Authorized documents : lecture notes and personal notes of this course.
Arguments should be concise and stated carefully, without neglecting partial answers.

Problem 1 (Ensta and Master students) : **SIR epidemic model**

Consider the state equation, for $t \in (0,T)$:

$$\begin{aligned} \dot S(t) &= -u(t)S(t)I(t) - \kappa v(t)S(t), \\ \dot I(t) &= u(t)S(t)I(t) - \alpha I(t), \end{aligned} \tag{1}$$

with given initial conditions

$$S(0) = S_0 > 0; \quad I(0) = I_0 > 0. \tag{2}$$

The state $S(t)$ (resp. $I(t)$) is the size of the susceptible (resp. infected) population. There are two control variables, the confinement rate $u(t)$ and the vaccination rate $v(t)$. Two given parameters are $\alpha > 0$, and $\kappa \geq 0$ (vaccination efficiency). The control constraint are, for some $0 < \check u < \hat u$ and $0 < \hat v$:

$$\check u \leq u(t) \leq \hat u; \quad 0 \leq v(t) \leq \hat v, \quad \text{for a.a. } t \in (0,T). \tag{3}$$

We admit that, given a feasible control, the state equation has a unique bounded, positive solution $y = (S, I)$. Assuming that the death rate is proportional to $I(t)$, minimizing the death toll up to some horizon $T > 0$ amounts to minimizing

$$J(u,v,y) := \int_0^T I(t)\mathrm{d}t. \tag{4}$$

1/ Give the expression of the pre-Hamiltonian and of the costate equation, including the final condition. Check that the multiplier β can be taken equal to 1. We denote the costate components by p_S, p_I.

Answer: *Since the final state is free, $\beta = 1$ so that $H = I - p_S(uSI + \kappa vS) + p_I(uSI - \alpha I)$. The costate equation is therefore*

$$\begin{aligned} -\dot p_S(t) &= H_S = (p_I(t) - p_S(t))u(t)I(t) - \kappa p_S(t)v(t) \text{ a.e.}; \quad p_S(T) = 0; \\ -\dot p_I(t) &= H_I = 1 + (p_I(t) - p_S(t))u(t)S(t) - \alpha p_I(t) \text{ a.e.}; \quad p_I(T) = 0. \end{aligned} \tag{5}$$

2/ Compute the partial derivatives H_u and H_v. When can we deduce from these expressions the control as function of state and costate ?

Answer: *We have that*

$$\begin{aligned} H_u(t) &= (p_I(t) - p_S(t))S(t)I(t) \text{ a.e.}; \\ H_v(t) &= -\kappa p_S(t)S(t) \text{ a.e.} \end{aligned} \tag{6}$$

So, since $S(t)I(t) > 0$, $u(t) = \check u$ (resp. $\hat u$) if $p_I(t) > p_S(t)$ (resp. $p_I(t) < p_S(t)$), and since $S(t) > 0$, $v(t) = 0$ (resp. $\hat v$) if $p_S(t) < 0$ (resp. > 0).

3/ Show that $\dot{p}(T)$ exists and compute it. Deduce, for t close to T, the signs of $q(t) := p_I(t) - p_S(t)$ and the value of $u(t)$.

Answer: *The state is bounded since $\dot{S}(t) + \dot{I}(t) \leq 0$, implying $S(t) + I(t) \leq S(0) + I(0)$. Therefore, so is the costate, solution of a linear ODE with bounded coefficients. It follows that $\dot{p}$ is bounded. Since $p(T) = 0$, this implies $|p(t)| = O(T-t)$, so that $\dot{p}(t) = (0\ -1)^{\dagger} + O(T-t)$. Consequently, p has derivative at T : $\dot{p}_S(T) = 0$, $\dot{p}_I(T) = -1$. So, $q(T) = 0$ and $\dot{q}(T) < 0$. Whence, for t close to T, $q(t) > 0$ and $u(t) = \check{u}$.*

4/ For t close to T, evaluate the sign of $p_S(t)$, and deduce the value of $v(t)$.

Answer: *We have that for t close to T :*

$$-\dot{p}_S(t) = p_I(t)u(t)I(t) + o(T-t) = (T-t)u(t)I(T) + o(T-t) > 0. \tag{7}$$

Since $I(T) > 0$ and $p_S(T) = 0$, we deduce that $p_S(t) > 0$, so that $v(t) = \hat{v}$ for t close to T.

Problem 2 (Master students only) : **Model with limited hospital capacities**

We consider the same problem, but with the additional state constraint, for some given $I_M > 0$:

$$I(t) \leq I_M. \tag{8}$$

1/ What is the costate equation ?

Answer: *Dynamics and transversality conditions for p_S unchanged ; those for p_I are*

$$-\mathrm{d}p_I(t) \quad = (1 + (p_I(t) - p_S(t))u(t)S(t) - \alpha p_I(t))\mathrm{d}t + \mathrm{d}\mu(t).\ a.e.; \quad p_I(T) = 0. \tag{9}$$

2/ In the sequel we assume that for some $0 < t_a < t_b \leq T$, the state constraint is active over (t_a, t_b). Give the expression of $u(t)$ for $t \in (t_a, t_b)$. Is $u(t)$ decreasing or increasing ?

Answer: *For $t \in (t_a, t_b)$, $0 = \dot{I}(t) = (u(t)S(t) - \alpha)I(t)$; since $u(t) = \alpha/S(t)$, and $S(t)$ is decreasing since it has a negative derivative, $u(t)$ is increasing.*

3/ Show that $p_I(t) = p_S(t)$, for all $t \in (t_a, t_b)$.

Answer: *Since $u(t)$ is increasing, it is out of bounds over (t_a, t_b). By the Hamiltonian inequality, $H_u(t) = 0$. Since $S(t)I(t) > 0$, the result follows.*

4/ Show that the state constaint multiplier μ, restricted to (t_a, t_b), belongs to $W^{1,\infty}(t_a, t_b)$, and give the expression of $\dot{\mu}$.

Answer: *In view of the previous question, over (t_A, t_b),*

$$0 = \mathrm{d}p_I(t) - \dot{p}_S(t)\mathrm{d}t = (H_S(t) - H_I(t))\mathrm{d}t - \mathrm{d}\mu(t). \tag{10}$$

Using that $p_I(t) = p_S(t)$, we deduce that there exists,

$$\dot{\mu}(t) = H_S(t) - H_I(t) = -\kappa p_S(t)v(t) - 1 + \alpha p_S(t) = p_S(t)(\alpha - \kappa v(t)) - 1. \tag{11}$$

Note that we must have $0 \leq \dot{\mu}(t)$, meaning that

$$p_S(t)(\alpha - \kappa v(t)) \geq 1. \tag{12}$$

5/ We assume in the sequel that $\kappa\hat{v} < \alpha$. Check that $p_I(t) = p_S(t) > 0$ over an arc where the state constraint is active.

ANSWER: *Consequence of previous question, especially* (12)*, which implies also* $p_I(t) \geq 1/(\alpha - \kappa v(t)) \geq 1/\alpha$.

6/ Check that $v(t) = \hat{v}$ over an arc where the state constraint is active.

ANSWER: *In view of the previous question, over this arc,* $H_v < 0$*. The conclusion follows.*