Optimal Control
Cours A8-2 - Ensta Paris Tech
Séances 1 à 3

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CHAPTER 1

First step: calculus of variations

1. Calculus in \( \mathbb{R}^n \) and in Banach spaces

1.1. Some notations. We denote by \( \mathbb{R}^n \) the Euclidian space of dimension \( n \), whose elements are identified with vertical vectors, with norm \( |x| := (\sum_{i=1}^{n} x_i^2)^{1/2} \). The dual space \( \mathbb{R}^n^* \) is identified with the set of \( n \) dimensional horizontal vectors. We denote by \( A^\dagger \) the transpose of a matrix \( A \), and by \( x \cdot y := \sum_{i=1}^{n} x_i y_i \) the scalar product of \( x \) and \( y \). Other useful norms in \( \mathbb{R}^n \) are the \( \ell^{s} \) norm \( |x|_s := (\sum_{i=1}^{n} |x_i|^s)^{1/s} \), for \( s \in [1, \infty] \), and the uniform norm \( |x|_\infty := \max\{|x_i|, 1 \leq i \leq n\} \). Note that norms in \( \mathbb{R}^n \) are denoted with a single bar.

Let \( X \) be a Banach space (a normed vector space that is complete, i.e., every Cauchy sequence has a limit), with norm denoted by \( \|x\|_X \) or \( \|x\| \) if there is no ambiguity. If \( Y \) is another Banach space, we denote by \( L(X,Y) \) the set of linear continuous mappings \( X \to Y \). Endowed with the norm

\[
\|A\| := \sup\{\|Ax\|_Y; \ \|x\|_X \leq 1\},
\]

\( L(X,Y) \) is a Banach space. Note that a linear mapping \( A : X \to Y \) is continuous iff the above r.h.s. is finite.

Exercice 1.1. Prove that, similarly, if \( Z \) is a third Banach space and \( a : X \times Y \to Z \) is bilinear, then it is continuous iff

\[
\|a\| := \sup\{\|a(x,y)\|_Z; \ \|x\|_X \leq 1, \ \|y\|_Y \leq 1\},
\]

and that the set of continuous bilinear forms endowed with this norm is a Banach space.

The set of linear continuous forms over \( X \) (linear and continuous applications \( X \to \mathbb{R} \)) is denoted by \( X^* \), and the action of \( x^* \in X^* \) over \( x \in X \) is denoted by \( \langle x^*, x \rangle_X \). When endowed with the dual norm

\[
\|x^*\|_* := \sup\{\langle x^*, x \rangle_X; \ \|x\|_X \leq 1\},
\]

\( X^* \) is a Banach space. We say that the Banach space \( X \) is a Hilbert space if there exists a symmetric continuous bilinear form \( a(\cdot, \cdot) \) over \( X \) such that

\[
\|x\|^2 = a(x, x), \text{ for all } x \in X.
\]

An Hilbert space \( H \) is endowed with the scalar product

\[
\langle x, x' \rangle_X := a(x, x'), \text{ for all } x, x' \text{ in } X.
\]

By the Fréchet-Riesz representation theorem, if \( X \) is a Hilbert space, we have that

Exercice 1.2. Prove the Fréchet-Riesz representation theorem by showing that we can take for \( x' \) the unique solution of the optimization problem below:

\[
\min_{x \in X} \frac{1}{2}\|x\|_X^2 - \langle x^*, x \rangle_X.
\]

In the sequel \( X \) is a Banach space. If \( A \) and \( B \) are subsets of \( X \), their Minkowski sum and difference are

\[
A + B := \{a + b; \ a \in A, b \in B\}; \quad A - B := \{a - b; \ a \in A, b \in B\}.
\]
If $E \subset \mathbb{R}$ we define the product
\begin{equation}
(1.9) \quad EA := \{ea; e \in E, a \in A\}.
\end{equation}

If $f : X \to Y$, $A \subset X$ and $B \in Y$, we set
\begin{equation}
(1.10) \quad f(A) := \{f(a); a \in A\}; \quad f^{-1}(B) := \{x \in \mathbb{R}^n; f(x) \in B\}.
\end{equation}

The interior and boundary of $A$ are denoted by
\begin{equation}
(1.11) \quad \text{int}(A) := \{x \in X; y \in A \text{ if } y \text{ is close enough to } x\}; \quad \partial A := A \setminus \text{int}(A).
\end{equation}

The segment $[x, y]$, where $x$ and $y$ belong to $X$ is
\begin{equation}
(1.12) \quad [x, y] := \{\alpha x + (1 - \alpha)y; \alpha \in [0, 1]\}.
\end{equation}

We say that $A$ is convex if
\begin{equation}
(1.13) \quad [x, y] \subset A, \quad \text{for all } x, y \text{ in } A.
\end{equation}

By $\{0\}_X$ we denote the set having for unique element the zero of $X$. The positive (resp. negative) cone of $\mathbb{R}^n$ is the set of element of $\mathbb{R}^n$ whose all coordinates are nonnegative (resp. nonpositive), and is denoted by $\mathbb{R}^n_+$ (resp. $\mathbb{R}^n_-$). More generally, if $X$ is a space of real valued functions (perhaps defined only a.e.), we call positive (resp. negative) cone of $X$ the set of nonnegative (perhaps only a.e.) functions of $X$.

### 1.2. Differential calculus
Let $A, B$ be $n$ dimensional symmetric matrices. We write $A \succeq B$ if $A - B$ is semi positive definite, and $A \succ B$ if $A - B$ is positive definite. Let $f : X \to Y$, where $X$ and $Y$ are Banach spaces. We say that $f$ is differentiable, or Fréchet differentiable, at $x \in \mathbb{R}^n$, if there exists a linear mapping $X \to Y$, denoted by $Df(x)$ or $f'(x)$, and called derivative of $f$ at $x$, such that
\begin{equation}
(1.14) \quad \|f(x + h) - f(x) - f'(x)h\|_Y = o(h)\|h\|_X).
\end{equation}

When $X = \mathbb{R}^n$ and $Y = \mathbb{R}^p$, we identify $f'(x)$ with the usual $p \times n$ Jacobian matrix. When $Y = \mathbb{R}$, and $X$ is an Hilbert space, we denote by $\nabla f(x) := f'(x)^\dagger$ the gradient of $f$ at $x$. If in addition $X = \mathbb{R}^n$, then $\nabla f(x)$ is the element of $\mathbb{R}^n$ whose coordinates are equal to the partial derivatives of $f$ at $x$.

If $Z$ is another Banach space and $f : X \times Y \to Z$, we denote by e.g. $D_xf(x, y)$ or $f_x(x, y)$ its partial derivatives. If in addition $Z = \mathbb{R}$, we denote by $\nabla_x f(x, y)$ its partial gradient, and then, if $X = \mathbb{R}^n$ and $Y = \mathbb{R}^q$, $f_{xy}(x, y)$ is identified with the $n \times q$ matrix with general term $\partial f(x, y)/\partial x_i \partial y_j$.

We recall the Taylor expansion with integral term: if $f : X \to Y$ is $(n + 1)$ times continuous differentiable, with $n \geq 0$, then
\begin{equation}
(1.15) \quad f(x + h) = f(x) + \cdots + \frac{1}{n!}D^n f(x)(h)^n + \int_0^1 (1 - t)^n \frac{n!}{n!} D^{n+1} f(x + th)(h)^{n+1} dt.
\end{equation}

We use this expression mainly for $n = 0$ and $1$. We obtain
\begin{equation}
(1.16) \quad f(x + h) = f(x) + \int_0^1 Df(x + th)h dt,
\end{equation}
\begin{equation}
(1.17) \quad f(x + h) = f(x) + Df(x)h + \int_0^1 (1 - t)D^2 f(x + th)(h)^2 dt.
\end{equation}
1.3. Equality constraints. Let $X$ and $Y$ be Banach spaces, $f : X \to \mathbb{R}$ and $g : X \to Y$ be of class $C^1$. We discuss here the optimization problem
\[
\min_{x \in X} f(x); \quad g(x) = 0.
\]
Let $\bar{x} \in g^{-1}(0)$. We say that $\bar{x}$ is qualified if it satisfies the following regularity condition:
\[
Dg(\bar{x}) \text{ is onto.}
\]

**Remark 1.3.** If $Y = \mathbb{R}^p$, then (1.19) is equivalent to the classical condition of linear independence of the family $\{g_1^\prime(\bar{x}), \ldots, g_p(\bar{x})\}$.

The Lagrangian of problem (1.18) is $L : X \times Y^* \to \mathbb{R}$ defined by
\[
L(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle_Y.
\]

We next state the primal form of first order necessary conditions.

**Theorem 1.4.** Let $\bar{x}$ be a qualified, local solution of (1.18). Then $f'(\bar{x})h = 0$, for all $h \in \text{Ker } g'(\bar{x})$.

**Proof.** Let, on the contrary, assume that there exists $h \in \text{Ker } g'(\bar{x})$ such that $f'(\bar{x})h \neq 0$. Changing if necessary $h$ into $-h$, we may assume that $f'(\bar{x})h < 0$. By the metric regularity theorem [1.41] for $\sigma > 0$ small enough, there exists $x_\sigma \in g^{-1}(0)$ such that $\|x_\sigma - (\bar{x} + \sigma h)\| = O(\|g(\bar{x} + \sigma h)\|) = o(\sigma)$. Since $\bar{x}$ is a local solution, we have that
\[
0 \leq \lim_{\sigma \downarrow 0} \frac{f(x_\sigma) - f(\bar{x})}{\sigma} = f'(\bar{x})h,
\]
contradicting our hypothesis. \qed

We now state the dual form of first order necessary conditions.

**Theorem 1.5.** Let $\bar{x}$ be a qualified, local solution of (1.18). Then there exists a unique Lagrange multiplier $\lambda \in Y^*$ such that
\[
0 = D_x L(\bar{x}, \lambda) = f'(\bar{x}) + g'(\bar{x})^\dagger \lambda.
\]

**Proof.** By theorem 1.4, $f'(\bar{x}) \in (\text{Ker } g'(\bar{x}))^\perp$. Since $g'(\bar{x})$ is surjective, by proposition 1.43 there exists $\lambda \in Y^*$ such that $-f'(\bar{x}) = g'(\bar{x})^\dagger \lambda$, as was to be proved. \qed

**Remark 1.6.** In a finite dimensional setting, the orthogonal to the kernel of a linear mapping is always equal to the image of its transpose. This is no mode true in a Banach space setting, as shows example 1.12.

**Exercise 1.7.** Consider the problem $\min_{x \in \mathbb{R}} \{f(x) : x^2 = 0\}$. Show that $\bar{x} = 0$ is solution. Is there a Lagrange multiplier when (i) $f(x) = x$, (ii) $f(x) = x^2$?

**Definition 1.8.** Let $\bar{x} \in g^{-1}(0)$. We say that $\lambda = (\lambda_0, \lambda_E) \in \mathbb{R}_+ \times Y^*$ is a generalized Lagrange multiplier associated with $\bar{x}$, if setting $L^g(x, \lambda) := \lambda_0 f(x) + \langle \lambda, g(x) \rangle_Y$, we have that
\[
0 = D_x L^g(\bar{x}, \lambda) = \lambda_0 f'(\bar{x}) + g'(\bar{x})^\dagger \lambda_E, \quad \lambda_0 \neq 0.
\]

**Remark 1.9.** The set $\Lambda^g(\bar{x})$ of generalized Lagrange multipliers associated with $\bar{x}$ (defined to be empty if $\bar{x} \not\in g^{-1}(0)$) is a cone. When $\lambda_0 \neq 0$ we can identify the generalized Lagrange multiplier $\lambda^g$ with the ‘classical’ one $\lambda_0^g / \lambda_0$.

**Theorem 1.10.** Let $\bar{x}$ be a local solution of (1.23), with $Y = \mathbb{R}^p$. Then $\Lambda^g(\bar{x})$ is nonempty.

**Proof.** If $\{g_1(\bar{x}), \ldots, g_p(\bar{x})\}$ is linearly independent, then $\bar{x}$ is qualified and the result follows from theorem 1.5. Otherwise, take $\lambda_0 = 0$ and for $\lambda_E$ the nonzero coefficients of a zero linear combination of the $g_1(\bar{x})$. \qed

**Exercise 1.11.** Consider the problem $\min_{x \in \mathbb{R}} \{f(x) : x^2 = 0\}$. Show that $\bar{x} = 0$ is solution. What is the set of (generalized) Lagrange multipliers when (i) $f(x) = x$, (ii) $f(x) = x^2$?
It may happen that no generalized Lagrange multiplier exists when $Y$ is a Banach space.

**Example 1.12.** Take $X = L^2(0, 1), Y = L^1(0, 1), A \in L(X, Y)$ the canonical injection, $g(x) := Ax$, and $f(x) := \langle x^*, x \rangle_H$ for some $x^* \in X \setminus L^\infty(0, 1)$. Since $Y^* = L^\infty(0, 1)$, we have that $L^0 g(x, \lambda) = \lambda_0(x^*, x)_H + (\lambda^E, Ax) = \int_0^1 (\lambda_0 x^*(t) + \lambda^E(t)) x(t) dt$. So, $D_x L^0(\bar{x}, \lambda) = 0$ means that $\lambda_0 x^* + \lambda^E = 0$. Since $\lambda^E \in L^\infty(0, 1)$ and $x^* \in X \setminus L^\infty(0, 1)$, we cannot satisfy this relation with $\lambda \neq 0$. Observe that $A$ has a dense range, but is not surjective.

Note also that $A$ is injective, so that the orthogonal of its kernel is the space $X = L^2(0, 1)$, whereas the image of $A^\dagger$ is $L^\infty$.

2. Sobolev spaces

2.1. Approximation by smooth functions. Let $T > 0$. We need some approximation results of integrable functions by smooth ones. A first result is the following:

**Lemma 1.13.** The set $C([0, T])$ is a dense subset of $L^s(0, T)$, for every $1 \leq s < \infty$.

**Proof.** We say that a measurable function is simple if its range is finite. The space of such functions is a dense subset of $L^s(0, T)$, since the elements of the latter are constructed as limits of simple functions. Any simple function is a linear combinations of characteristic functions of integrable functions. So it is enough to show that characteristic functions are limits in $L^s(0, T)$ of continuous functions.

This holds if $C$ is a closed subset of $[0, T]$, since then its characteristic function $\chi_C$ is a.e. the limit of the continuous functions $(1 - \text{dist}(t, C)/\epsilon)_+$, where $\epsilon \downarrow 0$, and by the dominated convergence theorem, also the limit in $L^s(0, T)$. More generally, if $A$ is a measurable subset of $[0, T]$, for any $k > 0$, there exists a compact $C_k$ of $[0, T]$ such that $C_k \subset A$ and $\text{mes}(A \setminus C_k) < 1/k$. Changing $C_k$ into $\cup_{\ell \leq k} C_\ell$ if necessary we may assume that $C_k$ is nondecreasing, and therefore $\chi_{C_k} \to \chi_A$ in $L^s(0, T)$ by dominated convergence. The conclusion follows.

We denote (whenever it exists) the time derivative of a function $f$ of time $t \in [0, T]$ by $\dot{f}(t)$. Let $D(0, T)$, where $T > 0$, be the set of $C^\infty$ functions with compact support over $[0, T]$. We remind that the support of a continuous function is the set of points where it is nonzero. A function having compact support over $[0, T]$ is for some $\epsilon > 0$, equal to zero over $[0, \epsilon[\cup[0,T]-\epsilon,T]$. A useful way of approximating an integrable function by an element of $D(0, T)$ is the following approximation by a regularizing kernel. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be $C^\infty$, with support in the closed unit ball and integral 1. Set, for $\epsilon > 0$, $\rho_\epsilon(s) := \rho(s/\epsilon) / \epsilon$. This function has integral 1 and support over a closed ball of radius $\epsilon$. The convolution of two integrable functions $f, g$ over $\mathbb{R}$ is the bilinear, symmetric operator $L^1(\mathbb{R}) \times L^1(\mathbb{R}) \to L^1(\mathbb{R})$ defined by

\[
(1.24)\quad f * g(t) := \int_\mathbb{R} f(s) g(t - s) ds = \int_\mathbb{R} f(t - s) g(s) ds.
\]

Note that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. When $g$ is nonnegative with integral equal to 1, we can interpret $f * g(t)$ as an averaging of the values of $f$. In particular, $f_\epsilon := f * \rho_\epsilon$ is a $C^\infty$ function and

\[
(1.25)\quad D^n f_\epsilon = f * D^n \rho_\epsilon, \quad \text{and} \quad f_\epsilon \to f \text{ in } L^1(\mathbb{R}).
\]

If $f \in L^1(0, T)$, we can approximate it by a function of $D(0, T)$. Define first

\[
(1.26)\quad f^\epsilon(t) := f(t) \text{ if } t \in [2\epsilon, T - 2\epsilon], \quad \text{and} \quad 0 \text{ otherwise}.
\]

Then $f^\epsilon * \rho_\epsilon$ (defined by extending $f^\epsilon$ by 0 outside of $[0, T]$), belongs to $D(0, T)$ and converges to $f$ in $L^1(0, T)$.

2.2. Weak derivatives. Let $f \in L^1(0, T)$. If there exists $g \in L^1(0, T)$ tel que

\[
(1.27)\quad \int_0^T g(t) \varphi(t) dt = - \int_0^T f(t) \dot{\varphi}(t) dt, \quad \text{for all } \varphi \in \mathcal{D}(0, T),
\]

we say that $g$ is the weak derivative of $f$. This derivative is uniquely defined, in view of the following lemma.
Lemma 1.14. Let $g \in L^1(0, T)$ be such that
\begin{equation}
\int_0^T g(t)\varphi(t)dt = 0, \text{ for all } \varphi \in \mathcal{D}(0, T).
\end{equation}
Then $g = 0$ a.e.

Proof. Consider the following truncation of $g$, $h(t) := \max(-1, \min(1, g(t)))$. By \[1.13\], there exist a sequence $h_k$ in $\mathcal{D}(0, T)$ converging to $h$ in $L^1(0, T)$. Let $F : C^\infty : \mathbb{R} \to \mathbb{R}$ be such that $F(y) = y$ for all $y \in [-1, 1]$, and $|F(y)| \leq 2$ for all $y$. Since $h$ has value in $[-1, 1]$, changing if necessary $h_k$ into $F(h_k)$, we may assume that $h_k$ is bounded in $L^\infty(0, T)$. Since $|g(t)h_k(t)| \leq 2|g(t)|$ and $g(t)h_k(t) \to \min(g^2(t), |g(t)|)$ a.e., by the dominated convergence theorem, we have that
\begin{equation}
0 = \lim_k \int_0^T g(t)h_k(t)dt = \int_0^T \min(g^2(t), |g(t)|)dt,
\end{equation}
proving that $g = 0$ a.e., as was to be shown. \hfill \Box

Remark 1.15. If $f \in C^1([0, T])$, then the weak and classical derivatives of $g$ coincide a.e.

In view of the above remark we will also denote the weak derivative of $f \in L^1(0, T)$, whenever it exists, by $\dot{f}$.

Remark 1.16. Define $L^1_{loc}(0, T)$ as the set of measurable functions over $[0, T]$ whose restriction over $[\varepsilon, T - \varepsilon]$ is, for any $\varepsilon \in [0, \frac{1}{2}T]$, integrable. It is easily checked that the above lemma still holds if we only assume that $g \in L^1_{loc}(0, T)$.

The weak derivative is obviously a linear operator (over the set of functions having weak derivatives). We must be careful, however, when computing it for sums, compositions etc. Here is a result for products with a specific structure.

Lemma 1.17. Let $g \in L^1(0, T)$ be the weak derivative of $f \in L^1(0, T)$, and let $\eta \in C^\infty([0, T])$. Then $\eta f$ has a weak derivative equal to $\dot{\eta} f + \eta \dot{f}$.

Proof. Let $\varphi \in \mathcal{D}(0, T)$. Then $\psi(t) := \eta(t)\varphi(t)$ also belongs to $\mathcal{D}(0, T)$, $\dot{\psi} = \dot{\eta}(t)\varphi(t) + \eta(t)\dot{\varphi}(t)$, and so, skipping time arguments
\begin{equation}
0 = \int_0^T g\psi dt + \int_0^T f\dot{\psi} dt = \int_0^T (\eta g + \dot{\eta} f)\varphi dt + \int_0^T \eta f \dot{\varphi} dt.
\end{equation}
The conclusion follows. \hfill \Box

Definition 1.18. We say that $f : [0, T] \to \mathbb{R}$ is absolutely continuous if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all sequence $[a_k, b_k]$ of disjoint intervals, included in $[0, T]$, we have that $\sum_k |f(b_k) - f(a_k)| < \varepsilon$ if $\sum_k (b_k - a_k) < \delta$.

We first admit the following result:

Lemma 1.19. Let $f \in L^1(0, T)$. Then $f$ is the primitive of a function $g \in L^1(0, T)$ iff $f$ is absolutely continuous. If $f$ is Lipschitz, then it is the primitive of some $g \in L^\infty(0, T)$.

We next show that when defining the weak derivative, instead of $\mathcal{D}(0, T)$, it is equivalent to choose the larger set of test functions
\begin{equation}
V_0 := \{ \varphi : [0, T] \to \mathbb{R} : \text{ Lipschitz; } \varphi(0) = \varphi(T) = 0 \}.
\end{equation}

Exercice 1.20. Check that $V_0$ is a Banach space when endowed with the norm $\|\varphi\| := \|\dot{\varphi}\|_\infty$.

Lemma 1.21. Let $f \in L^1(0, T)$. Then $g \in L^1(0, T)$ is the weak derivative of $f$ iff we have that
\begin{equation}
\int_0^T g(t)\varphi(t)dt = -\int_0^T f(t)\dot{\varphi}(t)dt, \text{ for all } \varphi \in V_0.
\end{equation}
Proof. (a) Since \( \mathcal{D}(0,T) \subset V_0 \), (1.32) implies (1.27). Conversely, let (1.27) hold. If \( \varphi \in V_0 \) vanishes close to 0 and \( T \), taking \( \varphi_\varepsilon := \varphi * \rho_\varepsilon \) (that belongs to \( \mathcal{D}(0,T) \)) in (1.27) and making \( \varepsilon \downarrow 0 \) we obtain that the equality in (1.33) holds for this particular \( \varphi \).

(b) In the general case, set, for \( \varepsilon > 0 \) small enough, \( \varphi_\varepsilon(t) := \varphi(-\varepsilon + (1+2\varepsilon)t/T) \). Then \( \varphi_\varepsilon(t) \) vanishes close to 0 and \( T \), and converges as well as its derivative in \( L^1(0,T) \) to \( \varphi \) and its derivative, resp. By step (a) the equality in (1.33) holds for \( \varphi_\varepsilon \). We conclude by making \( \varepsilon \downarrow 0 \).

Lemma 1.22. Let \( f \in L^1(0,T) \) have weak derivative 0, i.e.,

\[
(1.33) \quad \int_0^T f(t)\dot{\varphi}(t)dt = 0, \quad \text{for all } \varphi \in \mathcal{D}(0,T).
\]

Then \( f \) is constant.

Proof. We follow the idea of Example 9.5. Consider the optimization problem

\[
(1.34) \quad \min_{\varphi \in Lip(0,T)} \int_0^T f(t)\dot{\varphi}(t)dt; \quad \varphi(0) = \varphi(T) = 0,
\]

where \( Lip(0,T) \) is the Banach space of Lipschitz functions, endowed with the norm

\[
(1.35) \quad \| \varphi \| := \| \varphi(0) \| + \| \dot{\varphi} \|_\infty.
\]

By lemma 1.21, \( \varphi = 0 \) is solution of this problem, whose constraints are qualified. By theorem 1.15 there exists a Lagrange multiplier \( (\lambda, \mu) \in \mathbb{R}^2 \) such that, for all \( \varphi \in Lip(0,T) \), with derivative \( \psi \), since \( \varphi(T) = \varphi(0) + \int_0^T f(t)\dot{\varphi}(t)dt \):

\[
(1.36) \quad 0 = \int_0^T f(t)\dot{\varphi}(t)dt + \lambda \varphi(0) + \mu \varphi(T) = (\lambda + \mu)\varphi(0) + \int_0^T (f(t) + \mu)\psi(t)dt.
\]

This must hold for any \( \varphi(0) \in \mathbb{R} \), and so, \( \lambda + \mu = 0 \). We deduce that \( \int_0^T (f(t) + \mu)\psi(t)dt = 0 \), for all Lipschitz function \( \psi \), and conclude that \( f(t) = -\mu \) with lemma 1.14. The conclusion follows.

Lemma 1.23 (Du Bois-Raymond). We have that \( g \in L^1(0,T) \) is the weak derivative of \( f \in L^1(0,T) \) iff \( f \) is the primitive of \( g \), i.e., iff for all \( \tau \in [0,T] \), \( f(\tau) = f(0) + \int_0^\tau g(t)dt \).

Proof. (a) Let \( f \) be the primitive of \( g \). We may assume that \( f(0) = 0 \). Then by Fubini’s theorem

\[
(1.37) \quad \int_0^T f(t)\dot{\varphi}(t)dt = \int_0^T \left( \int_0^T g(s)ds \right) \dot{\varphi}(t)dt = \int_0^T g(s) \left( \int_s^T \dot{\varphi}(t)dt \right) ds = -\int_0^T g(s)\varphi(t)dtds
\]

and so, \( g \) is the weak derivative of \( f \), as was to be proved.

(b) Let \( g \) be the weak derivative of \( f \). Denote by \( F \) a primitive of \( g \). By point (a), \( F - f \) has weak derivative 0. By lemma 1.22 \( F - f \) is constant. The result follows.

For \( s \in [0,\infty] \), let us denote by \( W^{1,s}(0,T) \) the set of functions of \( L^s(0,T) \) with weak derivative in \( L^s(0,T) \), with associated norm, for \( s \neq 2 \):

\[
(1.38) \quad \| y \|_{1,s} := \| y \|_{L^s(0,T)} + \| \dot{y} \|_{L^s(0,T)}.
\]

By the Du Bois-Raymond lemma 1.23 this coincides with the space of primitives of functions of \( L^s(0,T) \). Let \( y \in W^{1,s}(0,T) \). Then \( y \) is continuous, and we can extend this function as a continuous function \( \tilde{y} \) over \( \mathbb{R} \) by setting \( \tilde{y}(t) = y(t) \) over \( [0,T] \), \( \tilde{y}(t) = y(0) \) if \( t < 0 \), and \( y(t) = y(T) \) if \( t > T \). By the regularizing kernel technique applied to these extensions we check that \( C^\infty([0,T]) \) is a dense subset. When \( s = 2 \) we obtain an Hilbert space structure by taking the norm

\[
(1.39) \quad \| y \|_{1,2} := \left( \int_0^T (y^2(t) + (\dot{y})^2(t)) dt \right)^{1/2}.
\]
Note that (by lemma 1.19) \( W^{1,\infty}(0, T) \) is the set of Lipschitz functions over \([0, T]\). We have the integration by parts formula:

**Lemma 1.24.** Let \( f \in W^{1,s}(0, T) \) and \( g \in W^{1,s'}(0, T) \) where \( s \in [1, \infty] \), \( 1/s + 1/s' = 1 \). Then

\[
\int_{0}^{T} \left( \dot{f}(t)g(t) + f(t)\dot{g}(t) \right) dt + f(0)g(0) - f(T)g(T) = 0.
\]

**Proof.** (a) Let \( a(f, g) \) denote the bilinear form in the l.h.s. of (1.40); it is well-defined (the integrands are summable), continuous over \( W^{1,s}(0, T) \times W^{1,s'}(0, T) \), and vanishes over \( (C^\infty([0, T]))^2 \). So, the result follows by density if \( s \in [1, \infty] \).

(b) Assume that say \( s = \infty \) and \( s' = 1 \). Let the sequence \( g_k \) in \( C^\infty([0, T]) \) converge to \( g \) in \( W^{1,1}(0, T) \). Since \( L^\infty(0, T) \subset L^s(0, T) \) for all \( s \in [1, \infty] \), (1.40) holds the pair \((f,g_k)\). The result follows by passing to the limit in \( k \).

\[\square\]

3. Classical calculus of variations

3.1. Setting. Consider the optimization problem

\[
\min \int_{0}^{T} \ell(t, y(t)) dt; \quad y(0) = a; \quad y(T) = b.
\]

Here \( y : [0, T] \to \mathbb{R}^n \) is integrable, \( \dot{y} \) is the weak derivative of \( y \), and \( \ell : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \) is a \( C^2 \) function. We will restrict the admissible set to \( \mathcal{Y} := W^{1,\infty}(0, T)^n \), so that \( u = \dot{y} \) belongs to \( L^\infty(0, T)^n \). In that case the cost function \( J(y) := \int_{0}^{T} \ell(t, y(t)) dt \) is well-defined.

**Definition 1.25.** We say that \( \dot{y} \) is a weak solution of (1.41) if it is admissible and satisfies, for some \( \varepsilon > 0 \):

\[
J(\dot{y}) \leq J(y) \quad \text{if} \quad \|y - \dot{y}\|_\infty + \|\dot{y} - \ddot{y}\|_\infty < \varepsilon,
\]

and is a strong solution of (1.41) if it is admissible and satisfies, for some \( \varepsilon > 0 \):

\[
J(\ddot{y}) \leq J(y) \quad \text{if} \quad \|y - \ddot{y}\|_\infty < \varepsilon.
\]

3.2. A simple example. We start by discussing the example when \( n = 1 \) and, for some \( f \in L^1(0, T) \):

\[
\ell(t, u, y) := \frac{1}{2}(u^2 + y^2) - f(t)y.
\]

In that case \( J(y) \) is quadratic. Let \( \ddot{y} \) be a weak solution. Set \( \ddot{y} := \dot{y} \), and \( z \in \mathcal{Y} \) be such that \( z(0) = z(T) = 0 \). For all \( \theta \in \mathbb{R} \), \( \dot{y} + \theta z \) is admissible, and expanding the quadratic cost function:

\[
J(\dot{y} + \theta z) = J(\dot{y}) + \theta \int_{0}^{T} \ell(\dot{y}(t) + \theta z(t)) dt + O(\theta^2).
\]

We deduce that

\[
0 = \lim_{\theta \to 0} \frac{J(\dot{y} + \theta z) - J(\dot{y})}{\theta} = \int_{0}^{T} \ell(\dot{y}(t) + \theta z(t)) dt.
\]

So, we can write by definition of a weak derivative:

\[
\dot{y} - \ddot{y} = f.
\]

Note that, here, \( \ddot{y} \) means the weak derivative of \( \dot{y} \). We can solve the ODE (1.48), with the boundary conditions \( \dot{y}(0) = a \) and \( \dot{y}(T) = b \), by the variations of constant formula, or by an appropriate numerical method. Here we have a decoupling property, these equations need to be solved independently for each component \( i = 1 \) to \( n \).

**Example 1.26.** Take the particular case when \( f \) is zero, and \( n = 1 \). By (1.48), \( \dot{y}(t) = \alpha e^t + \beta e^{-t} \) for some \((\alpha, \beta)\); from the boundary conditions we obtain that \( \alpha = a \) and \( \beta = (b - ae^T)e^T \).
More generally, if $\ell$ is polynomial we easily check that, setting $\ell(t) := \ell(t, \bar{u}(t), \bar{y}(t))$:

$$J(\bar{y} + \theta z) = J(\bar{y}) + \theta \int_0^T (\ell_u(t)\dot{z}(t) + \ell_y(t)z(t)) \, dt + O(\theta^2).$$

We deduce that

$$0 = \lim_{\theta \to 0} \frac{J(\bar{y} + \theta z) - J(\bar{y})}{\theta} = \int_0^T (\ell_u(t)\dot{z}(t) + \ell_y(t)z(t)) \, dt$$

So, if the integrand is polynomial, a local solution satisfies the Euler-Lagrange equation:

$$\ell_y(\bar{y}(t), \bar{y}(t)) = \frac{d}{dt} \ell_u(\dot{\bar{y}}(t), \bar{y}(t)) \quad \text{in a weak derivative sense.}$$

### 3.3. General Euler-Lagrange equation

We next show that (1.51) holds also for non polynomial integrands. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ by of class $C^p$. By the first order Taylor equation (1.16), we have that the remainder

$$r(x, h) := f(x + h) - f(x) - Df(x)h$$

satisfies

$$|r(x, h)| \leq a(x, h)|h|, \quad \text{where } a(x, h) := \int_0^1 |Df(x + th) - Df(x)| \, dt.$$  

Since continuous functions are uniformly continuous over compact sets, it follows that $a(x, h) \to 0$ when $h \to 0$, uniformly over bounded sets. We easily deduce the following lemma.

**Lemma 1.27.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ by of class $C^p$. Then the mapping $F : L^\infty(0, T)^n \to L^\infty(0, T)^m$ defined by $F(y)(t) := F(y(t))$ a.e. is also of class $C^p$, with derivative satisfying $D^j F(y)(t) = D^j f(y(t))$ a.e., for all $j \leq p$.

By the above lemma, $J$ is $C^2$, and has the following Taylor expansion

$$J(\bar{y} + \theta z) = J(\bar{y}) + \theta \int_0^T (\ell_u(t)\dot{z}(t) + \ell_y(t)z(t)) \, dt + \frac{1}{2} \theta^2 Q(z) + O(\theta^3).$$

with

$$Q(z) := \int_0^T D^2\ell(t)(\dot{z}(t), z(t))^2 \, dt.$$  

**Theorem 1.28.** A weak minimum satisfies the Euler-Lagrange equation as well as the second order necessary condition

$$Q(z) \geq 0, \quad \text{for all } z \in V_0.$$

We can restate the Euler-Lagrange equation in a form that will help to make the link with general optimal control problems. Define the costate as

$$\dot{p}(t) := -\nabla_u \ell(t).$$

Then the Euler-Lagrange equation is equivalent to

$$-\dot{p}(t) := \nabla_y \ell(t); \quad \nabla_u \ell(t) + \dot{p}(t) = 0.$$  

Introducing the pre-Hamiltonian

$$\ell(u, y) + p \cdot u,$$

and the control variable

$$\bar{u} = \dot{y},$$

we can also write the Euler-Lagrange equation in the form

$$\dot{y}(t) := \nabla_p H(t); \quad -\dot{p}(t) := \nabla_y H(t); \quad 0 = \nabla_u H(t).$$
3.4. Other endpoints equality conditions. It may happen that \( y(0) \) or \( y(T) \) are free or subject to various constraints. In that case the Euler-Lagrange equations in the form \((1.61)\) still hold (by the same arguments) but we may hope to have some additional informations. We start with the case when only the initial condition is fixed, i.e., the problem

\[
\begin{align*}
\text{Min} & \quad J(y) + \phi(y(T)); \quad y(0) = a.
\end{align*}
\]

**Theorem 1.29.** Let \( \tilde{y} \) be a weak minimum in \( \mathcal{Y} \) of problem \((1.62)\). Then the Euler-Lagrange equation \((1.61)\) holds, as well as the boundary condition

\[
\begin{align*}
\bar{p}(T) = \nabla \phi(\tilde{y}(T)).
\end{align*}
\]

**Proof.** We already know that \((1.61)\) holds. Set \( J^0(y) := J(y) + \phi(y(T)) \). Let \( z \in W^{1,\infty}(0, T) \) be such that \( z(0) = 0 \). Then

\[
\begin{align*}
0 = DJ^0(y) = -\int_0^T (\bar{p}(t) \cdot \dot{z}(t) + \dot{\bar{p}}(t) \cdot z(t))dt + \phi'(\tilde{y}(T))z(T).
\end{align*}
\]

By lemma \(1.24\) the above integral is equal to \(-\bar{p}(T) \cdot z(T)\), and so, \((\nabla \phi(\tilde{y}(T)) - \bar{p}(T)) \cdot z(T) = 0\). Since \( z(T) \) may take arbitrary values, the result follows.

Consider now a more general constrained problem

\[
\begin{align*}
\text{Min } F(y) := \int_0^T \ell(y(t), y(t))dt + \phi(y(0), y(T)); \quad \Phi(y(0), y(T)) = 0,
\end{align*}
\]

with \( \phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), \( \Phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n_F} \), all functions being \( C^1 \). The **generalized pre-Hamiltonian**, is defined as

\[
H(u, y, \lambda_0, p) := \lambda_0 \ell(u, y) + p \cdot u.
\]

**Theorem 1.30.** Let \( \tilde{y} \) be a weak solution in \( \mathcal{Y} \) of problem \((1.65)\). Then there exists \( \lambda_i \), for \( i = 0, \ldots, n_F \), not all zero, such that \( \lambda_0 \geq 0 \),

\[
\begin{align*}
\dot{\tilde{y}}(t) := \nabla_p H(t); \quad \dot{\bar{p}}(t) := \nabla_y H(t); \quad 0 = \nabla_y H(t),
\end{align*}
\]

as well as the boundary conditions

\[
\begin{align*}
\begin{cases}
-\bar{p}(0) = \lambda_0 \nabla_{y_0} \phi(\tilde{y}(0), \tilde{y}(T)) + \sum_{i=1}^{n_F} \lambda_i \nabla_{y_0} \Phi_i(\tilde{y}(0), \tilde{y}(T)), \\
\bar{p}(T) = \lambda_0 \nabla_{y_T} \phi(\tilde{y}(0), \tilde{y}(T)) + \sum_{i=1}^{n_F} \lambda_i \nabla_{y_T} \Phi_i(\tilde{y}(0), \tilde{y}(T)).
\end{cases}
\end{align*}
\]

If in addition \( D\phi(\tilde{y}(0), \tilde{y}(T)) \) is onto, then the conclusion holds with \( \lambda_0 > 0 \).

**Proof.** Set \( \Phi^\lambda(y_0, y_T) := \lambda_0 \phi(y_0, y_T) + \sum_{i=1}^{n_F} \lambda_i \Phi_i(y_0, y_T) \). By theorem \(1.10\) there exists some nonzero \( \lambda_i \), for \( i = 0, \ldots, n_F \), not all zero, such that \( \lambda_0 \geq 0 \), and that the generalized Lagrangian

\[
\begin{align*}
\lambda_0 F(y) + \sum_{i=1}^{n_F} \lambda_i \Phi_i(y(0), y(T)) = \lambda_0 \int_0^T \ell(\tilde{z}(t), z(t))dt + \Phi(\tilde{y}(0), \tilde{y}(T))
\end{align*}
\]

has a zero derivative at \( \tilde{y} \), i.e., for all \( z \in \mathcal{Y} \):

\[
\begin{align*}
0 = \lambda_0 \int_0^T \ell'(t)(\tilde{z}(t), z(t))dt + D\Phi^\lambda(\tilde{y}(0), \tilde{y}(T))(z(0), z(T)).
\end{align*}
\]

Taking \( z \) in \( D(0, T)^n \), we first deduce first the variant \((1.67)\) of the Euler-Lagrange equations. Integrating by parts the coefficient of \( \tilde{z}(t) \), we deduce that

\[
\begin{align*}
\bar{p}(T) \cdot z(T) - \bar{p}(0) \cdot z(0) = D\Phi^\lambda(\tilde{y}(0), \tilde{y}(T))(z(0), z(T)).
\end{align*}
\]

Since we can give to \( z(0) \) and \( z(T) \) arbitrary values, \((1.68)\) follows.

Assume now that \( D\phi(\tilde{y}(0), \tilde{y}(T)) \) is onto. Then obviously the mapping \( \mathcal{Y} \to \mathbb{R}^{n_F} \) \( y \mapsto \phi(\tilde{y}(0), \tilde{y}(T)) \) has a surjective derivative at \( \tilde{y} \). We conclude with theorem \(1.35\) \( \square \).
Exercice 1.31. Assume that \( \varphi \) is the null function, and that the end point constraints reduce to the periodicity condition \( \Phi(y_0, y_T) = y_T - y_0 \). Show that the end point conditions for the costate reduce to \( \ddot{p}(T) = \lambda = -\dot{p}(0) \), i.e., to a periodicity condition for the costate. Detail the case when \( \ell(t, y, u) = \frac{1}{2}(y - t)^2 + \frac{1}{2}u^2 \).

3.5. Oleinik-Lax formula. If \( \ell \) depends only on its second variable we then write \( \ell(u) \) and observe that the Euler-Lagrange equation reduces to

\[
\ell_u(\dot{y}(t)) \text{ is a constant function of time.}
\]

Lemma 1.32. Let \( \ell(y, u) = \ell(u) \) be convex. If the calculus of variations problem has a solution (whatever the end point cost function and constraints are), then is has a solution, affine function of time, with the same initial and final values.

Proof. The result follows from the application below of Jensen’s formula: if \( y \) is admissible, then

\[
\int_0^T \ell \left( \frac{\dot{y}(T) - \dot{y}(0)}{T} \right) dt \leq \int_0^T \ell(\dot{y}(t))dt, \quad \text{for all admissible } y.
\]

Exercice 1.33. Set \( \mathcal{Y}(x) := \{ y \in \mathcal{Y}; y(0) = x \} \). Let the initial state be fixed, equal to \( x \), and the final one be free. Assume that we have the final cost \( \varphi(y(T)) \), and that \( \ell \) is convex and does not depend on its first argument. Prove the Oleinik-Lax formula formula

\[
\inf_{y \in \mathcal{Y}(x)} \int_0^T \ell(\dot{y}(t))dt + \varphi(y(T)) = \inf_{\xi \in \mathbb{R}^n} (T\ell(\xi) + \varphi(x + T\xi)).
\]

Exercice 1.34. Extend the Oleinik-Lax formula to the setting of problem \( 1.65 \).

4. Inequality constraints

4.1. Abstract setting. Let \( X \) be a Banach space, \( f : X \to \mathbb{R} \), \( g : X \to \mathbb{R}^{n_g} \) be \( C^1 \), \( n_g = n' + n'' \). Consider the optimization problem with finitely many equality and inequality constraints

\[
\text{Min}_{x \in X} f(x); \quad g_i(x) = 0, \quad 1 \leq i \leq n'; \\
g_i(x) \leq 0, \quad n' + 1 \leq i \leq n''.
\]

If \( x \) is admissible, we denote the set of active (equalities and inequalities) constraints by

\[
I^A(x) := \{ 1, \ldots, n' \} \cup \{ n' + 1 \leq i \leq n; g_i(x) = 0 \}.
\]

Let \( \bar{x} \) be a local solution. Then it is also a local solution of the equality constrained problem

\[
\text{Min}_{x \in X} f(x); \quad g_i(x) = 0, \quad i \in I^A(\bar{x}).
\]

By theorem 1.10, we obtain the existence of a generalized Lagrange multiplier (definition 1.8) whose components associated with non active inequality constraints are zero, or equivalently, satisfying the complementarity condition

\[
\lambda^E_i g_i(\bar{x}) = 0, \quad n' + 1 \leq i \leq n.
\]

However, the fact that the original problem has inequalities gives an additional restriction on the sign of the associated components of the multiplier. We denote the equality and inequality constraints by

\[
g^E(x) = (g_1(x), \ldots, g_{n'}(x)); \quad g^I(x) = (g_{n'+1}(x), \ldots, g_n(x)),
\]

and set for \( \lambda = (\lambda_0, \lambda^E, \lambda^I) \in \mathbb{R}_+ \times \mathbb{R}^{n'} \times \mathbb{R}^{n''} \) the generalized Lagragian

\[
L(x, \lambda) := \lambda_0 f(x) + \sum_{i=1}^{n'} \lambda^E_i g^E_i(\bar{x}) + \sum_{i=1}^{n''} \lambda^I_i g^I_i(\bar{x}).
\]
DEFINITION 1.35. Let \( \bar{x} \) be an admissible point of (1.75). We say that \( \lambda = (\lambda_0, \lambda^E, \lambda^I) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n''} \) is a generalized Lagrange multiplier associated with \( \bar{x} \) if it is nonzero, satisfies the complementarity condition (1.78), and is such that

\[
\lambda^I \geq 0; \quad D_xL(\bar{x}, \lambda) = \lambda_0 f'(\bar{x}) + Dg^E(\bar{x})\lambda^E + Dg^I(\bar{x})\lambda^I = 0.
\]

We denote by \( \Lambda(\bar{x}) \) the set of associated generalized Lagrange multipliers, and set

\[
(1.82) \quad \Lambda_k(\bar{x}) := \{ \lambda \in \Lambda(\bar{x}); \quad \lambda_0 = k \}, \quad k \geq 0.
\]

Theorem 1.36. With a local solution of (1.75) is associated a nonempty set of generalized Lagrange multipliers.

Proof. See section 5.2.

DEFINITION 1.37. Let \( \bar{x} \) be an admissible point of problem (1.75). We say that \( \bar{x} \) satisfies the Mangasarian-Fromovitz qualification condition if:

\[
(1.83) \quad \begin{cases} 
(i) \quad Dg_E(\bar{x}) \text{ is onto}, \\
(ii) \quad \text{There exists } \bar{h} \in \text{Ker } g'_E(\bar{x}) \text{ such that } g_i(\bar{x})\bar{h} < 0, \text{ for all } i \in I(\bar{x}).
\end{cases}
\]

Proposition 1.38. Let \( \bar{x} \) be a local solution of (1.75) satisfying the Mangasarian-Fromovitz qualification condition. Then \( \Lambda_0(\bar{x}) \) is empty, and \( \Lambda_1(\bar{x}) \) is nonempty and bounded.

Proof. This is a particular case of theorem 1.56.

We may and will rewrite the constraints in the compact form \( g(x) \in K[n', n''] \), with

\[
(1.84) \quad K[n', n''] := \{ 0 \}_R^n \times \mathbb{R}_+^{n''}.
\]

This set \( K[n', n''] \) is closed and convex.

4.2. Calculus of variations with inequality constraints. Consider the constrained calculus of variations problem (compare to (1.65))

\[
(1.85) \quad \min F(y) := \int_0^T \ell(y(t), y(t))dt + \phi(y(0), y(T)); \quad \Phi(y(0), y(T)) \in K[n', n''],
\]

with \( n' + n'' = n_g, \phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}, \Phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n_g} \), all functions being \( C^1 \).

Theorem 1.39. Let \( y \) be a weak solution in \( Y \) of problem (1.85). Then there exists \( \lambda_i \), for \( i = 0, \ldots, n_g \), not all zero, such that \( \lambda_0 \geq 0, (1.67)-(1.68) \) hold, as well as the complementarity condition (1.78) and the sign conditions \( \lambda_i \geq 0 \), for all \( i > n' \).

Proof. We apply theorem 1.36. The optimality system is the one for the corresponding equality constrained problem, obtained by changing active inequality constraints into equalities, obtained by applying theorem 1.29 with the additional sign conditions on the multipliers associated with inequalities. The result follows.

Example 1.40. Consider the problem of minimizing

\[
(1.86) \quad \min \frac{1}{2} \int_0^T \dot{y}(t)^2 dt; \quad \frac{1}{2} |y(T)|^2 \leq 1,
\]

with \( y(0) \) given, such that \( |y(0)| > 1 \). Any solution is qualified and so we may suppose that \( \lambda_0 = 1 \). By the Euler-Lagrange equations (1.61), we have that \( -\dot{u}(t) = \dot{p}(t) \) is constant, equal to its terminal value \( \dot{p}(T) = \lambda y(T) \). If \( \lambda = 0 \) then \( \dot{u}(t) = 0 \) for all \( t \) and the final condition cannot be satisfied, and so, \( \lambda > 0 \). The trajectory is a straight line ending at \( \dot{y}(T) = y(0)/|y(0)| \), and \( \dot{u}(t) = -\alpha y(0)/|y(0)| \) for some \( \alpha > 0 \) such that \( \alpha T = |y(0)| - 1 \).
5. Proofs of some abstract results

5.1. Open mapping theorem and metric regularity. Remember that \( B_X \) denotes the open unit ball of the Banach space \( X \). The Open mapping theorem is as follows.

**Theorem 1.41.** Let \( X \) and \( Y \) be Banach spaces, and \( A \in L(X,Y) \) be surjective. Then there exists \( \alpha > 0 \) such that \( \alpha B_Y \subset \text{AB}_X \).

**Proof.** See e.g. Brézis [10, Ch. 2]. \( \square \)

**Corollary 1.42.** Let \( X, Y, \) and \( A \) be as in the above theorem. Then

\[
\|A^* y^*\|_X \geq \alpha \|y^*\|_{Y^*}, \quad \text{for all } y^* \in Y^*.
\]

**Proof.** It suffices to observe that

\[
\|A^* y^*\|_X = \sup_{x \in B_X} \langle A^* y^*, x \rangle_Y = \sup_{x \in B_X} \langle y^*, Ax \rangle_Y \geq \sup_{y \in \alpha B_Y} \langle y^*, y \rangle_Y = \alpha \|y^*\|_{Y^*}.
\]

**Proposition 1.43.** Let \( X \) and \( Y \) be Banach spaces, and \( A \in L(X,Y) \). Then \( \text{Im} A^* \subset (\text{Ker} A)^\perp \), with equality if \( A \) is surjective.

**Proof.** (a) Let \( x^* \in \text{Im} A^1 \), i.e., \( x^* = A^1 y^* \), for some \( y^* \in Y^* \), and \( x \in \text{Ker} A \). Then \( \langle x^*, x \rangle_X = \langle y^*, Ax \rangle_Y = 0 \). Therefore, \( \text{Im} A^1 \subset (\text{Ker} A)^\perp \).

(b) Assume now that \( A \) is surjective. Let \( x^* \in (\text{Ker} A)^\perp \). For \( y \in Y \), set

\[
v(y) := \langle x^*, x \rangle, \quad \text{where } x \in X \text{ satisfies } Ax = y.
\]

Since \( x^* \in (\text{Ker} A)^\perp \), any \( x \) such that \( Ax = y \) gives the same value of \( \langle x^*, x \rangle_X \), and therefore \( v(y) \) is well-defined. It is easily checked that it is a linear function. By the open mapping theorem, there exists \( x \in \text{Ker} A \) such that \( Ax = y \), so that \( |v(y)| \leq \alpha \|y\|_Y \). So \( v \) is a linear and continuous mapping, i.e., there exists \( y^* \in Y^* \) such that \( v(y) = \langle y^*, y \rangle_Y \). For all \( x \in X \), we have therefore \( \langle x^*, x \rangle_X = \langle y^*, Ax \rangle_Y = \langle A^1 y^*, x \rangle_X \) so that \( x^* = A^1 y^* \), as was to be proved. \( \square \)

We next give a metric regularity theorem.

**Theorem 1.44.** Let \( X \) and \( Y \) be Banach spaces, \( g : X \to Y \) be of class \( C^1 \), and \( \bar{x} \in g^{-1}(0) \) be such that \( Dg(\bar{x}) \) is surjective. Then the metric regularity property holds at \( \bar{x} \), i.e., there exists \( \beta > 0 \) such that, if \( x \) is close to \( \bar{x} \), then there exists \( x' \in g^{-1}(0) \) such that \( \|x' - x\|_X \leq \beta \|g(x)\|_Y \).

**Proof.** Set \( A := Dg(\bar{x}) \), and consider a sequence \( x_k \) in \( X \) such that \( x_0 := x \) and \( g(x_k) + A(x_{k+1} - x_k) = 0 \). By the open mapping theorem, there exists such a sequence that satisfies \( \|x_{k+1} - x_k\| \leq \alpha^{-1} \|g(x_k)\| \). Since \( g \) is of class \( C^1 \) it has a Lipschitz constant \( L \) in \( B(\bar{x}, \varepsilon) \) for some \( \varepsilon > 0 \). We assume that \( x \in B(\bar{x}, \varepsilon') \) for some \( \varepsilon' < \varepsilon \). As long as \( x_k \in B(\bar{x}, \varepsilon) \), we have therefore

\[
\|x_{k+1} - x_k\|_X = \alpha^{-1} \|g(x_k)\| \leq \alpha^{-1} L \|x_k - \bar{x}\|.
\]

By the first order Taylor formula

\[
g(x_{k+1}) = \int_0^1 (Dg(x_k) + \theta (x_{k+1} - x_k) - A)(x_{k+1} - x_k) d\theta.
\]

Since \( Dg \) is a continuous function, reducing \( \varepsilon \) if necessary, we deduce that

\[
\|g(x_{k+1})\| \leq \frac{1}{2} \alpha \|x_{k+1} - x_k\| \leq \frac{1}{2} \|g(x_k)\|.
\]

So, if \( x_j \in B(\bar{x}, \varepsilon) \) for all \( j \leq k \):

\[
\|x_{k+1} - \bar{x}\| \leq \|x - \bar{x}\| + \sum_{j \leq k} \|x_{j+1} - x_j\| \leq \|x - \bar{x}\| + 2 \alpha^{-1} \|g(x)\| \leq \varepsilon'(1 + 2 \alpha^{-1} L).
\]

Now choose \( \varepsilon' := \varepsilon/(1 + 2 \alpha^{-1} L) \). Then the sequence \( x_k \) belongs to \( B(\bar{x}, \varepsilon) \), \( \|x_{k+1} - x_k\| \leq \alpha L \|g(x_k)\| \leq 2^{-k} \alpha L \|g(x)\| \), and so \( x_k \) converges to some \( x' \) such that \( g(x') = 0 \) and \( \|x' - \bar{x}\| \leq \|x - \bar{x}\| + 2 \alpha L \|g(x)\| = O(\|x - \bar{x}\|) \). The conclusion follows. \( \square \)
EXERCISE 1.45. Take \( X = Y = \mathbb{R} \) and \( g(x) = x^2 \). Show that the metric regularity property does not hold at \( \bar{x} = 0 \).

5.2. Separation of convex sets and convex duality. Let \( A \) and \( B \) be two subsets of a Banach space \( X \). Let \( x^* \in X^* \), \( x^* \neq 0 \). We say that \( x^* \) separates \( A \) and \( B \) if
\[
\langle x^*, a \rangle \leq \langle x^*, b \rangle, \quad \text{for all } a \in A \text{ and } b \in B.
\]
(1.94)

We start with the following geometric form of the Hahn-Banach theorem.

THEOREM 1.46. Let \( A \) and \( B \) be two subsets of a Banach space \( X \), with empty intersection. If \( A - B \) has a nonempty interior, then there exists \( x^* \) separating \( A \) and \( B \).

PROOF. (a) The result is obtained in [10, Ch. 1, Thm 1.6], assuming that either \( A \) or \( B \) is open.
(b) Set \( E := \text{int}(B - A) \). Since \( A \cap B = \emptyset \), \( E \) does not contain zero. By step (a), there exists \( x^* \) separating \( 0 \) and \( E \). Now let \( a \in A \) and \( b \in B \), and set \( e := b - a \). Then \( e \) is the limit of a sequence \( e_k \in E \) (take \( e_k := (1 - 1/k)e + (1/k)e_k \), with \( e_k \in E \)). Therefore, \( \langle x^*, b - a \rangle = \lim_k \langle x^*, e_k \rangle \geq 0 \), as was to be proved.

Consider now the “linear convex” optimization problem
\[
\text{Min}_{x} \langle x^*, x \rangle; \quad Ax = b; \quad Cx \in K,
\]
(1.95)
with \( X, Y_E, Y_I \) Banach spaces, \( A \in L(X, Y_E) \), \( C \in L(X, Y_I) \), \( b \in Y_E \), \( K \) closed convex subset of \( Y_I \). When in addition \( K \) is a cone, we speak of a linear conic problem.

DEFINITION 1.47. Let \( K \) be a closed convex subset of \( \mathbb{R}^n \) and \( \bar{x} \in K \). We say that \( y \in \mathbb{R}^n \) is a normal direction to \( K \) at \( \bar{x} \) if the following holds:
\[
y \cdot (x' - x) \leq 0, \quad \text{for all } x' \in K.
\]
(1.96)
The set of normal directions is called the **normal cone**. This is a closed and convex cone.

EXAMPLE 1.48. If \( K \) is a singleton, the normal cone is \( \mathbb{R}^n \). If \( x \in \text{int}(K) \), the normal cone is a singleton. If \( K \) is a vector subspace, the normal cone is the orthogonal space.

THEOREM 1.49. Assume that \( A \) is surjective and that \( K \) has a nonempty interior. Let \( \bar{x} \in X \) be optimal. Then there exists some nonzero \( \lambda = (\lambda_0, \lambda_E, \lambda_I) \in \mathbb{R}_+ \times Y_E^* \times Y_I^* \) such that
\[
(\text{i}) \quad \lambda_0 x^* + A^\dagger \lambda_E + C^\dagger \lambda_I = 0; \quad (\text{ii}) \quad \lambda_I \in N_K(C \bar{x}).
\]
(1.97)

PROOF. Since \( \bar{x} \) is optimal, the system below has no solution:
\[
\langle x^*, x - \bar{x} \rangle < 0; \quad A(x - \bar{x}) = 0; \quad C(x - \bar{x}) \in \bar{K},
\]
(1.98)
where \( \bar{k} := K - C \bar{x} \). Setting \( h := x - \bar{x} \) we see that this is equivalent to
\[
[-\infty, 0] \times \bar{K} \cap \{\langle x^*, h \rangle, Ch \}; \quad h \in \text{ker} A = \emptyset.
\]
(1.99)
We have two convex sets with empty intersection, the second one being nonempty. By theorem 1.46 there exists some nonzero pair \( (\lambda_0, \lambda_I) \in \mathbb{R}_+ \times Y_I^* \) such that, for all \( h \in \text{ker} A, \alpha < 0, \) and \( k \in K:\)
\[
\lambda_0 \alpha + \langle \lambda_I, k - C \bar{x} \rangle \leq \lambda_0 \langle x^*, h \rangle + \langle \lambda_I, Ch \rangle.
\]
(1.100)
Since we take \( h \) in the vector space \( \text{ker} A \) we deduce that
\[
(\lambda_0 x^* + C^\dagger \lambda_I, h) = 0, \quad \text{for all } h \in \text{ker} A.
\]
(1.101)
By proposition 1.43 \( \lambda_0 x^* + C^\dagger \lambda_I \in (\text{ker} A)^\perp = \text{Im} A^* \), and so, \( \langle x^*, h \rangle \) holds. The r.h.s. of (1.100) being equal to zero, we deduce by taking either \( \alpha \to 0 \) or \( k = C \bar{x} \) that \( \lambda_0 \geq 0 \) and \( \lambda_I \in N_K(C \bar{x}) \). The conclusion follows.

REMARK 1.50. Note that (1.97)(i) is nothing but the stationarity w.r.t. \( x \) of the Lagrangian function \( \lambda_0 \langle x^*, x \rangle + \langle \lambda_E, Ax \rangle + \langle \lambda_I, Cx \rangle \).
5.3. Finitely constrained problems. We show here how to prove theorem 1.36.

DEFINITION 1.51. Let \( \bar{x} \) be an admissible point of (1.75). We say that \( \bar{x} \) has degenerate equality constraints if \( Dg^E(\bar{x}) \) is not surjective.

Since \( Dg^E(\bar{x}) : X \to \mathbb{R}^n \), \( Dg^E(\bar{x}) \) is not surjective iff its orthogonal space is not reduced to 0, i.e., iff there exist some nonzero \( \lambda^E \in \mathbb{R}^n \) such that, for all \( h \in X \):

\[
0 = \lambda^E \cdot (Dg^E(\bar{x})h) = (Dg^E(\bar{x})\lambda^E, h)_X
\]

or equivalently \( Dg^E(\bar{x})^\dagger \lambda^E = 0 \). Note that this is nothing but the relation of linear dependence of the family \( \{Dg_i(\bar{x}), 1 \leq i \leq n'\} \), since

\[
Dg^E(\bar{x})^\dagger \lambda^E = \sum_{i=1}^{n'} \lambda^E_i Dg^E(\bar{x})^\dagger.
\]

We obtain the following statement:

\[
\text{Min}_{h \in X, \theta \in \mathbb{R}} \theta; \quad Df(\bar{x})h \leq \theta; \quad Dg_i(\bar{x})h = 0, \ 1 \leq i \leq n'; \quad Dg_i(\bar{x})h \leq \theta, \ i \in I(\bar{x}),
\]

(1.105)

where by \( I(\bar{x}) \) we denote the set of active inequality constraints.

(1.106)

\[
I(\bar{x}) := \{n' < i \leq n' + n''; \ g_i(\bar{x}) = 0\}.
\]

LEMMA 1.52. Let \( \bar{x} \), local solution of (1.75), have nondegenerate equality constraints. Then the linearized problem (1.105) has value \( \theta \).

PROOF. Problem (1.105) has the admissible point \((h, \theta) = (0, 0)\) with cost function zero, and so, has a nonpositive value. Let us now assume that \((h, \theta)\) is admissible, while \( \theta < 0 \). Changing \( \theta \) into \( \frac{1}{\theta} \) if necessary, we may assume that the inequality constraints are strict. As in the proof of theorem 1.34 by the metric regularity theorem 1.41 for \( \sigma > 0 \) small enough, there exists \( x_\sigma \in g^{-1}(0) \) such that

\[
||x_\sigma - (\bar{x} + \sigma h)|| = O(||g(\bar{x} + \sigma h)||) = o(\sigma).
\]

Since the linearized active inequalities are strict we have that for \( \sigma > 0 \) small enough:

\[
g_i^f(x_\sigma) = f(\bar{x}) + \sigma f'(\bar{x})h + o(\sigma) < f(\bar{x}) + \sigma \theta,
\]

for all \( i \in I^A(\bar{x}) \), and similarly for the expansion of the cost function. It follows that \((x_\sigma, \sigma \theta)\) is admissible for (1.75) with a smaller cost function than \( \bar{x} \). This contradicts the fact that \( \bar{x} \) is a local solution of (1.75).

PROOF OF THEOREM 1.36. If the equality constraints are degenerate, the conclusion holds in view of (1.104). Otherwise, apply theorem 1.49 to the linearized problem (1.105), which has a zero solution by lemma 1.52, with the help of remark 1.50. The Lagrangian function is

\[
L := \theta + \lambda_0(f'(\bar{x})h - \theta) + \lambda^E \cdot Dg^E(\bar{x})h + \sum_{i \in I} \lambda^f_i \cdot (g_i(\bar{x})h - \theta).
\]

The stationarity of the Lagrangian w.r.t. the primal variables \((h, \theta)\) which setting \( \lambda_i = 0 \) if \( i > n', i \notin I(\bar{x}) \), amount to

\[
f'(\bar{x}) + \sum_{i=1}^{n''} \lambda_i g_i'(\bar{x}) = 0; \quad 1 - \lambda_0 - \sum_{i>n'} \lambda_i = 0.
\]

(1.110)
The second relation implies that \( \lambda \neq 0 \). The conclusion follows. \( \Box \)

5.3.1. **Link to finitely constrained problems.**

**Lemma 1.53.** Let \( K = K[n', n''] \), and \( \bar{x} \in K \). Then

\[
N_K(\bar{x}) = \{ y \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}; \quad y_j \bar{x}_j = 0, \quad j = n' + 1, \ldots, n' + n'' \},
\]

**Proof.** Left as an exercise. \( \Box \)

Having this result in mind we see that we can express the set of Lagrange multipliers for the inequality constrained problem in the form, setting \( K = K[n', n''] \):

\[
\Lambda(\bar{x}) := \{ \lambda = (\lambda_0, \lambda_Y) \in \mathbb{R}_+ \times N_K(g(\bar{x})); \quad L_\bar{x}(\bar{x}, \lambda) = 0; \quad \lambda \neq 0 \}.
\]

This allows to express the set of multipliers in a compact way. This formula has also the interest of being valid, in certain cases, for more general convex sets, as we next see.

**5.4. Abstract constrained optimization problems.** Let \( X \) and \( Y \) be a Banach space, \( f : X \to \mathbb{R} \), and \( g : X \to Y \), both \( C^1 \), and let \( K \) be a nonempty, closed and convex subset of \( Y \). Consider “the abstract” constrained optimization problem

\[
\text{Min } f(x); \quad g(x) \in K.
\]

The Lagrangian of the problem is, for \( \lambda = (\lambda_0, \lambda_Y) \in \mathbb{R} \times Y^* \):

\[
L(x, \lambda) := \lambda_0 f(x) + \langle \lambda_Y, g(x) \rangle_Y.
\]

Let \( \bar{x} \) be admissible. We denote the set of Lagrange multipliers associated with \( \bar{x} \) by (the expression similar to (1.112)):

\[
\Lambda(\bar{x}) := \{ \lambda = (\lambda_0, \lambda_Y) \in \mathbb{R}_+ \times N_K(g(\bar{x})); \quad L_\bar{x}(\bar{x}, \lambda) = 0; \quad \lambda \neq 0 \}.
\]

In applications we often have the following product form of the constraints, where \( Y_E \) and \( Y_I \) are Banach spaces:

\[
\left\{ \begin{array}{l}
Y = Y_E \times Y_I; \\
K = \{0\} Y_E \times K_I; \\
K_I \text{ is a nonempty, closed and convex subset of } Y_I \text{ with nonempty interior.}
\end{array} \right.
\]

We denote by \( g(x) = (g_E(x), g_I(x)) \) the corresponding decomposition of the constraint mapping. An obvious particular case is the one of finitely many equality and inequality constraints.

**Theorem 1.54.** Let \( \bar{x} \) be a local solution of (1.113). If (1.116) holds and \( g_E(\bar{x}) \) is surjective, then \( \Lambda(\bar{x}) \) is nonempty.

**Proof.** After translation of \( K \) we may assume that \( \varepsilon B \subset K \) for some \( \varepsilon > 0 \). Consider the linearized problem

\[
\text{Min } \theta; \quad f'(\bar{x}) \leq \theta; \quad g'_E(\bar{x}) h = 0; \quad (1 - \theta) g(\bar{x}) + g'_I(\bar{x}) h \in K.
\]

Let us show that \( (h, \theta) = 0 \) is solution. Indeed, otherwise there would exist some admissible \( (h, \theta) \) with \( \theta < 0 \). By the metric regularity theorem 1.41, for \( \sigma \in ]0, 1[ \) small enough, there exists \( x_\sigma \in g^{-1}(0) \) such that \( \| x_\sigma - (\bar{x} + \sigma h) \| = O(\| g_E(\bar{x} + \sigma h) \|) = o(\sigma) \). We then have that

\[
g(x_\sigma) = g(\bar{x}) + \sigma g'(\bar{x}) h + o(\sigma)
\]

\[
= \sigma (1 - \theta) g(\bar{x}) + g'_I(\bar{x}) h + (1 - \sigma (1 - \theta)) g(\bar{x}) + o(\sigma)
\]

\[
\subset \sigma K + (1 - \sigma (1 - \theta)) g(\bar{x}) + o(\sigma).
\]

Since \( \varepsilon B \subset K \), for all \( k \in K \) and \( \alpha \in ]0, 1[ \), \( (1 - \alpha) k + \alpha \varepsilon B \subset K \). Take \( \alpha := -\sigma \theta/(1 - \sigma) \). Then for \( \sigma \) small enough

\[
(1 - \sigma (1 - \theta)) g(\bar{x}) + o(\sigma) = (1 - \sigma)((1 - \alpha) g(\bar{x}) + o(\sigma)) \in (1 - \sigma) K.
\]

It follows that \( g(x_\sigma) \in K \). But this contradicts the local optimality of \( \bar{x} \) since \( f(x_\sigma) < f(\bar{x}) \) for small enough \( \sigma \).

(c) So \( (h, \theta) \) is solution of (1.117), which is a problem with affine cost function and constraint mapping. We conclude by applying theorem 1.49. \( \Box \)
By qualification condition we mean an hypothesis allowing to obtain the existence of a regular multiplier, i.e., such that $\lambda_0 > 0$. The extended Mangasarian-Fromovitz qualification condition is as follows:

$$\begin{aligned}
\{ 
\text{(i)} & \quad Dg_E(\bar{x}) \text{ is onto}, \\
\text{(ii)} & \quad \text{There exists } h \in \text{Ker } g'_E(\bar{x}) \text{ such that } g_I(\bar{x}) + Dg_I(\bar{x})h \in \text{int}(K_I).
\end{aligned}$$

Remark 1.55. In the case of finitely many equality and inequality constraints, i.e., when $K = \{0\} \times \mathbb{R}_n^\prime$, setting $I(\bar{x}) := \{n' < i \leq n''; \ g_i(\bar{x}) = 0\}$, condition (1.120) is easily seen to be equivalent to the Mangasarian-Fromovitz qualification condition 1.37.

Set

$$\Lambda_k(\bar{x}) := \{ \lambda \in \Lambda(\bar{x}); \ \lambda_0 = k \}.$$

Theorem 1.56. Let $\bar{x}$ be a local solution of (1.113). If (1.116) and (1.120) hold, then $\Lambda_0(\bar{x})$ is empty and $\Lambda_1(\bar{x})$ is nonempty and bounded.

Proof. Set $\bar{y}_I := g_I(\bar{x}) + g'_I(\bar{x})\bar{h}$. By theorem 1.54, there exists $\lambda \in \Lambda(\bar{x})$. Since $\bar{h} \in \text{Ker } g'_E(\bar{x})$, we have that

$$0 = D_x L(\bar{x}, \lambda) = \lambda_0 f'(\bar{x})\bar{h} + \langle \lambda, g'_I(\bar{x})\bar{h} \rangle = \lambda_0 f'(\bar{x})\bar{h} - \langle \lambda_I, g_I(\bar{x}) - \bar{y}_I \rangle.$$

For $\varepsilon > 0$ small enough, $\bar{y}_I + \varepsilon z \in K_I$ whenever $\|z\|_Y \leq 1$, and since $\lambda_I \in N_K(g_I(\bar{x}))$, we have that $\langle \lambda_I, g_I(\bar{x}) - \bar{y}_I - z \rangle \leq 0$, i.e.,

$$\langle \lambda_I, z \rangle \leq -\langle \lambda_I, g_I(\bar{x}) - \bar{y}_I \rangle.$$

Maximizing the l.h.s. over $z$ such that $\|z\|_Y \leq 1$, and using (1.122), we obtain that

$$\varepsilon \|\lambda_I\| \leq -\langle \lambda_I, g_I(\bar{x}) - \bar{y}_I \rangle = -\lambda_0 f'(\bar{x})\bar{h}.$$

So $\lambda_0 = 0$ implies $\lambda_I = 0$, and then $0 = D_x L(\bar{x}, \lambda) = Dg_E(\bar{x})^\dagger \lambda_E$. Since $Dg_E(\bar{x})$ is surjective, $Dg_E(\bar{x})^\dagger$ is injective, proving that $\lambda_E = 0$. But then $\lambda = 0$ which contradicts the definition of a Lagrangre multiplier. So, $\lambda_0 \neq 0$. Assuming that $\lambda_0 = 1$ we obtain a bound over $\lambda_I = 0$ by (1.124). This gives a bound over $Dg_E(\bar{x})^\dagger \lambda_E = -\lambda_0 f'(\bar{x}) - g'_I(\bar{x})^\dagger \lambda_I$ and therefore on $\lambda_E$ by corollary 1.42. The conclusion follows. □
CHAPTER 2

Pontryagin’s principle

1. Setting

1.1. Controlled dynamical system and associated cost. Consider controlled dynamical systems of the type

\( (2.1) \) 

(i) \( \dot{y}(t) = f(u(t), y(t)) \), for a.a. \( t \in [0, T] \);  
(ii) \( y(0) = y^0 \).

The data are the horizon or final time \( T > 0 \), the dynamics \( f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \), assumed to be Lipschitz and of class \( C^\infty \), and the final condition \( y^0 \). We call \( \tilde{u}(t) \) and \( y(t) \) the control and state at time \( t \). The control and state spaces are

\( (2.2) \) 

\( \mathcal{U} := L^\infty(0, T; \mathbb{R}^m); \quad \mathcal{Y} := W^{1, \infty}(0, T; \mathbb{R}^n). \)

Then we may see \((2.1)\)(i) as an equality in \( L^\infty(0, T)^n \). By the Cauchy-Lipschitz theorem, for all \((u, y^0) \in \mathcal{U} \times \mathcal{Y} \), the state equation \((2.1)\) has a unique solution in \( \mathcal{Y} \), denoted by \( y[u, y^0] \) (or \( y[u] \) if \( y^0 \) is fixed). By Gronwall’s lemma, for some \( C_f \) depending only on the Lipschitz constant of \( f \):

\( (2.3) \) 

\( \|y[u', (y^0)'] - y[u, y^0]\|_\infty \leq C_f \left( \|u' - u\|_1 + |(y^0)' - y^0| \right). \)

We denote by \( z[v, z^0] \), or \( z[v] \) if \( z^0 = 0 \), the unique solution of the linearized state equation

\( (2.4) \) 

(i) \( \dot{z}(t) = D\tilde{f}(\tilde{y}(t), z(t)), \) for a.a. \( t \in [0, T] \);  
(ii) \( z(0) = z^0 \).

The mapping \( z[v, z^0] \) is well defined and continuous \( L^s(0, T; \mathbb{R}^m) \times \mathbb{R}^n \to W^{1,s}(0, T; \mathbb{R}^n) \), for any \( s \) in \([1, \infty]\).

**Proposition 2.1.** The mapping \( \mathcal{U} \times \mathbb{R}^n \to \mathcal{Y}, \) \((u, y^0) \mapsto y[u, y^0], \) is of class \( C^\infty \).

**Proof.** We apply the implicit function theorem (IFT) to the state equation. The mapping \( F \) from \( \mathcal{U} \times \mathcal{Y} \times \mathbb{R}^n \) to \( L^\infty(0, T; \mathbb{R}^n) \times \mathbb{R}^n \), that to \((u, y, y^0) \) associates the state equation \((2.1)\), is by lemma \(1.27\) of class \( C^\infty \). We have to check the invertibility of the partial derivative of \( F \) w.r.t. the state, which means that \( z \mapsto D_yF(u, y, y^0)z \) is bijective \( \mathcal{Y} \to L^\infty(0, T; \mathbb{R}^n) \times \mathbb{R}^n \). This means that, for any \((g, e) \in L^\infty(0, T; \mathbb{R}^n) \times \mathbb{R}^n \), the following variant of the state equation \((2.4)\) has a unique solution in \( \mathcal{Y} \):

\( (2.5) \) 

(i) \( \dot{z}(t) = D\tilde{f}(\tilde{y}(t), z(t)) + g(t), \) for a.a. \( t \in [0, T] \);  
(ii) \( z(0) = e, \)

which obviously is the case. The conclusion follows. \( \square \)

We say that \((u, y) \in \mathcal{U} \times \mathcal{Y} \) is a trajectory if \( y = y[u, y(0)] \). In the sequel we perform an analysis around the nominal trajectory \((\tilde{u}, \tilde{y})\). We denote

\( (2.6) \) 

\( \tilde{f}(t) := f(\tilde{u}(t), \tilde{y}(t)), \quad D\tilde{f}(t) := f'(\tilde{u}(t), \tilde{y}(t)) \)

the nominal dynamics and its derivative w.r.t. control and state, with a similar convention for derivatives at any order of \((\tilde{u}(t), \tilde{y}(t))\), the partial derivatives being denoted by e.g. \( f_u(t) \). By the above proposition, \( z[v, z^0] \) is the directional derivative of \( y[u, y^0] \) at the point \((\tilde{u}, \tilde{y})\) in direction \((v, z^0) \in \mathcal{U} \times \mathbb{R}^n \), i.e.,

\( (2.7) \) 

\( z[v, z^0] = \lim_{s \to 0} (y[\tilde{u} + sv, \tilde{y}^0 + sz^0] - y[\tilde{u}, \tilde{y}^0])/s. \)
With the controlled system (2.1) we associated the cost function

\[
J(u, y) := \int_0^T \ell(u(t), y(t))dt + \varphi(y(T)),
\]

sum of an integral cost, with integrand \(\ell\), and of a final cost \(\varphi\); both functions being of class \(C^\infty\). Since a composition of \(C^\infty\) mappings is \(C^\infty\), \(J : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}\) is \(C^\infty\), as well as the reduced cost

\[
J_R(u, y^0) := J(u, y[u, y^0]).
\]

In addition, for \((v, z^0) \in \mathcal{U} \times \mathbb{R}^n\), writing \(z = z[v, z^0]\), the expression of the derivative of the reduced cost is

\[
J'_R(\bar{u}, \bar{y}^0)(v, z^0) = \int_0^T \bar{\ell}(t)(v(t), z(t))dt + \varphi'(\bar{y}(T))z(T).
\]

Our next step is to consider abstract control constraints of the type

\[
u(t) \in U_{ad}, \quad \text{for a.a. } t \in [0, T],
\]

where \(U_{ad}\) is a closed subset of \(\mathbb{R}^m\). Typical examples are the case of bound constraints, and the closed unit ball and sphere:

\[
\Pi^m_{i=1}[a_i, b_i]; \quad \bar{B} := \{u \in \mathbb{R}^m; \sum_{i=1}^m u_i^2 \leq 1\}; \quad \partial \bar{B} := \{u \in \mathbb{R}^m; \sum_{i=1}^m u_i^2 = 1\}.
\]

2. Control constraints

2.1. Derivative of the cost function. Define the pre Hamiltonian \(H : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) by

\[
H(u, y, p) := \ell(u, y) + p \cdot f(u, y).
\]

With the trajectory \((\bar{u}, \bar{y})\) we associate the costate \(\bar{p} \in \mathcal{Y}\), solution of the costate equation

\[
\begin{cases}
\dot{p}(t) = \nabla_y H(\bar{u}(t), \bar{y}(t), \bar{p}(t)), & \text{for a.a. } t \in [0, T]; \\
\bar{p}(T) = \nabla \varphi(\bar{y}(T)).
\end{cases}
\]

Given \((\bar{u}, \bar{y})\), this equation is backwards (the final condition is given) and has a unique solution in \(\mathcal{Y}\). We adopt the notation \(\bar{H}(t)\) in the spirit of (2.6), i.e., for instance:

\[
\bar{H}(t) := H(\bar{u}(t), \bar{y}(t), \bar{p}(t)), \quad \nabla_y \bar{H}(t) := \nabla_y H(\bar{u}(t), \bar{y}(t), \bar{p}(t)).
\]

Note that the r.h.s. of the costate dynamics is \(\nabla_y \bar{H}(t) = \nabla_y \bar{\ell}(t) + \bar{f}(t)^\dagger \bar{p}(t)\), and therefore,

\[
\nabla_y \bar{H}(t) \cdot z(t) = \nabla_y \bar{\ell}(t) \cdot z(t) + \bar{p}(t)^\dagger \bar{f}(t)z(t).
\]

**Lemma 2.2.** The reduced cost \(J_R\), defined in (2.9), has a derivative at \((\bar{u}, \bar{y}^0)\) characterized by

\[
J'_R(\bar{u}, \bar{y}^0)(v, z^0) = \bar{p}(0) \cdot z^0 + \int_0^T \nabla_y \bar{H}(t) \cdot v(t)dt.
\]

**Démonstration.** Using (2.4) and (2.11)-(2.16), we get:

\[
\nabla \varphi(\bar{y}(T)) \cdot z(T) = \bar{p}(T) \cdot z(T)
\]

\[
= \bar{p}(0) \cdot z(0) + \int_0^T \frac{d}{dt}(\bar{p}(t) \cdot z(t))dt
\]

\[
= \bar{p}(0) \cdot z(0) + \int_0^T (\dot{\bar{p}}(t) \cdot z(t) + \bar{p}(t) \cdot \dot{z}(t))dt
\]

\[
= \bar{p}(0) \cdot z(0) + \int_0^T (\bar{p}(t)^\dagger \bar{f}(t)v(t) - \nabla_y \bar{\ell}(t) \cdot z(t))dt.
\]

The result follows using (2.10) and the expression of \(\nabla_y \bar{H}(t)\), similar to (2.16).
We say that \((\bar{u}, \bar{y}^0)\) is a local minimum point of \(J_R\) in \(\mathcal{U} \times \mathbb{R}^n\) if
\[
J_R(\bar{u}, \bar{y}^0) \leq J_R(u, y^0), \quad \text{if } |\bar{u} - u| + |\bar{y}^0 - y^0| \text{ is small enough.}
\]
We define in the same way local minima of arbitrary function, paying attention to the fact that the definition depends on the norm. In the case of the space \(\mathcal{U} \times \mathbb{R}^n\), local minimum points are also called weak minima. Since the derivative of a function (if it exists) vanishes at a local minimum point, we deduce from lemma 2.2 that:

**Theorem 2.3.** (i) If \(\bar{u}\) is a local minimum point of \(J_R(\cdot, y^0)\), then
\[
H_u(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = 0, \quad \text{for a.a. sur } [0, T].
\]
(ii) If \(\bar{y}^0\) is a local minimum point of \(J_R(\bar{u}, \cdot)\), then \(\bar{p}(0) = 0\).

**Example:** quadratic regulator. Consider the case with dynamics \(f(u, y) := Ay + Bu\), cost integrand \(\ell(u, y) := \frac{1}{2}(u^TRu + y^TQy)\), final cost \(\varphi(y) := \frac{1}{2}y^TQ_Ty\), and fixed initial state, the matrices \(R, Q, Q_T\) being symmetrical. The costate equation is
\[
\begin{aligned}
-\dot{p}(t) & = A^T\bar{p}(t) + Q\bar{y}(t), \quad \text{for a.a. } t \in [0, T], \\
\bar{p}(T) & = Q_T\bar{y}(T),
\end{aligned}
\]
and the local optimality condition (2.19) reads
\[
R\bar{u}(t) + B^T\bar{p}(t) = 0, \quad \text{for a.a. } t \in [0, T].
\]
If \(R\) is invertible, eliminating the control with the previous equation, we see that \((\bar{y}, \bar{p})\) is solution of the two point boundary value problem
\[
\begin{aligned}
\dot{y}(t) & = Ay(t) - BR^{-1}B^T\bar{p}(t), \quad \text{for a.a. } t \in [0, T], \\
-\dot{p}(t) & = A^T\bar{p}(t) + Qy(t), \quad \text{for a.a. } t \in [0, T], \\
y(0) & = y^0; \quad p(T) = Q_Ty(T).
\end{aligned}
\]

**2.2. Optimality conditions with control constraints.** We assume here that the initial state is given, and that there are only control constraints of type (2.11). Eliminating the initial state as argument of the reduced cost, we can write the reduced problem in the form
\[
\text{Min } J_R(u) \quad \text{s.t. } (2.11).
\]
The set of admissible, or feasible controls is
\[
\mathcal{U}_{ad} := \{u \in \mathcal{U}; \ u(t) \in U_{ad} \text{ for a.a.}\}
\]
Assume in addition that
\[
\text{U}_{ad} \text{ is a closed convex set.}
\]

**Lemma 2.4.** The set \(\mathcal{U}_{ad}\) is a closed convex subset of \(\mathcal{U}\).

**Proof.** The convexity of \(\mathcal{U}_{ad}\) follows from the one of \(U_{ad}\). Now let \(u_k \to \bar{u}\) in \(\mathcal{U}\), \(u_k \in \mathcal{U}_{ad}\). From any subsequence we can extract another subsequence converging a.e., for which, by the dominated convergence theorem, \(\int_0^T \text{dist}(\bar{u}(t); U)dt\) is the limit of \(\int_0^T \text{dist}(u_k(t); U_{ad})dt\), equal to 0. The result follows.

**Proposition 2.5.** Let \(\bar{u}\) be a weak minimum. Then
\[
\hat{H}_u(t)(u - \bar{u}(t)) \geq 0, \quad \text{for all } u \in U_{ad}, \text{ for a.a. } t \in [0, T].
\]

**Déémonstration.** (a) Let \(\{u^k\}\) be a dense sequence in \(U_{ad}\). We first check that (2.26) is equivalent to
\[
\hat{H}_u(t)(u^k - \bar{u}(t)) \geq 0, \quad \text{for all } k \in \mathbb{N}, \text{ for a.a. } t \in [0, T].
\]
Obviously (2.26) implies (2.27). Conversely, set \(I_k := \{t \in [0, T]; \hat{H}_u(t)(u^k - \bar{u}(t)) < 0\}\). If (2.27) holds, the set \(I := \cup_k I_k\) has null measure, being a countable union of null measure sets. Over \([0, T] \setminus I\), we have \(\hat{H}_u(t)(u^k - \bar{u}(t)) \geq 0\) for all \(k\), and so \(\hat{H}_u(t)(u - \bar{u}(t)) \geq 0\) for all \(u \in U_{ad}\)
since $u^k$ is a dense sequence in $U_{ad}$, and so, (2.26) holds.

(b) Let $s \in [0, 1]$. Then $u^s := \tilde{u} + s(u - \tilde{u})$ is admissible since it is a convex combination of $\tilde{u}$ and $u$, and hence, when $s$ is small enough, $J_R(u^s) \leq J_R(\tilde{u})$. By lemma 2.2:

$$0 \leq \lim_{s \to 0} \frac{J_R(u^s) - J_R(\tilde{u})}{s} = J'_{R}(\tilde{u})(u - \tilde{u}) = \int_0^T \tilde{H}_u(t)(u(t) - \tilde{u}(t))dt.$$ (2.28)

(c) If (2.27) does not hold, there exists $E \subset [0, T]$ measurable with positive measure and $k \in \mathbb{N}$ such that $\tilde{H}_u(t)(u^k - \tilde{u}(t)) < 0$ for a.a. $t \in E$. Define $u \in \mathcal{U}$ by $u(t) = u^k$ if $t \in E$, and $u(t) = \tilde{u}$ otherwise. Then

$$J'_R(\tilde{u})(u - \tilde{u}) = \int_E \tilde{H}_u(t)(u^k - \tilde{u}(t))dt < 0,$$

contradicting (2.28). The conclusion follows. □

**Remark 2.6.** If $U_{ad} = \mathbb{R}^m$, (2.26) boils down to (2.19).

We next explicit (2.26) for some specific choices of $U_{ad}$.

**Example 2.7.** Bound constraints: $U_{ad} = \Pi_{i=1}^m [a_i, b_i]$ with $b_i > a_i$. We have for a.a. $t$, and for $i = 1 \sim m:$ $\tilde{H}_u(t) \geq 0$ if $\tilde{u}_i(t) = a_i$, $\tilde{H}_u(t) \leq 0$ if $\tilde{u}_i(t) = b_i$, $\tilde{H}_u(t) = 0$ if $\tilde{u}_i(t) \in [a_i, b_i]$.

**Example 2.8.** Constraint on Euclidean norm: $U_{ad} = B$ (closed unit ball). We have for a.a. $t$, if $\tilde{H}_u(t) \neq 0$, then $\tilde{u}(t) = -\tilde{H}_u(t)/|\tilde{H}_u(t)|$.

### 2.3. Pontryagin’s minimum principle.

Consider again problem (2.23), with now $U_{ad}$ a nonempty closed set, possibly nonconvex. We say that the trajectory $(\tilde{u}, \tilde{y})$ is a Pontryagin extremal if the associated costate $\tilde{p}$ satisfies the Hamiltonian inequality

$$H(\tilde{u}(t), \dot{\tilde{y}}(t), \tilde{p}(t)) \leq H(u, \dot{y}(t), p(t)) \text{ for all } u \in U_{ad}; \text{ for a.a. } t \in [0, T],$$

or equivalently

$$H(\tilde{u}(t), \dot{\tilde{y}}(t), \tilde{p}(t)) = \inf_{p \in U_{ad}} H(u, \dot{y}(t), p(t)) \text{ for a.a. } t \in [0, T].$$

If the previous relation holds, we also say that $(\tilde{u}, \tilde{y}, \tilde{p})$ (or $(\tilde{y}, \tilde{p})$ if no ambiguity is possible) is a Pontryagin biextremal.

One easily deduces from the Hamiltonian inequality that if $U_{ad}$ is convex, a Pontryagin extremal satisfies the first order condition (2.26).

Pontryagin’s minimum principle (PMP) is (in our framework) the following statement:

**Theorem 2.9.** Let $\tilde{u}$ be solution of (2.23). Set $\tilde{y} := y[\tilde{u}]$. Then $(\tilde{u}, \tilde{y})$ is a Pontryagin extremal.

The proof uses the following lemma. In the sequel, for some $M \geq \|\tilde{u}\|$, set

$$U^M := \{u \in \mathcal{U}; \|u\| \leq M\}.$$ (2.32)

**Lemma 2.10.** There exists $c_M > 0$ such that, for all $u \in U^M$, we have that for some $r(u) \in \mathbb{R}$:

$$J_R(u) = J_R(\tilde{u}) + \int_0^T (H(u(t), \dot{y}(t), p(t)) - \tilde{H}(t))dt + r(u), \text{ with } |r(u)| \leq c_M \|u - \tilde{u}\|^2_1.$$ (2.33)

**Démonstration.** Integrating by parts and using the costate equation we obtain

$$J_R(u) = J_R(u) + \int_0^T \tilde{p}(t) \cdot (f(u(t), y(t)) - \dot{y}(t))dt = \Delta_1(u) + \Delta_2(u),$$ (2.34)

with $y = y[u]$ and

$$\Delta_1(u) := \varphi(y(T)) - \varphi'(y(T))y(T) + \tilde{p}(0) \cdot \tilde{y}(0),$$

$$\Delta_2(u) := \int_0^T (H(u(t), y(t), p(t)) - \tilde{H}(t)y(t))dt.$$
Therefore, $J_R(u) - J_R(\bar{u}) = \Delta_1(u) - \Delta_1(\bar{u}) + \Delta_2(u) - \Delta_2(\bar{u})$. We have that $u(t)$ and consequently $y(t)$ remain in bounded subsets of $U$ and $Y$, over which the derivatives of any order of the dynamics, integrand of the cost and final cost are uniformly Lipschitz. So, setting $\delta y(t) := y(t) - \bar{y}(t)$, that satisfies $\delta y(0) = 0$:

\[
\Delta_1(u) - \Delta_1(\bar{u}) = \varphi(y(T)) - \varphi(\bar{y}(T)) - \varphi'(\bar{y}(T))\delta y(T) = O((\|\delta y(T)\|)^2).
\]

In addition

\[
\Delta_2(u) - \Delta_2(\bar{u}) = \int_0^T (H(u(t), \dot{y}(t), \bar{p})(t) - \bar{H}(t))dt + \Delta_3,
\]

où

\[
\Delta_3 := \int_0^T (H(u(t), \dot{y}(t), \bar{p})(t) - H(u(t), \dot{y}(t), \bar{p})(t) - H_y(t)\delta y(t))dt.
\]

Since

\[
(2.36) \quad H(u(t), \dot{y}(t), \bar{p})(t) - H(u(t), \dot{y}(t), \bar{p})(t) = \left(\int_0^1 H_y(u(t), \dot{y}(t) + s\dot{y}(t), \bar{p})(t)ds\right)\delta y(t),
\]

we have that $|\Delta_3| = O(\|u - \bar{u}\| + \|\delta y\|\|\delta y\|\|\delta y\|\|\delta y\|\|\delta y\|\)$. We conclude by noting that $\|\delta y\|\|\delta y\| = O(\|u - \bar{u}\|)$. 

**Proof of theorem 2.9** (a) We first show that, if the trajectory $(u, y)$ is admissible, then

\[
(2.37) \quad H(\bar{u}(t), \dot{y}(t), \bar{p})(t) \leq H(u(t), \dot{y}(t), \bar{p})(t) \quad \text{for a.a. } u \in [0, T].
\]

Otherwise, it would exist $\varepsilon > 0$ and a measurable subset $E$ of $[0, T]$ with positive measure, such that

\[
(2.38) \quad H(u(t), \dot{y}(t), \bar{p})(t) \leq H(u(t), \dot{y}(t), \bar{p})(t) - \varepsilon \quad \text{a.e. over } I.
\]

Let $E'$ be a measurable subset of $I$. Define $u' \in \mathcal{U}$ by

\[
u(t) = u(t), \quad \text{if } t \in E', \quad \text{and } u(t) = \bar{u}(t) \quad \text{otherwise.}
\]

Since $\|u' - \bar{u}\| \leq 2M \text{mes}(E')$, lemma 2.10 implies

\[
(2.39) \quad J_R(u') \leq J_R(\bar{u}) - \varepsilon \text{mes}(E') + 4cM^2 \text{mes}(E')^2,
\]

which gives a contradiction by choosing $E'$ with a sufficiently small, positive measure.  

(b) Let $u^k$ be a dense sequence in $U_{a,d}$, and $u^k$ a suite in $\mathcal{U}$ defined by $u^0 := \bar{u}$, and

\[
(2.40) \quad u^k(t) = u^k \text{ if } H(u^k(t), \dot{y}(t), \bar{p})(t) < H(u^{k-1}(t), \dot{y}(t), \bar{p})(t), \text{ and } u^k(t) = u^{k-1}(t) \text{ otherwise.}
\]

The density property of $u^k$ implies that $H(u^k(t), \dot{y}(t), \bar{p})(t) \to \inf_{u \in U_{a,d}} H(u, \dot{y}(t), \bar{p})(t) \text{ a.e.}$ We conclude by taking $u = u^k$ in (2.37).

**Remark 2.11.** If $U_{a,d} = \mathbb{R}^m$, one easily obtains that any Pontryagin extremal satisfies the following **Legendre-Clebsch** condition:

\[
(2.41) \quad R_{uu}(t) \succeq 0 \quad \text{for a.a. } t \in [0, T].
\]

We next introduce a new notion of minimum.

**Definition 2.12.** A sequence $u^k$ of $\mathcal{U}$ is said to converge to $\bar{u}$ in **Pontryagin’s sense** if there exists $M > 0$ such that $\|u^k\| \leq M$, and that $u^k \to \bar{u}$ a.e. (by the dominated convergence theorem, $u^k \to \bar{u}$ also in $L^s(0, T)^m$ for all $s \in [1, \infty]$). We say that $\bar{u}$ is a **Pontryagin minimum** if $u^k \to \bar{u}$ in Pontryagin’s sense implies that $J_R(\bar{u}) \leq J_R(u^k)$ for $k$ large enough.

**Remark 2.13.** One easily checks that the proof of theorem 2.9 still holds when $\bar{u}$ is a Pontryagin minimum. To any Pontryagin minimum is therefore associated a Pontryagin extremal.
2.3.1. Double integrator, quadratic energy. Consider the following problem. The state equation is

\[ \dot{h} = v; \quad \dot{v} = u \]

with \( h \) the distance, \( v \) the velocity and \( u(t) \) the acceleration. The initial state is given, and the cost function is \( \frac{1}{2} \int_0^T u(t)^2 dt + \varphi(h(T), v(T)) \). The Hamiltonian is \( H = \frac{1}{2} u^2 + p_h v + p_v u \). The costate equation is

\[ -\dot{p}_h = 0; \quad -\dot{p}_v = p_h; \quad p_h(T) = \nabla_h \varphi(h(T), v(T)); \quad p_v(T) = \nabla_v \varphi(h(T), v(T)). \]

So \( p_h \) is constant and so \( u = -p_v \) is an affine function of time. It follows that the optimal state is a cubic function of time.

Consider the case when \( \varphi(h, y) := \frac{1}{2} h^2 \). Then

\[ p_h(t) = p_h(T) = h(T); \quad p_v(T) = 0 \Rightarrow u(t) = -p_v(t) = (t - T)h(T). \]

Therefore

\[ v(t) = v_0 + \left( \frac{1}{2} t^2 - tT \right) h(T); \quad h(t) = h_0 + tv_0 + \left( \frac{1}{6} t^3 - \frac{1}{2} t^2 T \right) h(T) \]

Setting \( t = T \) we determine \( h(T) \) solution of

\[ h(T) = h_0 + T v_0 - \frac{1}{3} h(T) \]

so that we have computed the optimal control.

**Exercice 2.14.** Extend the analysis to the case of \( n \) integrations.

3. Pontryagin’s principle with two point constraints

3.1. Setting, Lagrangian function and costate. We now allow the initial and final state to vary, subject to some constraints that may couple them (as in the case of periodic trajectories). So, consider the cost function

\[ J^{IF}(u, y) := \int_0^T \ell(u(t), y(t)) dt + \varphi(y(0), y(T)), \]

where ‘IF’ stands for initial-final, and the two point constraints

\[ \Phi(y(0), y(T)) \in K_{\Phi}, \]

Here, \( \varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( \Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n_\Phi} \), are \( C^\infty \) mappings, and \( K_{\Phi} \) is a nonempty, closed convex subset of \( \mathbb{R}^{n_\Phi} \). Consider the optimal control problem

\[ \text{Min } J^{IF}(u, y); \quad \text{s.t. (2.1), (2.11) and (2.48).} \]

The pre Hamiltonian, in the non qualified form, is the function \( H : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \)

\[ H(\beta, u, y, p) := \beta \ell(u, y) + p \cdot f(u, y). \]

Denoting by \( \Psi \in \mathbb{R}^{n_\Phi} \) the multiplier associated with two point constraints. The two point Lagrangian is

\[ L^{IF}(\beta, y^0, y_T, \Psi) := \beta \varphi(y^0, y_T) + \Psi \cdot \Phi(y^0, y_T). \]

The Lagrangian of problem \( (2.49) \) is

\[ L(\beta, u, y, p, \Psi) := \beta J^{IF}(u, y) + \int_0^T p(t) \cdot (f(u(t), y(t)) - \dot{y}(t)) dt + \Psi \cdot \Phi(y(0), y(T)) \]

\[ = \int_0^T \left( H(\beta, u(t), y(t), p(t)) - p(t) \cdot \dot{y}(t) \right) dt + L^{IF}(\beta, y(0), y(T), \Psi). \]
Let us express the condition of stationarity (i.e., zero partial derivative) of the Lagrangian w.r.t. the state. After an integration by parts (valid since we assume that \( p \in Y \)), this boils down to the fact that, for all \( z \in \mathcal{Y} \):

\[
\int_0^T \left( \nabla_y H(\bar{\beta}, \bar{u}(t), \bar{y}(t), \bar{p}(t)) + \dot{\bar{p}}(t) \right) \cdot z(t) \, dt
+ (\nabla_y L^{IF}(\bar{\beta}, \bar{y}(0), \bar{y}(T), \bar{\Psi}) + \bar{p}(0)) \cdot z(0)
+ (\nabla_{\nu t} L^{IF}(\bar{\beta}, \bar{y}(0), \bar{y}(T), \bar{\Psi}) - \bar{p}(T)) \cdot z(T) = 0.
\]

Restricting first \( z \) to \( D(0, T)^n \), we see that, by lemma 1.14 \( \nabla_y H(\bar{\beta}, \bar{u}(t), \bar{y}(t), \bar{p}(t)) + \dot{\bar{p}}(t) \) must be equal to zero. Then taking \( z \) arbitrary in \( \mathcal{Y} \), we obtain that the coefficients of \( z(0) \) and \( z(T) \) should be zero. This leads us to define the costate equation as

\[
\begin{cases}
(i) & -\dot{\bar{p}}(t) = \nabla_y H(\bar{\beta}, \bar{u}(t), \bar{y}(t), \bar{p}(t)), \text{ for a.a. } t \in [0, T], \\
(ii) & -\bar{p}(0) = \nabla_y L^{IF}(\bar{\beta}, \bar{y}(0), \bar{y}(T), \bar{\Psi}), \\
(iii) & \bar{p}(T) = \nabla_{\nu t} L^{IF}(\bar{\beta}, \bar{y}(0), \bar{y}(T), \bar{\Psi}).
\end{cases}
\]

The above two last line are called transversality conditions.

**Definition 2.15.** We call Pontryagin multiplier associated to the nominal trajectory \((\bar{u}, \bar{y})\), any triple \( \lambda := (\bar{\beta}, \bar{\Psi}, \bar{p}) \) verifying \([2.54]\), \( \bar{\beta} \geq 0 \), \( \bar{\Psi} \in N_{K\phi}(\Phi(\bar{y}(0), \bar{y}(T))) \), the non nullity relation

\[
\bar{\beta} + |\bar{\Psi}| > 0,
\]

and the Hamiltonian inequality that generalizes \([2.31]\):

\[
H(\bar{\beta}, \bar{u}(t), \bar{y}(t), \bar{p}(t)) = \inf_{u \in U_{ad}} H(\bar{\beta}, u, \bar{y}(t), \bar{p}(t)) \quad \text{for a.a. sur } [0, T].
\]

We say that \((\bar{u}, \bar{y})\) is a Pontryagin extremal if the set \( \Lambda_P(\bar{u}, \bar{y}) \) of associated Pontryagin multipliers is non empty.

**Definition 2.16.** If \( U_{ad} \) is convex, we may define the set \( \Lambda_L(\bar{u}, \bar{y}) \) of Lagrange multipliers by changing the Hamiltonian inequality \([2.56]\) into the first order condition (compare to \([2.26]\)), i.e.,

\[
H_u(\bar{\beta}, \bar{u}(t), \bar{y}(t), \bar{p}(t))(u - \bar{u}(t)) \geq 0, \text{ for all } u \in U_{ad}, \text{ for a.a. sur } [0, T].
\]

The set \( \Lambda_P(\bar{u}, \bar{y}) \) of Pontryagin multipliers is a cone (stable by multiplication by a positive number). Relation \([2.55]\) prevents a multiplier to be zero. If \( \Lambda_P(\bar{u}, \bar{y}) \) is not empty, it contains therefore an element which has either \( \bar{\beta} = 0 \) (singular multiplier) or \( \bar{\beta} = 1 \) (regular multiplier).

**Theorem 2.17.** A solution of \([2.49]\) is a Pontryagin extremal.

**Proof.** See section 5. \( \square \)

**Example 2.18.** Problem \([2.23]\), that had an fixed initial state and a free final state, is a particular case of the present setting where \( \varphi \) depends only on the final state, \( \Phi(y^0, y_T) = y^0 \), and \( K_\phi \) is the singleton \( \{y^0\} \). The two point conditions of the costate equation reduce then to

\[
\bar{p}(T) = \bar{\beta} \nabla \varphi(\bar{y}(T)); \quad \bar{p}(0) = -\bar{\Psi}.
\]

Te last relation expresses \( \bar{\Psi} \) as a function of \( \bar{p}(0) \), and we may see \( \bar{p}(0) \) as a free variable. If \( \bar{\beta} \) was equal to 0, we would have that \( \bar{p}(T) = 0 \), and then \( \bar{p} = 0 \) by the costate equation, so that \( \bar{\Psi} = 0 \), contradicting \([2.55]\). Therefore, there exists a multiplier with \( \bar{\beta} = 1 \). We recover the conclusion of theorem 2.9.

**Example 2.19.** More generally, if the initial state is fixed, we may write \( \varphi \) as function of \( y_T \) only, and assume the two point constraints of the form \( (y^0, \Phi_F(y(T))) \in \{y^0\}_{\mathbb{R}^n} \times K_F \), with \( K_F \) convex and closed. Denoting by \( \bar{\Psi}_F \) the multiplier associated with the final state constraint, we
see that, eliminating the multiplier associated with the initial state constraint, we may replace the transversality conditions \((2.54)\) by

\[
\begin{align*}
\dot{p}(T) &= \beta \nabla \varphi(\bar{y}(T)) + D\Phi_{F}(\bar{y}(T))\Psi_{F}; \\
\Psi_{F} &\in N_{K_{F}}(\Phi_{F}(\bar{y}(T))),
\end{align*}
\]

and the nullity condition \((2.55)\) by

\[
\beta + |\Psi_{F}| > 0.
\]

3.2. Conservation of the pre Hamiltonian. Let \((\bar{u}, \bar{y})\) be a trajectory, and \(p \in \mathcal{Y}\) satisfy \((2.54)\) \((i)\). We say that \((\bar{u}, \bar{y}, \bar{p})\) is a Pontryagin extremal over \([0, T]\) if the Hamiltonian inequality \((2.56)\) holds. Set \(h(t) := H(\beta, \bar{u}(t), \bar{y}(t), \bar{p}(t))\).

**Lemma 2.20.** Let \((\bar{u}, \bar{y}, \bar{p})\) be a Pontryagin extremal over \([0, T]\). Then \(h(t)\) is, up to a null measure set, constant over \([0, T]\).

**Demonstrations.**

a) Consider first the easy case when \(u(t)\) is differentiable over \([0, T]\), and the absence of control constraints. Then \(H_{u}(t) = 0\) for all \(t \in [0, T]\), and (using the state and costate equations) \(h(t)\) has a derivative satisfying

\[
\dot{h}(t) = H_{y}(t)\bar{y}(t) + H_{p}(t)\bar{p}(t) = -\dot{p}(t) \cdot \dot{\bar{y}}(t) + \dot{\bar{y}}(t) \cdot \dot{\bar{p}}(t) = 0,
\]

from which the result follows.

b) We now deal with the general case, inspired by the proof in [6, Part II, Section 1.4]. We first claim that \(h(t)\) is a Lipschitz function. Indeed, set \(U'_{ad} := U_{ad} \cap B(0, ||\bar{u}||)\). Then for all \(t \in [0, T]\),

\[
|h(t') - h(t)| \leq \sup_{u \in U'_{ad}} |H(\beta, u, \bar{y}(t'), \bar{p}(t')) - H(\beta, u, \bar{y}(t), \bar{p}(t))| \\
\leq c_{1} (|\bar{y}(t') - \bar{y}(t)| + |\bar{p}(t') - \bar{p}(t)|) \leq c_{2}|t' - t|,
\]

as was to be proved.

c) We know that a Lipschitz function is the integral of its derivative. So, it suffices to show that, if \(h\) is differentiable at some \(t_{0} \in [0, T]\), then \(\dot{h}(t_{0}) = 0\). Let \(H(\beta, \cdot, \bar{y}(t_{0}), \bar{p}(t_{0}))\) attain its minimum over \(U'_{ad}\) at some point \(u_{0}\). By the Hamiltonian inequality,

\[
\dot{h}(t_{0}) \leq \lim_{t \to t_{0}} \frac{H(\beta, u_{0}, \bar{y}(t), \bar{p}(t)) - H(\beta, u_{0}, \bar{y}(t_{0}), \bar{p}(t_{0}))}{t - t_{0}}
\]

\[
= H_{y}(\beta, u_{0}, \bar{y}(t), \bar{p}(t))\dot{\bar{y}}(t_{0}) + H_{p}(\beta, u_{0}, \bar{y}(t), \bar{p}(t))\dot{\bar{p}}(t_{0}) \\
= -\dot{p}(t_{0}) \cdot \dot{\bar{y}}(t_{0}) + \dot{\bar{p}}(t_{0}) \cdot \dot{\bar{y}}(t_{0}) = 0.
\]

By taking \(t \uparrow t_{0}\) we would prove the opposite inequality in the same way. The result follows.

4. Extensions

4.1. Decision variables. Consider, for \((u, y, \pi) \in \mathcal{U} \times \mathcal{Y} \times \mathbb{R}^{n_{y}}\), the cost function

\[
J(u, y, \pi) := \int_{0}^{T} \ell(u(t), y(t), \pi) dt + \varphi(y(T), \pi).
\]

Here \(\ell : \mathbb{R}^{m} \times \mathbb{R}^{n} \times \mathbb{R}^{n_{y}} \to \mathbb{R}\) and \(\varphi : \mathbb{R}^{n} \times \mathbb{R}^{n_{y}} \to \mathbb{R}\) are \(C^{\infty}\) mappings. Here \((u, y, \pi)\) must satisfy the state equation

\[
\dot{y}(t) = f(u(t), y(t), \pi), \quad \text{for a.a. } t \in [0, T], \ y(0) = y^{0},
\]

the initial state \(y^{0}\) being given. So the optimal control problem is

\[
\text{Min } J(u, y, \pi) \text{ s.t. (2.64).}
\]
We can reduce this problem to the general format (2.49) by interpreting \( \pi \) as an additional state variable denoted by \( \pi \), with zero dynamics. Then the augmented state is \( y^a := (y^1, \pi)^T \), and the state equation and initial-final constraints read
\[
\dot{y}^a(t) = \begin{pmatrix} f(u(t), y(t), \pi(t)) \\ 0 \end{pmatrix} \quad \text{for a.a.} \ t \in [0, T]; \quad y(0) = y^0.
\]
The associated cost function is
\[
J(u, y^a, \pi) := \int_0^T \ell(u(t), y(t), \pi(t))dt + \varphi(y(T), \pi(T)).
\]
We now discuss the optimality conditions of the resulting problem of minimizing (2.67) s.t. (2.66). With the augmented state \( y^a := (y, \pi) \) is associated the augmented costate \( p^a := (p, p_\pi) \).
It is enough to discuss the optimality system in qualified form (\( \beta = 1 \)). The Hamiltonian for the reformulated problem is 
\[
H(u, y, \pi, p) := \ell(u, y, \pi) + p \cdot f(u, y, \pi).
\]
Since \( \pi(t) = \pi \) for all \( t \), the costate equation (2.54) takes here the expression
\[
\begin{aligned}
\dot{p}_\pi(t) &= \nabla_y H(\bar{u}(t), \bar{y}(t), \bar{\pi}, \bar{p}(t)), \quad \text{for a.a.} \ t \in [0, T], \\
-\dot{p}_\pi(0) &= 0, \\
\dot{p}(T) &= \nabla_{y^a} \varphi(\bar{y}(T), \bar{\pi}), \\
-\dot{p}(0) &= \nabla_{y^a} \varphi(\bar{y}(T), \bar{\pi}).
\end{aligned}
\]
Since \( \dot{p}(0) = -\int_0^T \dot{p}_\pi(t)dt + \bar{p}_\pi(T) \), we deduce that
\[
0 = \dot{p}(0) = \int_0^T \nabla_{y} H(\bar{u}(t), \bar{y}(t), \bar{\pi}, \bar{p}(t))dt + \nabla_{\pi} \varphi(\bar{y}(T), \bar{\pi}).
\]
This is nothing that the condition of stationarity of the Lagrangian of the original problem (of minimizing (2.63) s.t. (2.64)) w.r.t \( \pi \). We also have the Hamiltonian inequality for the control variables. So, we have proved the following:

**Lemma 2.21.** When the optimal control problem includes decision variables, the PMP has the same expression as for the corresponding problem with decision variables set at their nominal value, with addition of stationarity of the Lagrangian w.r.t the decision variables.

**Exercise 2.22.** Show that we obtain the same result when in (2.67) we choose to express the initial-final cost function as \( \varphi(y(T), \pi(0)) \).

4.1.1. A design problem. Consider the following model of a ground vehicle with rectilinear trajectory:
\[
\begin{aligned}
\dot{h} &= v; \quad \dot{v} = (eu - D(\gamma, v))/m; \quad \dot{m} = -u,
\end{aligned}
\]
where \( h \) is the distance to the initial position, \( v \) is the velocity, \( m \) is the (positive) mass, the control is the mass flow \( u \in [0, U] \), with \( U > 0 \), \( e > 0 \) is the speed of mass ejection, taken below equal to 1, and the nonnegative and smooth function \( c(v, \gamma) \) represents the drag force, function of the speed and of a design variable (think of a parametrized shape of the vehicle) \( \gamma \in \mathbb{R} \). The initial state is given: \( h(0) = 0, v(0) = 0, m(0) = m_0 > 0 \), and we have the final state constraint \( m(T) \geq m_f \). We want to maximize the final distance, but have a cost \( c(\gamma) \) for the design, and so we decide to minimize \( r c(\gamma) - h(T) \) where \( r > 0 \) is a parameter.

The Hamiltonian is
\[
H = p_h v + p_v(u - D(\gamma, v))/m - p_m u.
\]
The costate equation is
\[
\begin{aligned}
-\dot{p}_h &= H_h = 0; \\
-\dot{p}_v &= H_v = p_h - p_v D_v/m; \\
-\dot{p}_m &= H_m = -p_v(u - D)/m^2; \\
p_h(T) &= -1, \\
p_v(T) &= 0, \\
p_m(T) &= -\Psi \leq 0.
\end{aligned}
\]
Obviously \( p_0(t) = 1 \) for all \( t \), and so
\[
(2.73) \quad \dot{p}_v = 1 + p_v D_v / m; \quad \tilde{p}_v(T) = 1.
\]
Since \( \dot{p}_v(T) = 1 \) and \( p_v(T) = 0 \), \( p_v(t) < 0 \) for \( t \) close to \( T \). As \( \dot{p}_v > 0 \) when \( p_v \) is close to zero, necessarily \( p_v(t) < 0 \) for all \( t \).

It follows that \( \tilde{p}_m \) has the same sign as \( D - u \). Since \( p_v(T) = 0 \), \( \tilde{p}_m(T) = 0 \), and so:
\[
(2.74) \quad H_u = p_v/m - p_m; \quad H_u(T) = \Psi \geq 0; \quad \dot{H}_u(T) = 1/m(T).
\]
if \( \Psi = 0 \), we then have that \( H_u(t) < 0 \) for \( t \) close to \( T \), and so, the trajectory ends with a full thrust arc, say over \((t_0, T)\). We assume that \( D \leq U \) over the trajectory. Then \( p_m \) decreases over that arc, and since it final condition is zero, then it is positive. Reminding that \( p_v(t) < 0 \) for all \( t \), we see that \( H_u(T) = -p_m(t_0) < 0 \) so that \( t_0 = 0 \): the optimal policy is full thrust if \( \Psi = 0 \).

If on the contrary \( \Psi > 0 \), for \( t \) close to \( T \), \( H_u > 0 \) and so \( u = 0 \): the trajectory ends with a zero thrust arc (over which the mass equals \( m_f \)) say \((t_0, T)\). Over this arc \( p_m \) increases.

Now the optimality condition w.r.t. \( \gamma \), i.e. the stationarity of the Lagrangian w.r.t. this variable gives
\[
(2.75) \quad c'(\gamma) = \int_0^T \frac{p_v(t)}{m(t)} D_v(\gamma, v(t)) dt = 0.
\]
We may assume that, for instance, \( D(\gamma, v) = \frac{1}{2} \gamma v^2 \), i.e., \( \gamma \) is the aerodynamic coefficient usually denoted by \( C_X \), and that \( c'(\gamma) < 0 \) (it is expensive to achieve a small \( C_X \)). Then the above equation reads
\[
(2.76) \quad c'(\gamma) = \frac{1}{2} \int_0^T \frac{p_v(t)}{m(t)} v(t)^2 dt.
\]
Observe that as expected both sides have the same negative sign.

### 4.2. Variable horizon.

In many problems, the final time is itself an optimization parameter. For instance, in the minimum time problem, we aim to reach a certain target in a minimum time. So, assume that \( T \) is a decision variable. We make the change of variable \( \tau := t/T \), so that \( \tau \in [0, 1] \); \( \tau \) is called the normalized time., and \( y^N(\tau) := y(\tau T) \) is the the normalized time state, with a similar conventions for other variables such as the control. Then the normalized control and state satisfy the (normalized) state equation
\[
(2.77) \quad \dot{y}^N(\tau) = T f(u^N(\tau), y^N(\tau)), \quad \tau \in [0, 1]; \quad y^N(0) = y^0.
\]
On the other hand the cost function \((2.47)\) can, in this setting, be expressed as
\[
(2.78) \quad J^N(u^N, y^N) := T \int_0^1 \ell(u^N(\tau), y^N(\tau)) d\tau + \varphi(y^0, y^N(1)).
\]
In the normalized optimal control problem of minimizing \((2.78)\) s.t. \((2.77)\), the horizon appears as a decision variable and we can apply the analysis of section 4.2. We see that we obtain the same optimality conditions as in the case of a fixed horizon (theorem 2.17), with in addition the stationarity of the Lagrangian of the normalized problem w.r.t. the variable \( T \), i.e.,
\[
(2.79) \quad \int_0^T H(\beta, \tilde{u}(t), \tilde{y}(t), \tilde{p}(t)) dt = 0.
\]
Since the pre Hamiltonian is constant by lemma 2.20, we obtain the following result.

**Lemma 2.23.** If the final time is free, then PMP has the same expression as in the case of a fixed final time, with in addition the condition that the pre Hamiltonian has constant value 0.

The previous lemma is of course valid only if the final time does not enter in the cost function, dynamics and constraints.
Example 2.24. Let the horizon enter in the initial-final constraint (think of a rendez-vous problem), whose expression is \( \Phi_f(\bar{y}(0), \bar{y}(T), T) = 0 \). Denoting by \( \bar{h} \) the constant value of the Hamiltonian, we see that the condition of stationarity of the Lagrangian w.r.t. \( T \) is expressed as
\[
\bar{h} + \bar{\Psi} \cdot D_T \Phi(\bar{y}(0), \bar{y}(T), T) = 0.
\]

Example 2.25. In the case of a simple integrator \( \dot{y} = \alpha y + u \), with \( y(t) \in \mathbb{R}^n \), consider the problem of minimizing the cost function \( \int_0^T (1 + \frac{1}{2}|u(t)|^2)dt \), the final time being free. Here we minimize a compromise of the horizon and the quadratic energy, with constraints \( y(0) = a \) and \( y(T) = b \). The pre Hamiltonian is \( H = 1 + \frac{1}{2}|u|^2 + p \cdot (\alpha y + u) \). The costate dynamics is \( \dot{p} = \alpha p \), and so, \( p(t) = e^{-\alpha t}p(0) = -u(t) \).

When \( \alpha = 0 \), the trajectories are straight lines from \( a \) to \( b \) with constant speed say \( \dot{u} = (b-a)/T \), \( p(t) = -\dot{u}, y(t) = a + t(b-a)/T \), and so since the time is free
\[
0 = H = 1 - \frac{1}{2}\dot{u}^2 = 1 - \frac{1}{2}|b-a|^2/T^2
\]
which determines the optimal horizon \( T = |b-a|/\sqrt{2} \). Observe that the optimal speed has always modulus \( \sqrt{2} \).

4.3. Interior point constraints. It may happen that constraints hold on the state at finitely many times other that the initial and final ones. Typical examples are (i) interpolation problems, (ii) successive rendez-vous (e.g. of several asteroids with a satellite). We may formalize these constraints as follows. Consider a subdivision of \([0, T]\):
\[
0 = t_0 < t_1 < \cdots < t_{n_{pi}} < t_{n_{pi}+1} = T,
\]
and the interior point constraints
\[
\Phi^i(\bar{y}(t_i)) \in K_i, \quad i = 1, \ldots, n_{pi},
\]
where \( K_i \) is a closed convex subset of \( \mathbb{R}^n \). Let us describe in general terms how to reduce these constraints to the setting \((2.49)\). As in the case of variable horizon we can reparametrize each interval \([t_i, t_{i+1}]\) so that it starts at time 0 and ends at time 1, putting as decision variable the duration \( T_i := t_{i+1} - t_i \). One can then write reformulated dynamics corresponding to each interval. Of course one must add the continuity condition that each final state of an interval is (except for the last one) equal to the initial state of the next interval.

4.4. Minimal time problems, geodesics. Minimal time problems are the particular case of problems with free final time, zero final cost, cost integrand \( \ell(u, y) = 1 \), fixed inital state, and final constraint \( \Phi_{f}(\bar{y}(T)) \in K_{f} \) (not to speak of variants where the initial state may be only subject to some constraints). The pre Hamiltonian \( \beta + p \cdot f(u,y) \) is equal to zero along a Pontryagin extremal since the final time is free, and the costate equation is (see \((2.59)-(2.60)\)):
\[
\begin{cases}
-\dot{p}(t) = f_{y}(t)^{\dagger} \bar{P}(t) & \text{for a.a. } t \in [0, T]; \\
\bar{P}(T) = D\Phi_{f}(\bar{y}(T))^{\dagger}\bar{\Psi}_{f},
\end{cases}
\]
with \( \bar{\Psi}_{f} \in N_{K_{f}}(\bar{y}(T)) \), so that \( \bar{p} \) vanishes at a given time iff it vanishes for all time. The main result of the theory is:

Lemma 2.26. If \( (\bar{u}, \bar{y}) \) is solution of the minimal time problem, or more generally a Pontryagin extremal of this problem, then \( \bar{\Psi}_{f} \neq 0 \), and if \( \Phi_{f}(\bar{y}(T)) \) is surjective, then \( \bar{p} \) does not vanish at any time.

Démonstration. If on the contrary \( \bar{\Psi}_{f} = 0 \), then \( \bar{p}(t) = 0 \) for all \( t \) in view of \((2.84)\). Since the pre Hamiltonian has zero value this implies \( \beta = 0 \), contradicting \((2.60)\). So \( \bar{\Psi}_{f} \neq 0 \). If \( \Phi_{f}(\bar{y}(T)) \) is surjective, then its transpose matrix is injective, and so \( \bar{p}(T) = \Phi_{f}(\bar{y}(T))^{\dagger}\bar{\Psi}_{f} \) is not zero. The result follows.

\[\Box\]
Remark 2.27. The Hamiltonian inequality (2.56) reduces here to
\[
(2.85) \quad \dot{p}(t) \cdot f(\bar{u}(t), \bar{y}(t)) = \inf_{u \in U_{ad}} \dot{p}(t) \cdot f(u, \bar{y}(t)).
\]
Since the pre Hamiltonian has zero value, the above l.h.s. is nonpositive, and equal to 0 iff \(\dot{\beta} = 0\).

Remark 2.28. We could formulate minimal time problems with a zero integral cost and final cost \(T\). The resulting optimality conditions are equivalent to those previously written. By pre Hamiltonian of a minimal time problem one often understands the one of this second formulation, i.e., \(p \cdot f(u, y)\).

Example 2.29. Case of linear dynamics: \(f(u, y) = Ay + Bu\) with \(A\) and \(B\) matrices of compatible dimensions. Then \(\dot{p}(t) = A^t\dot{p}(t)\), and so, \(\dot{p}(t) = e^{(T-t)A^t}\dot{p}(T)\) is an analytic function (i.e., has a power series expansion). Denote by \(p(t)^{(k)}\) the \(k\)th derivative of \(p(t)\). The control must minimize over \(U_{ad}\) the function \(\dot{p}(t)^tBu\). If the latter vanishes identically, then for \(k = 0\) to \(n - 1\), we have that
\[
0 = B^t\dot{p}(t)^{(k)} = (-1)^kB^t(A^t)^k\dot{p}(t),
\]
whence \(\dot{p}(t)\) is orthogonal to \(A^{(k)}B\). It is said that the pair \((A, B)\) is controllable, if (in the context of linear systems) the family \(A^{(k)}B, k = 0\) to \(n - 1\), has rank \(n\). If this condition holds then \(\dot{p}\) vanishes identically, and we know that this is impossible when \(\Phi_f(\bar{y}(T))\) is surjective.

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\]
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Observe that, if over some open subset \(E\) of \(\mathbb{R}^n\) the function \(F\) is constant, then \(\dot{p}\) and consequently \(u\) are also constant, so that the extremal trajectories are lines over \(E\).

Remark 2.31. The minimal energy transfert consists in minimizing \(\frac{1}{2}\int_0^T \|u(t)\|^2 dt\) with the above dynamics and final constraints. The pre Hamiltonian is \(\frac{1}{2}\dot{\beta}u^2 + F(y)p \cdot u\). We again can easily check that \(\dot{\beta} > 0\) and therefore assume that \(\dot{\beta} = 1\). The unique control minimizing the pre Hamiltonian being \(\dot{u}(t) = -F(\bar{y}(t))\dot{p}(t)\), the state-costate pair has dynamics
\[
(2.87) \quad \dot{y} = -F(\bar{y})\dot{p}/|p|; \quad \dot{p} = |p|\nabla F(\bar{y}).
\]
Observe that this dynamics is, at each point, proportional to the one in (2.87). Therefore, the ray orbits of the state-costate pair in geometrical optics coincide with those of the minimal energy transfer. Besides, in the minimal energy transfert problem, since the pre Hamiltonian has a constant value equal to \(-\frac{1}{2}\|u(t)\|^2\), the same property holds for \(|\dot{u}(t)| = F(\bar{y}(t))|\dot{p}(t)|\): the solution of an extremal of a minimal energy transfert problem has constant control norm.

In the case of geometric optics, the media is isotropic, that is, the speed at a given point does not depend on the direction of propagation. We next present an extension to the case of isotropic media where the speed is an affine function of the direction of propagation.
Example 2.32 (Riemannian geometry). Consider the minimal time problem with nominal dynamics \( \dot{y} = F(\dot{y})u \), the symmetric matrix \( F(y) \), of size \( n \), being invertible for all \( y \), and \( C^\infty \) function of \( y \), with the constraint \( u(t) \in B \) for a.a. time, which means that \( \dot{y}^T F(\dot{y})^{-1} \dot{y} \leq 1 \). We still can reduce to the case when \( \beta = 1 \) and show that \( p \) cannot vanish. The pre Hamiltonian \( 1 + p^TF(y)u \) attains, over a Pontryagin extremal, its minimum at \( -F(\dot{y})\dot{p}/F(\dot{y})\dot{p} \), and so, \( \dot{y} = -F^2(y)\dot{p}/F(\dot{y})\dot{p} \). In addition, for \( k = 1 \) to \( n \), setting \( F_k(y) := \partial F(y)/\partial y_k \), we have that \( \dot{p}_k = \dot{p}^TF_k(y)F(\dot{y})\dot{p}/F(\dot{y})\dot{p} \).

The corresponding minimal energy transfer problem consists in minimizing \( \frac{1}{2} \int_0^T u(t)^2 dt \) over the previous dynamics and constraints. The pre Hamiltonian \( \frac{1}{2}u^2 + p^TF(y)u \) attains its minimum at \( u = -F(y)p \), and so \( \dot{y} = -F^2(y)p \) and \( \dot{p}_k = \dot{p}^TF_k(y)F(\dot{y})p \). Here again, the orbits of the two problems coincide. One calls geodesics the orbits of the state. The pre Hamiltonian having the constant value \( -\frac{1}{2}|\dot{u}(t)|^2 \), the control norm \( |\dot{u}(t)| = |F(\dot{y}(t))\dot{p}(t)| \) is also constant.

5. Technical results

5.1. Ekeland’s principle. An optimization problem with finite value, for any \( \varepsilon > 0 \), possesses an \( \varepsilon \)-optimal solution \( x \). The following result demonstrates that it is possible then to construct another \( \varepsilon \)-optimal solution, close to \( x \), that becomes the minimizer of a slightly perturbed objective function.

Theorem 2.33. (Ekeland’s variational principle) Let \((E, \rho)\) be a complete metric space and \( f : E \to \mathbb{R} \cup \{+\infty\} \) a lower semicontinuous function. Suppose that \( \inf_{e \in E} f(e) \) is finite and let, for a given \( \varepsilon > 0 \), \( e \in E \) be an \( \varepsilon \)-minimizer of \( f \), i.e., \( f(e) \leq \inf_{e \in E} f(e) + \varepsilon \). Then for any \( k > 0 \), there exists a point \( \hat{e} \in E \) such that \( \rho(e, \hat{e}) \leq k^{-1} \) and

\[
(2.89) \quad f(\hat{e}) \leq f(e) - \varepsilon k \rho(e, \hat{e}),
\]

\[
(2.90) \quad f(\hat{e}) - \varepsilon k \rho(e, \hat{e}) < f(e), \quad \forall e \in E, \ e \neq \hat{e}.
\]

Proof. Consider the multifunction \( M : E \to 2^E \) defined by

\[
M(e) := \{ e' : f(e') + k\varepsilon \rho(e, e') \leq f(e) \}.
\]

It is not difficult to see that \( M(\cdot) \) is reflexive, i.e., \( e \in M(e) \), and transitive, i.e., \( e' \in M(e) \) implies \( M(e') \subset M(e) \). Consider the function \( v : \text{dom} f \to \mathbb{R} \) defined by

\[
v(e) := \inf\{ f(e') : e' \in M(e) \}.
\]

We have that \( \inf_{E} f \leq v(e) \leq f(e) \), and

\[
(2.91) \quad \varepsilon k \rho(e, e') \leq f(e) - v(e), \quad \forall e' \in M(e).
\]

Since \( f(\hat{e}) - v(\hat{e}) \leq f(\hat{e}) - \inf_{E} f \leq \varepsilon \), it follows that

\[
(2.92) \quad k \rho(e, \hat{e}) \leq 1, \quad \forall e \in M(\hat{e}).
\]

Consequently, the diameter (i.e., the supremum of distances between two points) of \( M(\hat{e}) \) is less than or equal to \( 2k^{-1} \).

Consider a sequence \( \{ e_n \} \) satisfying

\[
e_1 = \hat{e}, \ e_{n+1} \in M(e_n), \text{ and } f(e_{n+1}) \leq v(e_n) + \varepsilon 2^{-n}.
\]

(By definition of \( v(\cdot) \), such a sequence exists.) Since \( M(e_{n+1}) \subset M(e_n) \) (by transitivity of \( M(\cdot) \)), we have \( v(e_n) \leq v(e_{n+1}) \), and since \( v(e) \leq f(e) \), it follows that

\[
v(e_{n+1}) \leq f(e_{n+1}) \leq v(e_n) + \varepsilon 2^{-n} \leq v(e_{n+1}) + \varepsilon 2^{-n},
\]

and hence \( 0 \leq f(e_{n+1}) - v(e_{n+1}) \leq \varepsilon 2^{-n} \). Combining this with \( (2.91) \), we obtain that \( k \rho(e_{n+1}, e) \leq 2^{-n} \) for all \( e \in M(e_{n+1}) \), and hence the diameter of \( M(e_{n+1}) \) tends to zero as \( n \to \infty \). Since in addition \( e_{n+1} \in M(e_n) \) and \( M(e_{n+1}) \subset M(e_n) \), it follows that \( \{ e_n \} \) is a Cauchy sequence. By completeness of the space \( E \) it follows that \( \{ e_n \} \) converges to some \( \hat{e} \in \text{dom}(f) \), and since the diameters of \( M(e_n) \) tend to zero, \( \bigcap_{n=1}^{\infty} M(e_n) = \{ \hat{e} \} \). Since \( e_n \) is included in the set \( M(\hat{e}) \) and \( M(\hat{e}) \) is closed, we have by \( (2.92) \) that \( k \rho(\hat{e}, \hat{e}) \leq 1 \), and by the definition of \( M(\hat{e}) \) that
f(\ddot{e}) + \varepsilon k\rho(\dot{e}, \ddot{e}) \leq f(\ddot{e})\), so that (2.89) holds. In addition, since \(M(\ddot{e}) \subset M(e_n)\) for all \(n\), and \(\text{diam}(M(e_n)) \to 0\), we have \(M(\ddot{e}) = \{\ddot{e}\}\), which implies (2.90). \(\square\)

Note that condition (2.89) of the above theorem implies that \(\ddot{e}\) is an \(\varepsilon\)-minimizer of \(f\) over \(E\), and that condition (2.90) means that \(\ddot{e}\) is the unique minimizer of the “perturbed” function \(f(\cdot) + \varepsilon k\rho(\cdot, \ddot{e})\) over \(E\). In particular, by taking \(k = \varepsilon^{-1/2}\) we obtain that for any \(\varepsilon\)-minimizer \(\tilde{e}\) of \(f\) there exists another \(\varepsilon\)-minimizer \(\ddot{e}\) such that \(\rho(\tilde{e}, \ddot{e}) \leq \varepsilon^{1/2}\) and \(\ddot{e}\) is the minimizer of the function \(f(\cdot) + \varepsilon^{1/2}\rho(\cdot, \ddot{e})\).

### 5.2. Ekeland’s metric and the penalized problem.

Consider the set
\begin{equation}
U_{ad} := L^\infty(0, T; U_{ad})
\end{equation}
endowed with Ekeland’s metric
\begin{equation}
\rho_E(u, v) = \text{mes}\{t; \ u(t) \neq v(t)\}.
\end{equation}
It is not difficult to check that \((U_{ad}, \rho_E)\) is a complete metric space. Since the initial state is not fixed, we need to consider the product space \(U_{ad} \times \mathbb{R}^n\), endowed with the augmented Ekeland metric
\begin{equation}
\rho_A([u, y_0], [v, y_0]) := |y_0 - y_0| + \rho_E(u, v).
\end{equation}
Again, it is easily checked that \((U_{ad} \times \mathbb{R}^n, \rho_A)\) is a complete metric space.

We next introduce the composite cost function, defined for \(\varepsilon > 0\):
\begin{equation}
J_{R}(u, y_0) := (J_R(u, y_0) - \bar{J} + \varepsilon^2)^2 + d_{K_{\Phi}}[\Phi(y_0, y_T(u, y_0))]^2\Big]^{1/2}.
\end{equation}
In other words, \(J_{R}(u, y_0)\) is the Euclidean norm of the two dimensional vector
\begin{equation}
F_{\varepsilon}(u, y_0) := \left(\frac{\rho_A([u, y_0], [v, y_0])}{d_{K_{\Phi}}[\Phi(y_0, y_T(u, y_0))]}\right).
\end{equation}
Let us see first how to make an expansion of the composite cost. If \((\tilde{u}, \tilde{y})\) is a feasible trajectory, set \(\tilde{y}_0 := \tilde{y}_0\). Note that \(J_{R}(\tilde{u}, \tilde{y}_0) > 0\), otherwise \((\tilde{u}, \tilde{y})\) would contradict the optimality of \((\tilde{u}, \tilde{y})\). Let us define
\begin{equation}
\tilde{\beta} := \left[\frac{J_{R}(\tilde{u}, \tilde{y}_0) - \bar{J} + \varepsilon^2}{J_{R}(\tilde{u}, \tilde{y}_0)}\right]^2; \quad \tilde{\Psi} := \frac{[\Phi(\tilde{y}_0, \tilde{y}_T) - P_{K_{\Phi}}[\Phi(\tilde{y}_0, \tilde{y}_T)]]}{J_{R}(\tilde{u}, \tilde{y}_0)},
\end{equation}
and let \(\tilde{p} \in \mathcal{P}_\infty\) be the unique solution of the costate equation (where we have removed the initial condition on the costate)
\begin{equation}
\begin{cases}
-\tilde{p}_t = \nabla_y H(\tilde{\beta}, \tilde{u}, \tilde{y}_t, \tilde{p}_t) \text{ for a.a. } t \in (0, T); \\
\tilde{p}_T = \nabla_{y_T} LIF(\tilde{\beta}, \tilde{y}_0, \tilde{y}_T, \tilde{\Psi}).
\end{cases}
\end{equation}

**Remark 2.34.** We remind that the Euclidean distance to \(K_{\Phi}\), notée \(d_{K_{\Phi}}(\cdot)\), has a continuously differentiable square, and the expression of its derivative is, denoting by \(P_{K_{\Phi}}\) la projection orthogonale sur \(K_{\Phi}\)
\begin{equation}
\left(\frac{1}{2}d_{K_{\Phi}}^2(w)\right)' = w - P_{K_{\Phi}}(w), \quad w \in \mathbb{R}^{n_{\Phi}}.
\end{equation}

**Lemma 2.35.** Let \((\tilde{u}, \tilde{y})\) be a feasible trajectory, such that \(J_{R}(\tilde{u}, \tilde{y}_0) > 0\). Then the following expansion holds
\begin{equation}
J_{R}(u, y_0) = J_{R}(\tilde{u}, \tilde{y}_0) + \int_0^T \left(\frac{1}{2}d_{K_{\Phi}}^2(w)\right)' \text{d}t + (\tilde{p}_0 + D_{y_0} LIF(\tilde{\beta}, \tilde{y}_0, \tilde{y}_T, \tilde{\Psi})(y_0 - \tilde{y}_0) + o(||u - \tilde{u}||_1 + |y_0 - \tilde{y}_0|).
\end{equation}
PROOF. Observe that, since \( J_R^e(\tilde{u}, \tilde{y}_0) > 0 \), in the \( L^1 \) vicinity of \((\tilde{u}, \tilde{y}_0)\), when viewed as a function of \((J_R(u, y) \Phi(y_0, y_T))\), \( J_R \) is a composition of \( C^1 \) mappings: (i) \( s \mapsto s^2_+ \) with derivative \( 2s_+ \), (ii) \((y_0, y_T) \mapsto \frac{1}{2} d_{K_\Phi}(y_0, y_T)\), and (iii) the Euclidean norm whose derivative at point \( x \neq 0 \) is \( x/|x| \). Therefore, we have that

\[
J_R(u, y_0) = J_R(\tilde{u}, \tilde{y}_0) + \beta J(\tilde{u}, \tilde{y}_0) + \frac{\Phi(y_0, y_T) - \Phi(\tilde{y}_0, y_T)}{|y_0 - y_T|} + o(\|u - \tilde{u}\|_1 + |y_0 - y_T|).
\]

(2.102)

Observe that the remainder is of order \( o(\|u - \tilde{u}\|_1 + |y_0 - y_T|) \). The “first order term” is a difference at points \((u, y_0)\) and \((\tilde{u}, \tilde{y}_0)\) of cost functions defined by

\[
J^\beta(\tilde{u}, \tilde{y}_0)(u, y_0) := \beta \int_0^T \ell(u, y_0) dt + L^I(\beta, \tilde{y}_0, \tilde{y}_T, \Phi).
\]

The associated costate at the point \((\tilde{u}, \tilde{y}_0)\) is therefore \( \tilde{p} \). We conclude by checking, by arguments essentially similar to those in the proof of lemma 2.10, taking into account the fact that the initial state may vary and that it enters into the initial-final cost, that

\[
J^\beta(\tilde{u}, \tilde{y}_0)(u, y_0) = J^\beta(\tilde{u}, \tilde{y}_0) + \beta \int_0^T \ell(u, y_0) dt + L^I(\beta, \tilde{y}_0, \tilde{y}_T, \Phi).
\]

(2.103)

The proof of theorem 2.17. a) We first deal with the case when \( U_{ad} \) is bounded. Setting \( \bar{J} := J^{IF}(u, y) \), apply Ekeland’s principle to the perturbed problem

\[
\min_{u, y_0} J_R^e(u, y_0);\quad u \in U_{ad};\quad y_0 \in \mathbb{R}^n.
\]

(2.105)

Since \( J_R^e \) is a nonnegative function, and \( J_R^e(\tilde{u}, \tilde{y}_0) = \varepsilon^2 \), we have that \((\tilde{u}, \tilde{y}_0)\) is a \( \varepsilon^2 \)-minimum of \((P_e)\). Since \( U_{ad} \) is bounded, it is easily checked that the function \((u, y_0) \mapsto J_R^e(u, y_0)\) is continuous for the augmented metric. By theorem 2.33 there exists \((u^e, y^e) \in U \times Y\) such that

\[
|y_0 - \tilde{y}_0| + \rho_E(u^e, \tilde{u}) \leq \varepsilon,
\]

(2.106)

and let \( \beta^e = \frac{(J_R^e(u^e, y_0^e) - \bar{J} + \varepsilon^2)}{J_R^e(u^e, y_0^e)};\quad \Psi^e = \frac{\Phi(y_0^e, y_T^e) - P_{K_\Phi}(\Phi(y_0^e, y_T^e))}{J_R^e(u^e, y_0^e)}\),

(2.107)

and let \( p^e \in W^{1,\infty}(0, T; \mathbb{R}^{n\times}) \) be the unique solution of

\[
\begin{align*}
-p^e_t &= \nabla_y H(\beta^e, u^e, y_t, p_t^e), \quad p.p.\quad t \in (0, T); \\
-p^e_T &= \nabla_y L^I(\beta^e, y_0^e, y_T^e, \Psi^e).
\end{align*}
\]

Then \( \beta^e + |\Psi^e|^2 = 1 \).

(2.108)

By lemma 2.35 we have that

\[
J_R^e(u, y_0) = J_R^e(u^e, y_0^e) + \int_0^T (H(\beta^e, u^e, y_t^e, p_t^e) - H(\beta^e, u_t^e, y_t^e, p_t^e)) dt \\
+ (\tilde{p}_0 + D_{y_0} L^I(\beta^e, y_0^e, y_T^e, \Psi^e))(y_0 - \tilde{y}_0) + o(|u - u^e|_1 + |y_0 - y_0^e|).
\]

(2.109)

b) Setting \( u = u^e \) in (2.106), obtain

\[
J_R^e(u^e, y_0) \leq J_R^e(u^e, y_0) + \varepsilon|y_0 - y_0^e|, \quad \text{for all } y_0 \in \mathbb{R}^n.
\]

(2.110)
Substituting in (2.110) the expression of $J_R^\varepsilon(u^\varepsilon, y_0)$ given in (2.109) (note that, since $u = u^\varepsilon$, the integral term vanishes) we get

\[ 0 \leq (p_0^\varepsilon + D_{y_0} L^I(\beta, y_0, y_0^\varepsilon, \Psi))(y_0 - y_0^\varepsilon) + \varepsilon |y_0 - y_0^\varepsilon| + o(|y_0 - y_0^\varepsilon|). \]

Taking $y_0 = y_0^\varepsilon - \rho(p_0^\varepsilon + D_{y_0} L^I(\beta, y_0^\varepsilon, y_0^\varepsilon, \Psi))$, with $\rho \downarrow 0$, deduce that

\[ (p_0^\varepsilon + D_{y_0} L^I(\beta, y_0^\varepsilon, y_0^\varepsilon, \Psi)) \leq \varepsilon. \]

Equivalently, denoting $\bar{\beta}$, then

\[ B \text{ where } \frac{d}{dt} \bar{\beta} = g(\bar{\beta}, \bar{u}, \bar{y}^\varepsilon, p_t^\varepsilon), \]

we may then pass to the limit in (2.116) on the set $\{ t \in (0,T) \}$.

Therefore the costate equation is satisfied. Finally, let $y = y_0^\varepsilon - \varepsilon \rho_E(u, u^\varepsilon)$, for all $u \in U_\infty(u)$.

Substituting in (2.113) the expression of $J_R^\varepsilon(u, y_0^\varepsilon)$ provided by (2.109), get

\[ 0 \leq \int_0^T (H(\beta^\varepsilon, u, y^\varepsilon_t, p_t^\varepsilon) - H(\beta, u, y^\varepsilon_t, p_t^\varepsilon) + \varepsilon) \, dt + o(\varepsilon). \]

Equivalently, denoting $I_\varepsilon := \{ t \in (0,T); \; u_t \neq u^\varepsilon_t \}$, we have that

\[ 0 \leq \int_{I_\varepsilon}(H(\beta^\varepsilon, u, y^\varepsilon_t, p_t^\varepsilon) - H(\beta, u, y^\varepsilon_t, p_t^\varepsilon) + \varepsilon) \, dt + o(\varepsilon). \]

By techniques similar to those presented after (2.38), we obtain that

\[ H(\beta^\varepsilon, u^\varepsilon(t), y^\varepsilon(t), p^\varepsilon(t)) \leq H(\beta, u, y^\varepsilon(t), p^\varepsilon(t)) + \varepsilon, \]

for all $v \in U_{ad}$, for a.a. $t \in (0,T)$.

d) We now pass to the limit in (2.108)-2.16. Since $\rho_E(u, u^\varepsilon) \rightarrow 0$ and $U_{ad}$ is bounded, we have that $y^\varepsilon \rightarrow y$ uniformly. By (2.108), $(\beta, \Psi)$ has a unit norm limit point $(\bar{\beta}, \bar{\Psi})$, so that the Hamiltonian inequality holds. Also, passing to the limit in the relation (consequence of the definition of projection)

\[ \Psi_\varepsilon(k - K_{\bar{\beta}}(\Phi(y_0, y^\varepsilon_T))) \leq 0, \quad \text{for all } k \in K_{\bar{\beta}}, \]

and since $\Phi(y_0, y_T) \in K_{\bar{\beta}}$, we get that $\bar{\Psi} \in N_{K_{\bar{\beta}}}(\Phi(y_0, y_T))$. There is no difficulty in passing to the limit in (2.112) and (2.99) (the costate $p^\varepsilon$ remains bounded since $\beta^\varepsilon$ et $\Psi_\varepsilon$ are bounded). Therefore the costate equation is satisfied. Finally, let $\{ \varepsilon_k \}$ be a sequence associated with the limit-point $(\bar{\beta}, \bar{\Psi})$. For each $k$, (2.116) is satisfied on a set $\mathcal{J}_k$, of full measure dans $[0,T]$. We may then pass to the limit in (2.116) on the set $\cap_k \mathcal{J}_k$, that is also of full measure. The conclusion follows.

e) We end the proof by dealing with the case when $U_{ad}$ is not bounded. Set

\[ U_R := U_{ad} \cap B_R, \]

where $B_R$ is the ball of radius $R$ and center 0 in $\mathbb{R}^m$, and $U_R := L^\infty(0,T; U_R)$. If $R \geq \|\bar{u}\|_\infty$,

then $\bar{u}$ is solution of the problem

\[ \min J_R(u); \quad u \in U_R. \]

By point d), there exists $(\beta_R, \Psi_R) \in \mathbb{R}_+ \times N_{K_{\bar{\beta}}}(\Phi(y_0, y_T))$, with $(\beta, \Psi) \neq 0$, and $p^R \in \mathcal{Y}$, such that

\[ -\dot{p}^R_t = \nabla_y H(\beta_R, \bar{u}, \bar{y}_t, p^R_t), \quad \text{for a.a. } t, \]

\[ \bar{u}_t = \arg\min_{u \in U_R} H(\beta, u, \bar{y}_t, p^R_t), \quad \text{for a.a. } t, \]

\[ -p_0^R = \nabla_{y_0} L^I(\beta_R, y_0, y_T, \Psi_R); \]

\[ p_T^R = \nabla_{y_T} L^I(\beta_R, y_0, y_T, \Psi_R); \]

\[ \Psi_R \in N_{K_{\bar{\beta}}}(\Phi(y_0, y_T)). \]

Multiplying $(\beta_R, \Psi_R, p^R)$ by $(|\beta_R, \Psi_R|)^{-1}$, we may assume that $(|\beta_R, \Psi_R|) = 1$. It easily follows that $p^R$ is bounded in $\mathcal{Y}$. We conclude by passing to the limit in an extracted sequence for
which \((\beta_R, \Psi_R)\) converges, in \((2.119)-(2.123)\), using again the fact that a countable intersection of sets of full measure is of full measure.
State constrained problems

1. Pontryagin’s principle

Here we consider problems where additional inequality constraint on the state hold for each time. So the optimal control problem has now infinitely many inequality constraints, and so, we will need to rely on more general abstract tools than those previously introduced.

1.1. Setting. We recall the following definitions: the cost function

\[ J^F(u, y) := \int_0^T \ell(u(t), y(t)) \, dt + \varphi(y(0), y(T)), \]

the state equation

\[
\begin{align*}
(i) \quad & \dot{y}(t) = f(u(t), y(t)), \quad \text{for a.a. } t \in [0, T]; \\
(ii) \quad & y(0) = y^0,
\end{align*}
\]

the control constraint, where \( U_{ad} \) is a closed subset of \( \mathbb{R}^m \),

\[
\begin{align*}
& u(t) \in U_{ad}, \quad \text{for a.a. } t \in [0, T], \\
\end{align*}
\]

and the two point constraints

\[ \Phi(y(0), y(T)) \in K_\Phi. \]

We add state constraints of the form

\[ g_j(y(t)) \leq 0, \quad j = 1, \ldots, n_g, \quad \text{p.p. } t \in [0, T]. \]

All mappings are of class \( C^\infty \). We consider the following optimal control problem:

\[ \text{Min} \quad J^F(u, y) \quad \text{s.t. (3.2)-(3.5)}. \]

1.2. Duality in spaces of continuous functions. Pontryagin’s principle involves multipliers associated with constraints. In the case of the state constraint we must first choose the function space in which this constraint is expressed, and see what are the properties of the dual space, where the multipliers lives. It appears that a convenient choice is \( X_g := C([0, T])^{n_g} \), since it is a Banach space whose dual forms have a very specific structure.

Indeed, it is known that any continuous linear form on \( C([0, T]) \) is of the type \( h \mapsto \int_0^T h(t) \, d\mu(t), \) where \( d\mu \in M(0, T) \), the space of Borel measures over \([0, T]\). In addition, one can identify the action of such a measure \( d\mu \) with a Stieltjes integral of a bounded variation function denoted by \( \mu \), that we may assume to have value 0 at time \( T \). We recall that the variation of a function \( \mu \) over \([0, T]\) is

\[ \text{var}(\mu) := \sup \left\{ \sum_{i=0}^{n_p} |\mu(t_{i+1}) - \mu(t_i)|; \quad \text{where } (t_i) \text{ is a subdivision of } [0, T] \right\}, \]

(subdivisions were defined in (2.82)), and that the Stieltjes integral of \( h \in C([0, T]) \) associated with the bounded variation function \( \mu \) is

\[ \int_0^T h(t) \, d\mu(t) := \lim \sum_{i=0}^{n_p} h(\tau_i)(\mu(t_{i+1}) - \mu(t_i)) \]

where \( \tau_i \in [t_i, t_{i+1}] \), the limit being taken over subdivisions of vanishing maximal increment.
The Lagrangian of problem (3.6) is the sum of the one of problem (2.49) (defined in (2.52)), up to the addition of the contribution \( \Delta^g : \mathcal{Y} \times M(0,T)^n_y \to \mathbb{R} \) of the state constraint, i.e., skipping some arguments:

\[
\Delta^g(y, \mu) := \sum_{j=1}^{n_y} \int_0^T g_j(y(t)) d\mu_j(t).
\]

Note that, for all \( z \in \mathcal{Y} \):

\[
\Delta^g(y, \mu)z = \sum_{j=1}^{n_y} \int_0^T g_j^*(y(t))z(t) d\mu_j(t).
\]

Again, the costate equation is obtained by expressing the stationarity of the Lagrangian w.r.t. the state. We are looking now for the costate in the space \( \mathcal{P} := BV(0,T)^n \) of bounded variation functions. Since \( y \in \mathcal{Y} \), the following integration by parts formula is valid:

\[
\int_0^T p(t) \cdot \dot{y}(t) dt = p(T) \cdot y(T) - p(0) \cdot y(0) - \sum_{i=1}^n \int_0^T y_i(t) dp_i(t).
\]

Using it we easily express the directional derivative of the Lagrangian w.r.t. the state in a direction \( z \in \mathcal{U} \), and deduce that the costate equation is, \( L^{IF}(\cdot) \) being defined in (2.51):

\[
\begin{aligned}
\begin{cases}
- dp(t) & = \nabla_y H(\bar{\beta}, \bar{u}(t), \bar{y}(t), p(t)) dt + \sum_{j=1}^{n_y} \nabla g_j(\bar{y}(t)) d\bar{\mu}_j(t), \quad \text{p.p.} \ t \in [0,T], \\
- p(0) & = \nabla_y L^{IF}(\bar{\beta}, \bar{y}(0), \bar{y}(T), \bar{\Psi}), \\
p(T) & = \nabla_y L^{IF}(\bar{\beta}, \bar{y}(0), \bar{y}(T), \bar{\Psi}).
\end{cases}
\end{aligned}
\]

We call Pontryagin multiplier, associated with the nominal trajectory \( (\bar{u}, \bar{y}) \), any \( \lambda = (\bar{\beta}, \bar{\Psi}, \bar{\mu}, \bar{p}) \) satisfying (3.11), \( \bar{\Psi} \in N_{K^\alpha}(\Phi(\bar{y}(0), \bar{y}(T))) \), \( \bar{\beta} \geq 0 \), the Hamiltonian inequality (2.56), and the complementarity relations

\[
d\bar{\mu} \geq 0; \quad \int_0^T g_j^*(\bar{y}(t)) d\bar{\mu}_j(t) = 0, \quad j = 1, \ldots, n_y,
\]

and of non nullity

\[
\bar{\beta} + |\bar{\Psi}| + |\bar{\mu}| > 0.
\]

**Theorem 3.1.** Any \( (\bar{u}, \bar{y}) \) solution of (3.23), is a Pontryagin extremal.

**Proof.** See section 2. \( \square \)

### 1.3. Constraint order and junction conditions.

We consider the simple case when

\[
\text{(3.14)} \quad \text{there are no control constraint} \quad \text{and the extremal is qualified (} \bar{\beta} = 1 \).
\]

We denote the jumps of function of time having left and right limits at \( \tau \in [0,T] \) (this is the case for functions with bounded variation) by a bracket, e.g. \( [p(\tau)] := \bar{p}(\tau_+) - \bar{p}(\tau_-) \), where we set \( \bar{p}(0_-) := \bar{p}(0_+) \) and \( \bar{p}(T_-) := \bar{p}(T_+) \). We also set \( \bar{\nu}_j(\tau) := [\bar{\mu}_j(\tau)] \). In view of (3.11), we have, for all \( \tau \in [0,T] \):

\[
\begin{aligned}
- [p(\tau)] & = \sum_{j=1}^{n_y} \bar{\nu}_j(\tau) \nabla g_j(\bar{y}(\tau)).
\end{aligned}
\]

Let the total derivative de \( g_j \) (along the dynamics \( f \)) be defined by \( g_j^{(1)}(u,y) := g_j'(y)f(u,y) \). Along the trajectory \( (u,y) \), \( g_j^{(1)}(u,y) \) is the time derivative of \( g_j(y) \). If \( g_j^{(1)}(u,y) \) depends on \( u \), i.e., if \( D_u g_j^{(1)}(u,y) \) does not identically vanishes, we say that \( g_j \) is a first order state constraint.

Otherwise, writing \( g_j^{(1)} \) as function of \( y \) only, we may compute the total derivative of the latter, i.e., skipping some arguments:

\[
\begin{aligned}
g_j^{(2)}(u,y) & := D_y g_j^{(1)}(y)f(u,y) = g_j''(y)(f,f) + g_j'(y) f_y f.
\end{aligned}
\]
If \( g_j^{(2)}(u, y) \) depend on \( u \), we say that \( g_j \) is a second order state constraint, etc. We denote by \( g_j \) the order, assumed to be well-defined, of \( g_j \), by \( I_r \subset \{1, \ldots, n_q\} \) the set of constraints of order \( r \), et \( I_r(t) \) those active at time \( t \). Consider the hypothesis

\[
(3.17) \quad \begin{cases}
\dot{u} \text{ is continuous and } \{\nabla_u g_j^{(1)}(\bar{u}(t), \bar{y}(t)), \quad j \in I_1(t)\} \\
is \text{linearly independent, for all } t \in [0, T].
\end{cases}
\]

**Lemma 3.2.** Let \((u, y)\) be a Pontryagin extremal, and \((\bar{p}, \bar{\mu})\) an associated Pontryagin multiplier. Then (i) If \( H(\bar{\beta}, \cdot, \bar{y}(t), \bar{p}(t)) \) is (uniformly in time) strongly convex, \( \bar{u} \) is continuous. (ii) If \( (3.17) \) is satisfied, the components of \( \bar{\mu} \) associated with first order state constraints are continuous over \([0, T]\.\)

**Proof.** (i) Fix \( t \in [0, T]\.\) Set \( p^- := \bar{p}(t^-), \quad p^+ := \bar{p}(t^+) \), and for \( \sigma \in [0, 1] \),

\[
(3.18) \quad p^\sigma := (1 - \sigma)p^- + \sigma p^+.
\]

Since \( H(\bar{\beta}, \cdot, \bar{y}(t), \bar{p}(t)) \) is strongly convex, the control variable is a continuous function of the state and costate, and hence, has bounded variation, so that we may define in the same way \( u^\pm \) and \( u^\sigma \). We have that

\[
0 = \nabla_u H(\bar{\beta}, \bar{u}^+, \bar{y}_t, \bar{p}^+) - \nabla_u H(\bar{\beta}, \bar{u}^-, \bar{y}_t, \bar{p}^-) = \int_0^1 \left( H_{uu}(\bar{\beta}, \bar{u}^\sigma, \bar{y}_t, p^\sigma)[\bar{u}] + f_u(u^\sigma, \bar{y}_t)^\top [\bar{p}] \right) \, d\sigma.
\]

Using \( (3.15) \) and observing that \( g_{i,y} f_u = g_{i,u}^{(1)} \) if \( q_i > 1 \), we obtain that

\[
(3.20) \quad \int_0^1 H_{uu}(\bar{\beta}, \bar{u}^\sigma, \bar{y}_t, p^\sigma) \, d\sigma[\bar{u}] = \int_0^1 \sum_{i \in I_1} \nu_i \nabla_u g_i^{(1)}(u^\sigma, \bar{y}_t) \, d\sigma.
\]

Taking the scalar product of both sides of \( (3.20) \) by \([\bar{u}]\), by the strong convexity property of \( H(\bar{\beta}, \cdot, \bar{y}(t), \bar{p}(t)) \), we get that for some \( \alpha > 0 \):

\[
(3.21) \quad \alpha [\bar{u}]^2 \leq \sum_{i \in I_1} \int_0^1 \nu_i g_i^{(1)}(u^\sigma, \bar{y}_t) [\bar{u}] \, d\sigma = \sum_{i \in I_1} \nu_i [g_i^{(1)}(\bar{u}, \bar{y})].
\]

If \( \nu_i > 0 \), then \( g_i(\bar{y}(t)) = 0 \), and hence \( [g_i^{(1)}(\bar{u}, \bar{y})] \leq 0 \) since \( t \) is a local maximum of \( g_i(\bar{y}_t) \). Therefore, the right-hand side in \( (3.21) \) is nonpositive, implying \([\bar{u}] = 0\). Point (i) follows.

(ii) Obvious consequence of the following relation, itself implied by \( (3.15) \), noting that the contribution of state constraints of order greater than 1 to the sum below is zero:

\[
(3.22) \quad 0 = [\tilde{H}_u(t)] = [\bar{p}(t)] \cdot \tilde{f}_u(t) = \sum_{j=1}^{n_q} \tilde{p}_j(t) \nabla_u g_j^{(1)}(\bar{u}(t), \bar{y}(t)).
\]

We next show the link between a jump of the derivative of the control and those of \( \mu \) or of its derivative.

**Lemma 3.3.** Let \( (3.17) \) hold. Then (i) \( \tilde{H}_{uu}(t) \) is continuous, and (ii) if the jumps below are well defined, then, for all \( \tau \in [0, T]\.\):

\[
(3.23) \quad \tilde{H}_{uu}(\tau)[\dot{\bar{u}}(\tau)] = \sum_{j \in I_1(\tau)} [\dot{\bar{\mu}}_j(\tau)] \nabla_u g_j^{(1)}(\tau) - \sum_{j \in I_2(\tau)} \tilde{p}_j(\tau) \nabla_u g_j^{(2)}(\tau).
\]

**Proof.** (i) If \( j \in I_1(\tau) \), then \( \tilde{p}_j(\tau) = 0 \) by lemma 3.2 (ii), otherwise, \( g_j^{(1)}(u, y) \) does not depend on \( u \), so that, for \( 1 \leq i, k \leq m \):

\[
-\tilde{H}_{u_i u_k}(\tau) = -[\bar{p}(\tau)] \cdot \tilde{f}_{u_i u_k}(\tau) = \sum_{j=1}^{n_q} \tilde{p}_j(\tau) g_j^{(1)}(\tau) \tilde{f}_{u_i u_k}(\tau) = \sum_{j=1}^{n_q} \tilde{p}_j(\tau) D_{u_i u_k}^2 g_j^{(1)}(\tau, \bar{y}(\tau)) = 0.
\]
The result follows. (ii) Let $i \in \{1, \ldots, m\}$. Assuming for simplicity that $\ell = 0$ and skipping time arguments, we get
\[
\frac{d}{dt} H_{ui} = \frac{d}{dt} \langle \dot{p} \cdot \dot{f}_u \rangle = \dot{H}_{ui} \dot{u} + \dot{p} \cdot \dot{f}_{ui} + \dot{p} \cdot \dot{f}_u.
\]
By the costate equation and the fact that $\dot{g}_j \ddot{f}_u = 0$ if $j \notin I_1(\tau)$, we get
\[
(3.24) \quad \frac{d}{dt} H_{ui} = \dot{H}_{ui} \dot{u} + \dot{p} \cdot (\ddot{f}_{ui} - \ddot{y} \ddot{f}_u) - \sum_{j \in I_1} \dot{\mu}_j \dot{g}_j \ddot{f}_u.
\]
Since $\dot{H}_{ui} = \dot{\hat{H}}_{ui} = 0$ a.e., we have that $[\dot{H}_{ui}] = 0$. The control variable and $\dot{H}_{uu}$ being continuous, this implies
\[
(3.25) \quad \dot{H}_{ui} [\dot{u}] = -[\dot{p}] \cdot (\ddot{f}_{ui} - \ddot{y} \ddot{f}_u) + \sum_{j \in I_1} [\dot{\mu}_j] \dot{g}_j \ddot{f}_u.
\]
Now, if $j \notin I_1(\tau)$, we have that $0 = \left( \frac{\partial}{\partial y}(\dot{g}_j \ddot{f}_u) \right) \ddot{f} = \dot{g}''_j \ddot{f} \ddot{f}_u + \dot{g}'_j \ddot{y} \ddot{f}_u + \dot{g}_j \ddot{f}_u$, and therefore
\[
(3.26) \quad \dot{g}^{(2)}_{j,ui} = 2\dot{g}''_j \ddot{f} \ddot{f}_u + \dot{g}'_j \ddot{y} \ddot{f}_u + \dot{g}_j \ddot{f}_u.
\]
We conclude by replacing $[\dot{p}] = -\sum_{j=1}^{n_g} \nu_j \nabla \dot{g}_j$ in (3.25), and noting that $\nu_j = 0$ when $q_j = 1$, and $\dot{g}^{(2)}_{j,ui} = 0$ if $q_j > 2$. \[\square\]

**Example 3.4.** Let (3.17) hold, $\dot{H}_{uu}(\tau)$ be invertible, the state constraint being scalar. If it is of first order and $g^{(1)}_u(\tau) \neq 0$, then $\dot{H}_{uu}(\tau)[\dot{u}(\tau)] = [\dot{\mu}(\tau)] \nabla_{uy} g^{(1)}(\tau)$ and so, $\dot{u}(\tau)$ is discontinuous iff $\dot{\mu}$ is. If the constraint is of higher order, and $g^{(2)}_u(\tau) \neq 0$, then $\dot{H}_{uu}(\tau)[\dot{u}(\tau)] = -\ddot{v}(\tau) \nabla_{uy} g^{(2)}(\tau)$ and $\dot{u}(\tau)$ is discontinuous iff $\dot{\mu}$ is. If the constraint is not or order one or two, then $\dot{u}(\tau)$ is continuous. This illustrates the fact that the behavior depends in an essential way of the order of the state constraint.

**Example 3.5.** Consider a problem of equilibrium of an elastic string with length $T$, fixed at endpoints, with obstacle. Specifically, we minimize the sum of elastic and gravity energies, i.e., $E(y) := \int_0^T \left( \frac{1}{2} \dot{y}(t)^2 + y(t) \right) dt$, where $\dot{y}(t)$ represents the vertical deformation, with constraints $\dot{y}(0) = \dot{y}(T) = 1$ and $\dot{y}(t) \geq 0$ for all $t$.

We can reformulate this problem as an optimal control one with state equation $\dot{\dot{y}}(t) = \ddot{\dot{u}}(t)$, integral cost with integrand $\ell(u,y) := \frac{1}{2} u^2 + y$, the state constraint $g(y) = -y$, and the initial-final constraints $\Phi(y_0,y_T) := (y_0,y_T)$. For large enough $T$, the optimal state is, for some $t_0 \in [0,\frac{1}{2}T]$, equal to zero over $[t_0,t_1]$ with $t_1 = T-t_0$, and positive outside. The pre Hamiltonian is $H(u,y,p) = \frac{1}{2} u^2 + y + pu$, and the costate dynamics is $-d\ddot{p}(t) = dt - d\dot{\mu}(t)$. Since $\dot{\mu}(T) = 0$, it follows that $\ddot{p}(t) = t - t + \dot{\mu}(t) + \dot{p}(T)$. By the Hamiltonian inequality, $\ddot{u}(t) = -\ddot{p}(t)$. Over $[t_0,t_1]$, we have therefore $0 = \ddot{u}(t) = -\ddot{p}(t)$, which implies $\dot{\mu} = 1$. Over each unconstrained arc, we have that $\ddot{p}(t) = c - t$ for some $c \in \mathbb{R}$, and hence, $\ddot{u}(t) = t - c$. By lemma 3.2, both the control and state constraint multiplier are continuous at times $t_0$ and $t_1$, and so, the control is equal to $t - t_0$ over $[0,t_0]$ and $t - t_1$ over $[t_1,T]$. Observe that while $\ddot{u}$ has a jump of $-1$ at $t_0$ and $1$ at $t_1$, since $\dot{\mu} = 0$ out of $[t_0,t_1]$, the jump of $\dot{\mu}$ is $1$ at $t_0$ and $-1$ at $t_1$, so that (3.23) holds, as expected.

**Example 3.6.** Consider a problem of equilibrium of an elastic beam with length $T$, fixed at endpoints and with obstacle. We minimize the sum of elastic and gravity energies: $E(x) := \int_0^T \left( \frac{1}{2} \dot{x}(t)^2 + x(t) \right) dt$, where $\dot{x}(t)$ represent the vertical deformation, subject to $\dot{x}(0) = \dot{x}(T) = 1$ and $\dot{x}(t) \geq 0$, for all $t$. The equivalent optimal control problem has state equation $\dot{\dot{y}}_1(t) = \ddot{y}_2(t)$,
\[ \dot{y}_2(t) = \bar{u}(t), \]\n
is the integral cost function with integrand \( \ell(u, y) = \frac{1}{2}u^2 + y_1 \), the state constraint \( g(y) = -y_1 \), and the initial-final constraints \( y_1(0) = y_1(T) = 1 \). For large enough \( T \), the optimal state is, for some \( t_0 \in [0, \frac{1}{2}T] \), equal to zero over \( [t_0, t_1] \) with \( t_1 = T - t_0 \), and positive outside. The pre Hamiltonian is \( H(u, y, p) = \frac{1}{2}u^2 + y_1 + p_1 y_2 + p_2 u \), and the costate has dynamics \(-dp_1(t) = dt - d\mu(t)\), \(-\dot{p}_2(t) = \bar{p}_1(t)\). Since \( \mu(T) = 0 \), it follows that \( \bar{p}_1(t) = T - t + \mu(t) + \bar{p}(T) \). By the Hamiltonian inequality, \( \bar{u}(t) = -\bar{p}_2(t) \). Over \([t_0, t_1]\), we have that
\[ 0 = \bar{u}(t) = -\bar{p}_2(t) = \bar{p}_1(t), \]

implying \( \dot{\mu} = 1 \). Over each unconstrained arc, we have that \( \bar{p}_1(t) = c_1 - t \) for some \( c_1 \), and hence, \( \bar{p}_2(t) = \frac{1}{2}t^2 - c_1 t - c_2 \) for some \( c_2 \), so that \( \bar{u}(t) = c_1 + c_2 - \frac{1}{2}t^2 \). By lemma \( 3.2 \), the control is continuous at times \( t_0 \) and \( t_1 \), and is therefore of the form \(-c_1'(t-t_0) - \frac{1}{2}(t-t_0)^2 \) over \([0, t_0]\) and \( c_1''(t-t_1) - \frac{1}{2}(t-t_1)^2 \) over \([t_1, T]\). The curvature of the beam (i.e. the optimal control) being nonnegative, we have that \( [\bar{u}(t_0)] = c_1' \geq 0 \) and \( [\bar{u}(t_1)] = c_1'' \geq 0 \). By (3.23), the state constraint being of second order, since \( g^{(2)}_2(u)(t) = -1 \), we have that \( [\bar{u}(t_i)] = \bar{v}(t_i) \geq 0 \), \( i = 1, 2 \), which is compatible with the nonnegativity of the jumps of \( \bar{u} \), that we established just before.

\section{Proof of Pontryagin’s principle}

This section is devoted to the proof of theorem 3.1.

\subsection{Renormalization and distance to a convex set}

The proof uses the following lemmas. We recall that a Banach space is said to be separable if it contains a dense sequence.

\begin{lemma}
Let \( X \) be a separable Banach space. Then it has equivalent norm, say \( N \), such that the corresponding dual norm is strictly convex.
\end{lemma}

\begin{proof}
Let \( x_k, k \in \mathbb{N} \), be a dense sequence in \( X \). Set \( y_k := x_k/\|x_k\| \) (we may assume the \( x_k \) to be nonzero). Consider the following norm on \( X^* \):
\begin{equation}
N^*(x^*) := \|x^*\| + N^*_1(x^*) = \left( \sum_k 2^{-k} \langle x^*, y_k \rangle^2 \right)^{1/2}.
\end{equation}

Since \( \langle x^*, y_k \rangle \leq \|x^*\| \), we have that \( \|x^*\| \leq N^*(x^*) \leq (1 + \sqrt{2})\|x^*\| \) so that \( N^* \) is equivalent to the dual norm. Since the sum of a two convex functions, whose one is strictly convex, is strictly convex, if suffices to check that \( N^*_1 \) is strictly convex. We can write \( N^*_1(x^*) = N^*_2(Ax^*) \) where \( N^*_2(\cdot) \) is the strictly convex norm of \( \ell^2 \) (the Hilbert space of square summable sequences) and \( (Ax^*)_k = \langle x^*, y_k \rangle \) is linear continuous and injective. It is easily checked that the composition of an injective linear mapping by a strictly convex function is strictly convex. So \( N^* \) is a strictly convex norm. The norm \( N^* \) appears to be dual to the norm over \( X \) defined by
\begin{equation}
N(x) := \sup \{ \langle x^*, x \rangle; \ N^*(x^*) \leq 1 \},
\end{equation}

which is easily proved to be an equivalent norm. The conclusion follows.
\end{proof}

The next lemma uses the following notion: if \( f : X \to \mathbb{R} \) we define the subdifferential (in the sense of convex analysis) of \( f \) at \( x \in X \) by
\begin{equation}
\partial f(x) := \{ x^* \in X^*; \ f(x') \geq f(x) + \langle x^*, x' - x \rangle, \ \text{for all} \ x' \in X \}.
\end{equation}

It is known (e.g. [1] Ch. 2), that a nonexpansive, (i.e., with Lipschitz constant 1) convex function has at any \( x \in X \) a nonempty subdifferential, included in the closed (dual) unit ball, and that it is Gâteaux differentiable at \( x \) iff \( \partial f(x) \) is a singleton (which is the Gâteaux derivative).

\begin{lemma}
Let \( X \) be a separable Banach space endowed with the above norm \( N \), and let \( K \) be a closed convex subset of \( X \). Then the distance function \( d_K \) is Gâteaux differentiable at any point \( x \notin K \).
\end{lemma}

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Proof. It easily checked that $d_K$ is convex and nonexpansive. Let $x \in X \setminus K$. For any $y \in K$, we have that
\begin{equation}
0 = d_K(y) \geq d_K(x) + \langle x^*, y - x \rangle.
\end{equation}
Let $y_k \in K$ be such that $\|y_k - x\| \to d_K(x)$. Then
\begin{equation}
d_K(x) \leq -\langle x^*, y_k - x \rangle \leq \|x^*\|\|y_k - x\| \to \|x^*\|d_K(x),
\end{equation}
proving that $\|x^*\| = 1$. Therefore every element of the subdifferential has a unit norm. Since the dual norm is strictly convex it follows that $\partial d_K(x)$ is a singleton. The conclusion follows. \qed
We say that a sequence $x_k^*$ in $X^*$ weakly converges to $x^* \in X$ if
\begin{equation}
\langle x_k^*, x \rangle \to \langle x^*, x \rangle, \quad \text{for all } x \in X.
\end{equation}

Lemma 3.9. Let $X$ be a separable Banach space. Then any bounded sequence on $X^*$ has a weakly convergent subsequence.

Proof. See e.g. [10], Cor. 3.30. \qed

Lemma 3.10. Let $X$ be a Banach space, $K$ a convex subset of $X$ with nonempty interior, $x_k \in K$ converge to $\bar{x}$, $x_k^*$ weakly converge to $x^*$ in $X^*$, be such that
\begin{equation}
\langle x_k^*, x - x_k \rangle \leq 0 \quad \text{for all } x \in K.
\end{equation}
If $\liminf_k \|x_k^*\| > 0$, then $x^* \neq 0$.

Proof. We may assume that $B(0, \varepsilon) \subset \text{int}(K)$, for some $\varepsilon > 0$. Maximizing over $x \in B(0, \varepsilon)$ the l.h.s. of (3.33) we get that
\begin{equation}
\varepsilon \|x_k^*\| \leq \langle x_k^*, x_k \rangle \to \langle x_k^*, \bar{x} \rangle.
\end{equation}
The above l.h.s. having a positive lower limit, this proves that $\langle x_k^*, \bar{x} \rangle > 0$. The result follows. \qed

2.2. Proof of theorem 3.1. It suffices to discuss the case when the initial state is fixed, and the final state is free. Set $K := C([0, T])^{-\sigma}$. In the spirit of the proof of theorem 2.17 we introduce, for $\varepsilon > 0$, the ‘penalized’ cost function (compare to (2.96)): 
\begin{equation}
J^\varepsilon_R(u) := (|J_R(u) - J + \varepsilon^2|^2_+ + d_K^2([g(y|u)])^{1/2}.
\end{equation}
Here we take the distance function after renormalization of the separable space $C([0, T])^{-\sigma}$. Thanks to the above lemmas, we can compute the directional derivatives of $J^\varepsilon_R(u)$.

For $R \geq \|u\|$, set $U_R := L^\infty(0, T, U_{ad} \cap \overline{B}(0, R))$. Since $J^\varepsilon_R$ is continuous over this set endowed with Ekeland’s metric (2.94), and $u$ is an $\varepsilon^2$ minimizer of $J^\varepsilon_R$ over $U_R$, by Ekeland’s principle (theorem 2.33), there exists $u^\varepsilon \in U_R$ such that $\rho_E(u^\varepsilon, u) \leq \varepsilon$ and
\begin{equation}
J^\varepsilon_R(u^\varepsilon) \leq J^\varepsilon_R(u) + \varepsilon \rho_E(u, u^\varepsilon), \quad \text{for all } u \in U.
\end{equation}
We have that $J^\varepsilon_R(u^\varepsilon) > 0$ (otherwise this would contradict the fact that $\bar{u}$ is a solution of the problem). Denote by $y^\varepsilon$ the associated state. Set
\begin{equation}
\left\{
\begin{array}{ll}
\beta^\varepsilon := (J(u^\varepsilon) - J(u) + \varepsilon^2)_+ / J^\varepsilon_R(u^\varepsilon);

\eta^\varepsilon := d_K(g(y^\varepsilon)) / J^\varepsilon_R(u^\varepsilon)
\end{array}
\right. \quad \text{if } g(y^\varepsilon) \notin K, \quad \text{sinon}.
\end{equation}
Since $\|dK(g(y^\varepsilon))\| = 1$, we have that $\beta^\varepsilon + \|\eta^\varepsilon\|^2 = 1$. In addition, since $d_K(\cdot)$ is a convex function, we have
\begin{equation}
\langle \eta^\varepsilon, y - g(y^\varepsilon) \rangle \leq 0, \quad \text{for all } y \in C[0, T].
\end{equation}
Let $\eta^\varepsilon \in BV(0, T)$ be such that $d\eta^\varepsilon = \eta^\varepsilon$ and $\eta^\varepsilon(T) = 0$. Let $p^\varepsilon \in BV([0, T])$ be solution of (note that, since $\ell = 0$, the Hamiltonian does not depend on $\beta$):
\begin{equation}
\begin{aligned}
-dp^\varepsilon_t &= \nabla_y H(u^\varepsilon, y^\varepsilon, p^\varepsilon)dt + \sum_{i=1}^n \nabla_y g_i(y^\varepsilon)d\xi^\varepsilon_{it}, \quad \text{for a.a. } t \in [0, T],

p^\varepsilon_T &= \beta^\varepsilon \nabla\phi(y^\varepsilon_T).
\end{aligned}
\end{equation}
By arguments similar to those in the proof of theorem 2.17 (taking advantage of the integration by parts formula (3.10)) we deduce that

\[ H(\beta, u^\varepsilon, \dot{y}^\varepsilon, p^\varepsilon_t) \leq H(\beta, u, \dot{y}_t, p_t) + \varepsilon, \quad \text{for all } u \in U_R, \text{ and a.a. } t \in (0, T). \]

Let us take \( R = R_\varepsilon = \max(\|\bar{u}\|_\infty, 1/\sqrt{\varepsilon}) \). Since \( \rho_E(u^\varepsilon, \bar{u}) \leq \varepsilon \), we have that, for small enough \( \varepsilon \), \( \|u^\varepsilon - \bar{u}\|_1 \leq 2\varepsilon R_\varepsilon = 2\sqrt{\varepsilon} \) proving the uniform convergence of \( y^\varepsilon \) towards \( y \). Next, since \( \beta^2 + \|\eta^\varepsilon\|^2 = 1 \), in view of (3.38) we can extract a subsequence \( \varepsilon_k \downarrow 0 \), such that (denoting by \( \star \) the \( \star \)weak convergence)

\[ \left\{ \begin{array}{l}
\beta_{\varepsilon_k} \rightarrow \tilde{\beta} \in [0, 1]; \\
\eta_{\varepsilon_k} \rightharpoonup \bar{\eta}; \\
\langle \bar{\eta}, y - g(\bar{y}) \rangle \leq 0, \quad \text{for all } y \in K.
\end{array} \right. \]

We deduce from the costate equation that \( p_{\varepsilon_k} \) converges in the weak \( \star \) topology of \( M([0, T], \mathbb{R}^{n*}) \) to \( \bar{p} \) solution of the costate equation (3.11). This implies \( p_{\varepsilon_k} \rightarrow \bar{p} \) a.e. Passing to the limit in (3.40), we obtain the Hamiltonian inequality. Passing to the limit in the last relation in (3.41) we obtain (3.12). Finally, if \( \beta = 0 \), then \( \|\eta_{k}\| \rightarrow 1 \), and then by lemma 3.10 the non nullity relation (3.13) follows.

**Remark 3.11.** Under classical qualification conditions it can be checked that \( \tilde{\beta} > 0 \). In that case, dividing \( \bar{\eta} \) and \( \bar{p} \) by \( \tilde{\beta} \), we obtained the **qualified** form of Pontryagin’s principle, i.e., with \( \tilde{\beta} = 1 \), and we may remove \( \tilde{\beta} \) from the statement of Pontryagin’s principle.
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