Application of convex lexicographical optimization to the balance of GRTgaz gas grid

Version of Nov. 28, 2009

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November 28, 2009

Abstract

Shippers are daily users of the French gas grid. Differences between planned and effective gas demand unbalance the grid. To restore the balance, GRTgaz computes every day amounts of gas transiting on the grid. Amounts injected or withdrawn from the storages, balancing tolerances use rates are also computed. Finally, if the grid is still unbalanced, amounts of gas (associated with penalties) bought or sold to shippers are computed too. To minimize billed penalties to shippers, GRTgaz uses all these balancing facilities in a certain order. We solve a four stages lexicographical (or hierarchical) optimization program. The cost function to be minimized at each stage is convex quadratic. Lagrange multipliers are interpreted as pressures; flows try to balance pressures over the network. In the subset of nodes with zero pressure, a careful formulation of the previous stages problems

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is necessary in order to guarantee the robustness of computations. A numerical illustration is given.

Keywords Multiple objective programming, OR in energy, quadratic programming, lexicographical optimization, duality theory.

1 Introduction

Before the opening of the French gas market, the grid operator had to deal with one shipper. When the shipper had an excess or a lack of gas, the imbalance was absorbed by the grid operator. Now several gas shippers are acting on the French grid. Every day, GRTgaz (the grid operator) has to share the shippers's gas on its grid in order to balance it. When the grid needs to be balanced, the shippers responsible for the imbalance have to pay penalties.

The shippers are offered by GRTgaz the possibility to subscribe to a daily balancing service that reduces their imbalances. The problem of computing the reduced shippers's imbalances, described in the next section, is a multi-criteria problem. The other European grid operators offer different balancing services (see the gas balancing alert system on the UK gas grid, the cumulated imbalance system in Belgium and the Dutch "OLB" service.

Our problem is modeled as a lexicographic optimisation problem: for references on such problems, see Nemhauser and Wolsey [3], the overview in Floudas and Zlobec [2] including a duality theory and the link to sensitivity), and the theory of piecewise lexicographic programming, see Premoli and Ukovich [4] and Ukovich et al. [5]. Note also that lexicographic optimization may be viewed as the limit of the minimization of a weighted combination of the costs with weights equal to different powers of a small parameter "epsilon"; basically, for the case of two cost functions, this is what Danskin's theorem says (see e.g. Bonnans and Shapiro [1, Theorem 4.13]. The numerical minimization of such a composite cost function, however, does not look promising from the numerical point of view. Indeed minimization of a cost function with different orders of magnitude can lead to bad conditioned problems and convergence troubles. So the choice of the weights of the criteria in a multi-criteria objective function is a sensitive problem by itself.

Lexicographical modeling allows us to ignore this problem, that is why lexicographical optimization has been preferred to a single objective function model. Furthermore it is more readable, because each criterion is clearly isolated from the others. This is a key point because the balancing method had to be validated by the Commission de Régulation de l'Energie (regulator for the French electricity and gas markets) and understood by the shippers. The robustness of the method was also a crucial point because the algorithm is used every day, especially to bill the shippers for their imbalances. An important economical point is thus at stake.

Assume that we have, as in our problems, linear constraints and convex quadratic costs, the latter being a strongly convex function of a subset of the variables. A possible approach for solving lexicographical optimization problem of that type is to start by solving the first problem, and then to fix the first set of variables to its (well-defined) optimal value. The next problem is then again linear quadratic, and so on. This method can lead to numerical difficulties due to the numerical errors involved in the computation of each set of variables. The resulting second problem, or one to be solved later could be inconsistent. This source of trouble can be avoided by fixing only variables that are set to a bound, and otherwise taking the optimality system (or a equivalent subset of it) as a constraint. That ensures that the cost function of the next step problem is minimized over the solution set of the current step. In this way we avoid numerical difficulties. Parts 2 to 5 of this paper describe our problem, give the optimality systems and explain how to solve them. Part 6 gives performances features and numerical illustrations.

2 Setting of the problem

GRTgaz gas grid is used by several shippers. The grid is divided into "balancing zones", which correspond to large regions of France. Each day, each shipper must be balanced on each zone: all the gas brought by the shipper to a zone must be consumed or put in a storage. In order to balance the shippers, GRTgaz can make the gas circulate between zones through the connecting arcs. If despite these gas flows, a shipper has too much gas or has a lack of gas in a zone, he has to pay penalties. However shippers can subscribe to a balancing service to help them reduce their penalties. This service is made up of two parts. The first one is a bigger or smaller storage use than what the shipper had scheduled. For each storage, this use is bounded by flexibilities around the scheduled quantity (flexibilities are contractually set). The second part is the use of balancing tolerances, defined for each balancing zone. These balancing tolerances are like virtual storages where gas excesses can be put. Gas can also be withdrawn from a virtual storage to compensate a lack of gas. These virtual storages can contain a negative amount of gas. The contract between the shipper and the grid operator sets the size of balancing tolerances (min and max amount of gas in each virtual storage). Flexibilities of the storages must be used prior to balancing tolerances, in order to absorb the potential gas lack/excess. If these mechanisms are not sufficient, penalties are due by the shipper on every zone where there is an imbalance.

We identify zones with the vertices of a graph, whose edges correspond to the possibility of having nonzero flows between two zones. At last, arcs between zones must be used in order to maximize the share of the gas between these zones. A quadratic criterion is then used. To balance the grid, recourse to the offered facilities is minimized in this order: first the penalties, then the use of the balancing tolerances followed by the storage flexibilities and at last the use of the between-zones links (this use is also limited by contractual terms). This problem corresponds to a lexicographical optimization. In the next part, we briefly introduce the lexicographical optimization. Then we give the model formulation.

3 Lexicographical optimization

The lexicographical order in $\mathbb{R}^p$ is as follows. The approximation order of y and z in $\mathbb{R}^p$ is

$$I(y,z) := \max\{i; y_i = z_i\}. \tag{3.1}$$

We say that y lexicographically precedes z, and denote $y \preceq_{\text{lex}} z$ if

$$y = z \quad \text{or} \quad y_j < z_j, \text{ with } j = 1 + I(y,z). \tag{3.2}$$

Let f be a mapping from a set X into $\mathbb{R}^p$. The lexicographical minimization problem is

$$\text{Find } \bar{x} \in X \text{ such that } f(\bar{x}) \preceq_{\text{lex}} f(x), \quad \text{for all } x \in X.$$

This problem will be denoted as

$$\operatorname*{Min}_x f(x); \quad x \in X \tag{LEX}$$

The latter is equivalent to the sequence of (ordinary) optimization problems, for $i = 2, \dots, p$:

$$\text{Min } f_1(x); \quad x \in X \tag{L_1}$$

$$\text{Min } f_i(x); \quad x \in S(L_{i-1}) \tag{L_i}$$

where by $S(L_k)$ we denote the solution set of problem (L_k). The latter will be called the k *stage problem*.

4 Model formulation

The balancing problem applied to GRTgaz gas grid is written as follows. Given a grid, let be a zone and I be the set of the zones. For each zone i, let b_i be the shipper's (given) initial imbalance, D_i be the final imbalance imputed to the shipper on the zone, EBC_i be the use of its balancing tolerance, DPM_i the recourse to the storage of the zone. Moreover, being i and j two zones, let QJR_{ij} be the gas transit from zone i to zone j. For each point $i \in I$, the following balance equation must be respected:

$$D_i + EBC_i - DPM_i + \sum_{j \in Le_i} QJR_{ij} - \sum_{j \in Ar_i} QJR_{ji} = b_i \tag{4.3}$$

where Le_i (resp. Ar_i) denote the set of arcs leaving (resp. arriving at) node i, and b_i is a given data. Variables Di are unbounded, and EBC_i, DPM_i and QJR_{ij} are bounded as follows:

$$\begin{cases} \underline{EBC}_i \leq & EBC_i \leq & \overline{EBC}_i; \\ \underline{DPM}_i \leq & DPM_i \leq & \overline{DPM}_i; \\ 0 \leq & QJR_{ij} \leq & \overline{QJR}_{ij} \end{cases} \tag{4.4}$$

where the bounds are the given contractual min/max bounds on the balancing tolerances, the storages and the arcs. Given constants C_{qi}, $q = 1, \dots, 3$, and C_{4ij}, for all i and j in I (the imbalance repartition coefficients), the problem is to lexicographically minimize the cost function

$$f(D, EBC, DPM, QRJ) := \\ \tfrac{1}{2} \left(\sum_{i \in I} C_{1i} D_i^2, \ \sum_{i \in I} C_{2i} EBC_i^2, \ \sum_{i \in I} C_{3i} DPM_i^2, \ \sum_{i,j \in I} C_{4ij} QJR_{ij}^2 \right).$$

The imbalance repartition coefficients allow GRTgaz to share the imbalances according to zone storages capacities and arcs capacities, and are computed as follows:

$$C_{1i} = C_{2i} = \left(\overline{EBC}_i\right)^{-1}; \quad C_{3i} = 1; \quad C_{4ij} = \left(\overline{QJR}_{ij}\right)^{-1}. \tag{4.5}$$

The cost function for each stage is separable and quadratic convex, and the constraints are linear. By induction, using that the set of the solutions of a convex problem is convex, we deduce that each problem of the stream is convex. Moreover, the strict convexity of each cost function with respect to vectors D, EBC, DPM, QJR, resp., implies that each of these variables will be uniquely determined when solving the corresponding stage.

5 Analysis of the first problem

5.1 Optimality system

In this section, we focus on the first optimization problem, namely

$$\text{Min } \tfrac{1}{2} \sum_{i \in I} C_{1i} D_i^2; \quad \text{subject to } (4.3) - (4.4). \tag{P_1}$$

Let us denote by λ, (resp. $s \geq 0$, $t \geq 0$) the Lagrange multiplier associated with the balance equation (resp. with the lower and upper bound on variables subject to the bounds (4.4)). The problem being convex quadratic and linearly constrained, existence of a Lagrange multiplier holds. The optimality system involves the primal constraints (4.3) and (4.4), the dual constraints (for each $i \in I$)

$$C_{1i} D_i + \lambda_i = 0 \tag{5.6}$$

$$\lambda_i + t_{EBC_i} - s_{EBC_i} = 0, \tag{5.7}$$

$$\lambda_i + t_{DPM_i} - s_{DPM_i} = 0, \tag{5.8}$$

and with respect to each flow from region i to region j

$$\lambda_i - \lambda_j + t_{QJR_{ij}} - s_{QJR_{ij}} = 0, \tag{5.9}$$

and finally the slackness conditions

$$t_{EBC_i}(EBC_i - \overline{EBC}_i) = 0; \quad s_{EBC_i}(EBC_i - \underline{EBC}_i) = 0, \tag{5.10}$$

$$t_{DPM_i}(DPM_i - \overline{DPM}_i) = 0; \quad s_{DPM_i}(DPM_i - \underline{DPM}_i) = 0, \tag{5.11}$$

$$t_{QJR_{ij}}(QJR_{ij} - \overline{QJR}_{ij}) = 0; \quad s_{QJR_{ij}} QJR_{ij} = 0. \tag{5.12}$$

With each stage is associated what we may call the set of *variables entering nonlinearly*, that appear in the quadratic cost function. Since the latter is strictly convex w.r.t. the variables entering nonlinearly, their values are constant over the solution set In addition, if the Lagrange multipliers associated with some bound constraints are non zero, then the corresponding variables may be set to this bound on the solution set (and therefore are also uniquely determined). We may interpret the multipliers λ_i as *potentials* associated with each region. The flows QJR_{ij} allows the gas to communicate between two storages. Amounts EBC and DPM tend to bring potentials to 0, as expressed by (5.7) and (5.8).

5.2 Interpretation

Nodes of the network linked by an arc are said to be adjacent. A maximal connected subset of nodes of the graph with the same potential in each node is called a *communicating subnetwork*. Communicating subnetworks are a partition of the set of nodes of the graph. A communicating subnetwork with nonzero (resp. zero) potential is said saturated (resp. non saturated). By (5.8), flows between nodes of different communicating subnetworks are set to a bound.

Flows QJR_{ij} between different communicating subnetworks are set to a bound and therefore are no more optimization variables. Therefore, once the potential associated with one of the four stages is known, the subsequent stages split into independent subproblems for each communicating subnetwork. We therefore restrict the discussion to the analysis of a single communicating subnetwork.

5.3 Characterization of the solution of a communicating subnetwork

Let $\bar{\lambda} \in I\!R$ denote the common potential of the subnetwork $\bar{I} \subset I$. Using (5.6), we can eliminate D_i in the balance equation. That leads to:

$$-C_{1i}^{-1}\bar{\lambda} + EBC_i - DPM_i + \sum_{j \in Le_i} QJR_{ij} - \sum_{j \in Ar_i} QJR_{ij} = b_i, \ i \in \bar{I}. \tag{5.13}$$

Knowing by (5.9) that Lagrange multipliers associated with the bounds of the flows are equal to zero, the only equations regarding flows to be satisfied are:

$$0 \le QJR_{ij} \le \overline{QJR}_{ij}. \tag{5.14}$$

Regarding variables EBC_i and DPM_i, two cases have to be analyzed:

a) If the potential is non zero, variables EBC and DPM are fixed to a known bound and so can be eliminated.

b) If the potential $\bar{\lambda}$ equals zero, it can be replaced by 0 in the balance equation (5.13), and the following bounds constraints have to be taken into account:

$$\begin{cases} \underline{EBC}_i \le \quad EBC_i \le \quad \overline{EBC}_i, \quad i \in \bar{I}. \\ \\ \underline{DPM}_i \le \quad DPM_i \le \quad \overline{DPM}_i, \quad i \in \bar{I}. \end{cases} \tag{5.15}$$

In practice, a zero potential is detected by the fact that all EBC_i and DPM_i variables are at one of their bounds.

6 Stream

We next discuss the higher stage problems. We have to distinguish two cases.

a) The easiest case is the one of a nonzero potential: EBC and DPM are then equal to one of their bound, depending on the sign of the Lagrange multiplier. So, the second and third stage problems are trivial and we just have to solve the last one, under constraints (5.13) and (5.14) having in mind that $\bar{\lambda}$ is an unknown of this last problem.
b) In the case of a null potential subnetwork, the second criterion has to be minimized under constraints (5.13) to (5.15). Here $\bar{\lambda}$ equals zero and hence then can be eliminated, so that the balance equation reduces to

$$EBC_i - DPM_i + \sum_{j \in Le_i} QJR_{ij} - \sum_{j \in Ar_i} QJR_{ij} = b_i, \ i \in \bar{I}. \tag{6.16}$$

Denote by λ^2 the Lagrange multiplier associated with the balance constraint (6.16). In the optimality system we obtain relations analogous to (5.9):

$$\lambda_i^2 - \lambda_j^2 + t_{QJR_{ij}} - s_{QJR_{ij}} = 0. \tag{6.17}$$

Similarly to the first stage, we may partition each "first stage" communicating subnetwork into "second stage" communicating subnetworks defined as (maximal) connected subnetworks with an equal value denoted $\bar{\lambda}^2$ of the "second stage" potential. Flows between different second stage communicating subnetworks points are fixed to bounds. The problem decomposes in two sub-cases:

b1) When $\bar{\lambda}^2$ is non zero, DPM is binded to a bound and hence can be eliminated. In that case the third stage problem is trivial and it remains then to solve the fourth stage one. The stationarity of the Lagrangian w.r.t. EBC variables gives

$$C_{2i}EBC_i + \bar{\lambda}^2 + t_{EBC_i} - s_{EBC_i} = 0, \quad i \in \bar{I}^2. \tag{6.18}$$

We can assume that it is possible to detect whether $bar\lambda^2 = 0$ or not. In any case, we can eliminate EBC_i, which equals a bound if t_{EBC_i} or s_{EBC_i} is non zero, and equals $-C_{2i}^{-1}\bar{\lambda}^2$ otherwise. The balance equation (on each point of a subnetwork) is written (depending if the associated Lagrange multipliers are zero or not) either as

$$-C_{2i}^{-1}\bar{\lambda}^2 + \sum_{j \in Le_i} QJR_{ij} - \sum_{j \in Ar_i} QJR_{ji} = b_i^2 \tag{6.19}$$

or

$$\sum_{j \in Le_i} QJR_{ij} - \sum_{j \in Ar_i} QJR_{ji} = b_i^3 \tag{6.20}$$

Here $b_i^2 = b_i + DPM_i$ and $b_i^3 = b_i + DPM_i - EBC_i$ are known. On the countrary, $\bar{\lambda}^2$ is an unknown.

b2) On a subnetwork where the second stage potential equals zero, we have to solve the third stge problem with the following balance equation:

$$-DPM_i + \sum_{j \in Le_i} QJR_{ij} - \sum_{j \in Ar_i} QJR_{ji} = b_i^4. \tag{6.21}$$

Note that $b_i^4 = b_i - EBC_i$ is known, since each EBC_i is either zero or equal to a bound. Denote by λ^3 the Lagrange multiplier associated with the balance equation. The optimality equation for DPM variables is as follows:

$$C_{3i}DPM_i + \bar{\lambda}^3 + t_{DPM_i} - s_{DPM_i} = 0. \tag{6.22}$$

Again we have to deal with communicating subnetworks denoted $\bar{I}^3$ with a constant value of $\bar{\lambda}^3$. As before, we can eliminate the DPM_i variables, that are equal either to $-C_{3i}^{-1}\bar{\lambda}^3$, or to a bound. Depending on the case, for each point, the fourth stage balance equation can be written either as:

$$-C_{3i}^{-1}\bar{\lambda}^3 + \sum_{j\in Le_i} QJR_{ij} - \sum_{j\in Ar_i} QJR_{ji} = b_i^5, \quad i \in \bar{I}^3, \tag{6.23}$$

or

$$\sum_{j\in Le_i} QJR_{ij} - \sum_{j\in Ar_i} QJR_{ji} = b_i^6, \quad i \in \bar{I}^3. \tag{6.24}$$

Finally $\bar{\lambda}^3$ is a variable, except if we know that is is equal to zero.

7 Synthesis of the method

For each of the four stages, a convex quadratic program has to be solved, and the resulting communicating subnetworks have to be identified. For those with a nonzero potential, all optimization variables are set to a given bound, and hence, we can move on to the last stage. For those with a zero potential, then for the first stage, the corresponding D type variable is also equal to 0 by (5.6). For the second and third stages, we have that either the optimization variable of the corresponding stage is on a bound (if the corresponding Lagrange multiplier is nonzero) or is to be computed by either (6.18) or (6.22). In this way we obtain a stable computation of the minimum of the cost function. A tool implementing this method has been developed. It has been used on a daily basis GRTgaz, until a recent change in the model. Numerical results given in the next parts come from this tool.

8 Numerical results

Our numerical example corresponds to a model of the GRTagz gas grid. There are 5 balancing zones (Est, Ouest, Sud, Nord H, Nord B) and since the “Centre” storage is linked to two balancing regions ‘(Ouest and Sud) , we add a virtual “Centre” balancing zone linke to these two zones. So there are 6 zones and 7 links between them. Since each zone has three varaibles and each link has two, there is a total of 32 variables and 6 constraints for each balancing problem (the use of the optimality systems method can add constraints along the stages of the optimization).

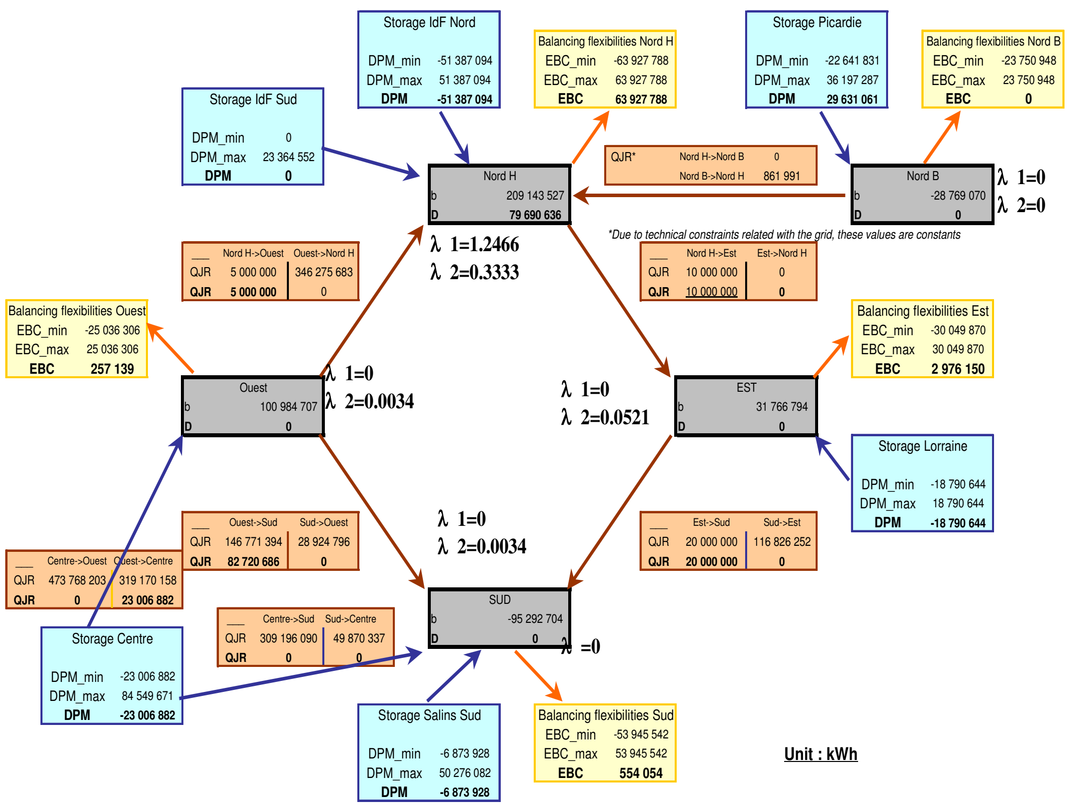
A balancing problem is associated with one shipper, therefore every day there are as many problems solved as there are shippers on the grid. The algorithm has to run fast. The calculation takes on average twenty milliseconds by shipper on a 2.80 GHz PC with 1 Go RAM. Tests have shown that without the optimality systems methods, five balancing calculations would have failed in 2005. Knowing that a shipper's daily penalties can reach hundreds of thousands of euros, defaults of billing must absolutely be avoided.

Balancing examples Here is an example of absorption of a shipper's imbalance. Figure 1 gives a view of the tool. This example illustrates the spread of the potentials and identifies classes and sub classes.

Zone Nord B is isolated because there is no transportation capacity arriving at this zone. Zone Nord H belongs to a saturated class. Zones Est, Sud and Ouest belong to the same non saturated class. Into this last class, Zone Ouest and Sud belong to the same 0.0034 sub potential class. Zone Est belongs to another sub class.

9 Conclusion

Robustness and clarity of the model are useful, if not compulsory, features of industrial models, in particular for gas transportation. Balancing a gas grid is one of this kind. A method clearly explainable was found to solve it: the lexicographical optimization. The second feature was obtained by using a method based on an optimality systems approach: the Lagrange multiplier may be interpreted as a potential; flows try to make equal potential over the network. In the subset of nodes with zero potential, called the zero potential subset, a careful formulation of the higher stages problems is necessary in order to guarantee the robustness of computations. Tests have confirmed that such a method stabilizes the problem. The balancing tool was used every day by GRTgaz and we estimate that about five balancing calculations a year would have failed without this method. That represents hundreds of thousands of euros. In this period of markets opening, the European competition bodies and the national regulators pay a lot of attention to grid operators' behaviors. So it is very important that the tool is reliable because GRTgaz has to be equally fair towards all the shippers. This robust lexicographical method can be very useful in the gas scope where multi-criteria optimization



Storage IdF Nord
DPM_min -51 387 094
DPM_max 51 387 094
DPM -51 387 094
Balancing flexibilities Nord H
EBC_min -63 927 788
EBC_max 63 927 788
EBC 63 927 788
Storage Picardie
DPM_min -22 641 831
DPM_max 36 197 287
DPM 29 631 061
Balancing flexibilities Nord B
EBC_min -23 750 948
EBC_max 23 750 948
EBC 0
Storage IdF Sud
DPM_min 0
DPM_max 23 364 552
DPM 0
QJR* Nord H->Nord B 0
Nord B->Nord H 861 991
Nord H
b 209 143 527
D 79 690 636
Nord B
b -28 769 070
D 0
λ 1=0
λ 2=0
*Due to technical constraints related with the grid, these values are constants
λ 1=1.2466
λ 2=0.3333
Nord H->Ouest Ouest->Nord H
QJR 5 000 000 346 275 683
QJR 5 000 000 0
Nord H->Est Est->Nord H
QJR 10 000 000 0
QJR 10 000 000 0
Balancing flexibilities Ouest
EBC_min -25 036 306
EBC_max 25 036 306
EBC 257 139
Balancing flexibilities Est
EBC_min -30 049 870
EBC_max 30 049 870
EBC 2 976 150
Ouest
b 100 984 707
D 0
λ 1=0
λ 2=0.0034
λ 1=0
λ 2=0.0521
EST
b 31 766 794
D 0
Storage Lorraine
DPM_min -18 790 644
DPM_max 18 790 644
DPM -18 790 644
Ouest->Sud Sud->Ouest
QJR 146 771 394 28 924 796
QJR 82 720 686 0
λ 1=0
λ 2=0.0034
Est->Sud Sud->Est
QJR 20 000 000 116 826 252
QJR 20 000 000 0
Centre->Ouest Ouest->Centre
QJR 473 768 203 319 170 158
QJR 0 23 006 882
Centre->Sud Sud->Centre
QJR 309 196 090 49 870 337
QJR 0 0
SUD
b -95 292 704
D 0
λ =0
Storage Centre
DPM_min -23 006 882
DPM_max 84 549 671
DPM -23 006 882
Storage Salins Sud
DPM_min -6 873 928
DPM_max 50 276 082
DPM -6 873 928
Balancing flexibilities Sud
EBC_min -53 945 542
EBC_max 53 945 542
EBC 554 054
Unit : kWh

often occurs (optimization of a physical criterion, economical criterion).

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