Nov. 14, 2019 ; PART I ONLY

Authorized documents : lecture notes and personal notes of this course.
Arguments should be concise and stated carefully, without neglecting partial answers.

Problem 1 (Ensta and Master students) : **Optimal fish harvesting**

Consider the state equation

$$\dot{y}(t) = r(y(t)) - u(t), \quad t \in (0,T); \quad y(0) = y_0 > 0. \tag{1}$$

Here $y(t) \in \mathbb{R}$ is the number of fish, $u(t)$ is the harvesting rate, $y_0 > 0$ and $T > 0$ are given, $r : \mathbb{R} \to \mathbb{R}$ is of class C^1, and satisfies $r(0) = 0$ and $r'(0) > 0$. The control constraint is, for some $u_M > 0$:

$$u(t) \in [0, u_M], \quad \text{for a.a. } t \in (0,T). \tag{2}$$

We minimize the opposite of total harvesting :

$$J(u,y) := -\int_0^T u(t)\mathrm{d}t, \tag{3}$$

with final constraint, for some $y_T > 0$:

$$y(T) \geq y_T. \tag{4}$$

We will ignore the implicit state constraint $y(t) \geq 0$ (always satisfied in view of the state equation and final constraint).

1/ Give the expression of the pre-Hamiltonian and of the costate equation (including the final condition and the conditions on the multiplier). We denote the costate by p, and the multiplier associated with the initial and final constraint by Ψ_0 and Ψ_F, resp.

Answer: *We have that $H = -\beta u + p(r(y) - u)$. The costate equation is therefore*

$$-\dot{p}(t) = p(t)r'(y(t)) \text{ a.e.}; \quad p(T) = -\Psi_F; \quad \Psi_F \geq 0; \quad \Psi_F(y(T) - y_T) = 0. \tag{5}$$

2/ Show that the nontriviality condition (for the Pontryagin multiplier) is equivalent to

$$\beta + |\Psi_F| > 0. \tag{6}$$

Answer: *The nontriviality condition is*

$$\beta + |\Psi_0| + |\Psi_F| > 0. \tag{7}$$

It is enough to check that (7) implies (6), and for this to check that $\Psi_0 = 0$ if $\Psi_F = 0$. By the costate equation we then have $p(t) = 0$ for all t and in particular $0 = -p(0) = \Psi_0$.

3/ Compute the solution in the case when $\Psi_F = 0$.

ANSWER: *If $\Psi_F = 0$, then $\beta = 1$ in view of (6), and (by the costate equation) $p(t) = 0$ for all t, so that $H_u(t) = -1$, so that $u(t) = u_M$ for all t. In that case, a maximal harvesting is compatible with the final state constraint.*

4/ **In the sequel we assume that** $\Psi_F > 0$. Show that $p(t) < 0$, for all $t \in (0,T)$.

ANSWER: *If $p(t_0) = 0$ for some $t_0 \in (0,T)$, then by the Cauchy-Lipschitz theorem, $p(t) = 0$ for all t, contradicting the condition $p(T) = -\Psi_F \neq 0$. So $p(t)$ remains negative for all t.*

5/ Discuss the case when $\beta = 0$.

ANSWER: *Then, in view of the previous question, $H_u(t) = -p(t)$ is always positive, so that $u(t) = 0$ for all t, meaning that the final state constraint implies to have a zero harvesting rate.*

6/ Let $(t_1, t_2) \subset (0,T)$ be such that $r'(y(t)) > 0$ (resp. < 0), for all $t \in (t_1, t_2)$. Show that, over (t_1, t_2), $p(t)$ is increasing (resp. decreasing) and that the control variable has value in $\{0, u_M\}$ with at most one switch (give the value of the control before and after the switch).

ANSWER: *Since $p(t) < 0$ for all t, $\dot{p}(t) = -p(t)r'(y(t))$ has the same sign as $r'(y(t))$. The strict monotonicity of p follows. Therefore $H_u(t) = -1 - p(t)$ is also strictly monotonous. If $r'(y(t)) > 0$, $H_u(t)$ is decreasing so that the control switches from 0 to u_M. If $r'(y(t)) < 0$, $H_u(t)$ is increasing so that the control switches from u_M to 0.*

7/ Can we compute the optimal control when $r(y) = r_1(y)$, where $r_1(y) := y/(1+y)$.

ANSWER: *Then we always have $r'(y(t)) > 0$. By the previous question, either the control has constant value 0 or u_M, or there exists $t_0 \in (0,T)$ such that $u(t) = 0$ for all $t < t_0$, and $u(t) = u_M$ for all $t < t_0$. The value of t_0 can be estimated by checking that $y(T) = y_T$.*

8/ Assume that we have a *singular arc*, i.e., $H_u(t) = 0$ for all $t \in (t_1, t_2)$ with $0 \leq t_1 < t_2 \leq T$. Show that over the singular arc, $p(t)$ has a constant value (to be determined) and that $r'(y(t)) = 0$.

ANSWER: *An obvious consequence of the condition $-1 - p(t) = H_u(t) = 0$ is that $p(t) = -1$ over the singular arc. Consequently $0 = \dot{p}(t) = -p(t)r'(y(t))$. Since $p(t) < 0$ for all t, this implies $r'(y(t)) = 0$.*

9/ In the next questions we assume that $r(y) = r_2(y)$, with $r_2(y) = y(2-y)$. What is the value of the state and control over a singular arc ? Is this compatible with the control constraint ?

ANSWER: *We have that $r'(y) = 2(1-y)$ so that the state is equal to 1 over a singular arc. By the state equation, $u(t) = r(1) = 1$. So a singular arc cannot occur if $u_M < 1$.*

10/ We assume that $y_0 \in (0,1)$, $y_T \in (1,2)$ and we look for a trajectory satisfying Pontryagin's principle, of the following form : there exists $0 < t_1 < t_2 < T$ such that $u(t) = 0$ if $t < t_1$, (t_1, t_2) is a singular arc, and $u(t) = 0$ if $t \in (t_2, T)$. Explain how to compute the state and costate and check all conditions in Pontryagin's principle.

ANSWER: *Since the control is determined over each of the three arcs, we compute t_1 as the time for which the state has value 1 when integrating the state equation starting from $t = 0$. We proceed similarly for t_2, integrating backwards with final condition $y(T) = y_T$. We need to check that $t_1 < t_2$.*

Over the singular arc, $p(t) = -1$. Then we integrate p over (t_2, T) and $(0, t_1)$ (the latter backwards) with "initial" conditions $p(t_1) = p(t_2) = -1$.
We next discuss the Hamiltonian inequality. By construction, $H_u(t) = 0$ over the singular arc. Over $(0, t_1)$, $y(t) < 1$ so that $r'(y(t)) > 0$, and $p(t)$ is increasing, so $H_u(t)$ is decreasing (with zero value at time t_1), and has therefore positive values which is compatible with the zero value for the control. This compatibility also holds over (t_2, T) using similar arguments. Since $-\Psi_F = p(T) < 0$, all conditions of Pontryagin's principle are satisfied.

Problem 2 (Master students only) : **Optimal fish harvesting with state constraint**

We consider the same problem with the additional state constraint

$$y(t) \leq y_M, \quad \text{for all } t \in [0, T], \tag{8}$$

and $y_0 < y_T < y_M$. We call "state constrained arc" an interval (t_1, t_2) with $0 < t_1 < t_2 < T$ such that $y(t) = y_M$, for all $t \in [t_1, t_2]$.

1/ Give the expression of the costate equation and of the Hamiltonian inequality.

Answer: *The costate equation is*

$$-\mathrm{d}p(t) = p(t)r'(y(t)) + \mathrm{d}\mu(t), \tag{9}$$

with the same initial and final conditions for p, and

$$\mathrm{d}\mu(t) \geq 0, \quad y(t) \leq y_M, \quad \int_0^T (y(t) - y_M)\mathrm{d}\mu(t) = 0. \tag{10}$$

The Hamiltonian inequality gives : if $-1 - p(t) > 0$, then $u(t) = 0$; if $-1 - p(t) < 0$, then $u(t) = u_M$.

2/ In the sequel we assume that $\beta = 1$. Give the value of $u(t)$ along a state constrained arc and show that if such an arc exists, then $r(y_M) \leq u_M$.

Answer: *Along a state constrained arc, w must have $0 = \dot{y}(t) = r(y_M) - u(t)$ so that $u(t) = r(y_M)$. Necessarily $u(t) \leq u_M$, whence the conclusion.*

3/ In the sequel we assume that $r(y_M) < u_M$. Give the value of $p(t)$ and $\dot{\mu}(t)$ along a state constrained arc and show that if such an arc exists, then $r'(y_M) \geq 0$.

Answer: *Over a state constrained arc we have that $u(t) \in]0, u_M[$ implying that $0 = H_u(t) = -1 - p(t)$. Therefore $p(t) = -1$, so that $0 = -\mathrm{d}p(t) = -r'(y_M)\mathrm{d}t + \mathrm{d}\mu(t)$ and so, there exists $\dot{\mu}(t) = r'(y_M)$. Since μ is nondecreasing, the conclusion follows.*

4/ Find the sign of the jumps $[u(\tau)]$ (whenever they exist).

Answer: *We have that*

$$[H_u(\tau)] = -[p(\tau)] = [\mu(\tau)] \geq 0, \tag{11}$$

and by the Hamiltonian inequality, $[H_u(\tau)]$ and $[u(\tau)]$ have opposite sign. So, $[u(\tau)] \leq 0$.

5/ Show that either $\mu(t)$ or $u(t)$ is continuous at any time τ where $[u(\tau)]$ exists.

Answer: *Assume on the contrary that $\nu := [\mu(\tau)] > 0$, then $y(t)$ has a maximum at time τ, so that $[\dot{y}(\tau)] \leq 0$, implying $[u(\tau)] \geq 0$, and the opposite inequality is provided by the previous question.*