times t' and t'', a firstly minimal control would violate the state constraint for some $t > t'$ since $A(x) > 0$ when $x > 0$, and a maximal control for $t < t''$ close to t'' would contradict $x(t'') = x_M$ in view of (17). The conclusion follows.*

7/ Explain how to compute t_1 and t_2, and the control over (t_1, t_2).

ANSWER: *The time t_1 is the unique one for which the solution of the ODE $\dot{x}(t) = A(x(t))$, with initial condition $x(0) = x_0$ takes value x_M. Over (t_1, t_2), we have that $0 = \dot{x}(t) = A(x_M) - aux_M$ so that the control has the constant value $u_c := A(x_M)/(ax_M)$. Then, by the final state constraint*

$$u_c(t_2 - t_1) + u_M(T - t_2) = Z_M, \tag{18}$$

which gives t_2 as function of t_1 (well defined since, by our hypotheses, $u_c < u_M$).

8/ Show that, over (t_1, t_2), p is constant, and μ is affine.

ANSWER: *Over (t_1, t_2) we must have $0 = \bar{H}_u(t) = \Psi - ax_M p(t)$, so that $p(t) = p_c := \Psi/(ax_M)$. Then by the costate equation*

$$0 = \mathrm{d}p(t) = (A'(x_M) - au_c)p_c\mathrm{d}t + \mathrm{d}\mu(t). \tag{19}$$

The conclusion follows.

9/ Do p and μ have jumps at time t_1 and t_2 ?

ANSWER: *Since μ is nondecreasing, and p and μ have jumps of opposive value, the jumps of p are negative. If p had a jump at time t_1, we would have*

$$[H_u(t_1)] = -ax(t_1)[p(t_1)] > 0. \tag{20}$$

Since $H_u(t) = 0$ over (t_1, t_2), this would imply, for $t < t_1$ close to t_1, $H_u(t) < 0$ and therefore $u(t) = u_M$ which gives a contradiction with the previous results. By a similar argument at time t_2 we show that μ and p have no jump.
NOTA : it easily follows that μ and p have essentially bounded derivatives.