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# A boundary Pontryagin's principle for the optimal control of state-constrained elliptic systems

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**Abstract** Pontryagin's principle, originally devised for the optimal control of ordinary differential equation, has recently been extended to the optimal control of semilinear elliptic systems and variational inequalities in the case of a distributed control. In this paper we show that if the control is also active at the boundary of the domain, then a boundary Hamiltonian satisfying a boundary maximum Pontryagin's principle appears in a natural way.

## 1 Statement and discussion of the main result

In this paper we consider the following optimal control problem. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $C^{1,1}$ boundary $\Gamma$. The state equation is

$$\begin{cases} -\Delta y = F(y(x), u_d(x)), & x \in \Omega, \\ \alpha y + \dfrac{\partial y}{\partial n} = f(y(\sigma), u_b(\sigma)), & \sigma \in \Gamma. \end{cases} \tag{1}$$

Here $\alpha > 0$, $F$ and $f$ are continuous mapping from $\mathbb{R}^2$ into $\mathbb{R}$, differentiable with respect to $y$, and such that $F_y$ and $f'_y$ are continuous. We define $u = (u_d, u_b)$ and the value function as

$$J(y,u) := \int_\Omega L(y(x), u_d(x))dx + \int_\Gamma \ell(y(\sigma), u_b(\sigma))d\sigma. \tag{2}$$

We also assume that $L$ and $\ell$ are continuous, with continuous derivative with respect to $y$. Let $g(x,y)$ be a mapping $\mathbb{R}^2 \to \mathbb{R}$ with continuous derivative with respect to $y$, and finally let $K_d$, $K_b$ be two bounded subsets of $\mathbb{R}$. We define the set of controls as

$$U := \{u = (u_d, u_b) \text{ measurable }; \ u_d(x) \in K_d, \text{ a.e. } x \in \Omega \ ; \ u_b(\sigma) \in K_b, \text{ a.e. } \sigma \in \Gamma\}.$$

We consider the optimal control problem

$$\min J(y,u) \text{ s.t. } (1), \ u \in U, \ g(x, y(x)) \leq M, \ \forall x \in \Omega. \tag*{$(P)_M$}$$

In order to obtain well-possedness of (1), we assume the following :

$$F'_y \leq 0 \text{ and } f'_y \leq 0. \tag{3}$$

Then the following holds :

**Proposition 1** *Under the above hypotheses, for each $u \in U$, equation (1) admits a unique solution*

$$y = y_u \in Y := H^1(\Omega) \cap C(\bar{\Omega}).$$

We now define the distributed Hamiltonian (here and after we will omit the subscripts $d$, $b$ on $u_d$, $u_b$ when no confusion is possible)

$$H(y,u,p) := L(y,u) + pF(y,u),$$

and the boundary Hamiltonian

$$h(y,u,p) := \ell(y,u) + pf(y,u).$$

We say that $(P)_M$ is strongly stable if

$$\inf(P)_M \geq \inf(P)_{M'} + 0(M - M').$$

The hypotheses is of strong stability is generic in the following sense. We note that $M \to \inf(P)_M$ is decreasing ; let $\bar{M}$ be the smallest number such that $\inf(P)_M < \infty$ if $M > \bar{M}$.

It can be checked that $\inf(P)_M > -\infty$. Now as $\inf(P)_M$ is decreasing,is it differentiable a.e. $M \geq \bar{M}$, hence $(P)_M$ is strongly stable a.e. $M \geq \bar{M}$.

We now state our main result.

**Theorem 1** *(Extension of Pontryagin's principle). Let $\bar{u}$ be a solution of $(P)_M$ with $(P)_M$ strongly stable, and $\bar{y}$ the associated state. Then there exists $\bar{p}$ in $W^{1,\sigma}(\Omega)$, for all $\sigma < n/(n-1)$, $\bar{\lambda}$ in $M(\bar{\Omega})$, such that (denoting by $\bar{\lambda}_d, \bar{\lambda}_b$ the restriction of $\bar{\lambda}$ to $M(\Omega)$ and $M(\Gamma)$) :*

$$\left\{ \begin{array}{lcl} -\Delta\bar{p} & = & F'_y(\bar{y},\bar{u})\bar{p} + L'_y(\bar{y},\bar{u}) + g'_y(x,\bar{y}(x))\bar{\lambda}_d \text{ in } \Omega, \\ \alpha\bar{p} + \dfrac{\partial\bar{p}}{\partial n} & = & f'_y(\bar{y},\bar{u})\bar{p} + \ell'_y(\bar{y},\bar{u}) + g'_y(\bar{x},\bar{y}(x))\bar{\lambda}_b \text{ on } \Gamma, \end{array} \right. \tag{4}$$

$$\left\{ \begin{array}{l} \bar{\lambda} \geq 0, \quad g(\bar{x}, y(\bar{x})) \leq M, \\ \int_\Omega (g(x,\bar{y}(x) - M) d\bar{\lambda}(x) = 0, \end{array} \right. \tag{5}$$

*and*

$$H(\bar{y}(x),\bar{u}(x),\bar{p}(x)) = \min_{v \in K_d} H(\bar{y}(x), v, \bar{p}(x)), \quad a.e.\ x \in \Omega, \tag{6}$$

$$h(\bar{y}(\sigma),\bar{u}(\sigma),\bar{p}(\sigma)) = \min_{v \in K_b} h(\bar{y}(\sigma), v, \bar{p}(\sigma)), \quad a.e.\ \sigma \in \Gamma. \tag{7}$$

We compare this result to the litterature. Some first-order conditions for the control of state-constrained elliptic systems can be found in Mackenroth [12] in the convex case and the authors [2] [3], the last reference dealing with ill-posed systems. A derivation of Pontryagin's principle for problems without state constraints was obtained in Bonnans-Casas [4] ; Bonnans-Tiba [6] extended this approach to the control of elliptic variational inequalities (V.I.). The result for state constrained problem was obtained in Bonnans [1] and extended by the authors [5]. However in the previous references it is assumed that the state satisfies an homogeneous Dirichlet condition at the boundary and so only the distributed Hamiltonian appears. The novelty here lies in the boundary Hamiltonian, satisfying a natural extension of Pontryagin's principle.

The paper is as follows. In section 2 we connect strong stability to exact penalization and show how to regularize the problem with exact penalty. In section 3, using an Hamiltonian formulation of the variation of the cost, spike perturbations, and Ekeland's principle, we derive some optimality conditions for the approximate problem. Finally in section 4 we show how to pass to the limit in order to get the main result.



## 2 Exact penalization and regularization

Here we give easy extensions of some results of Bonnans [1], Bonnans-Casas [5]. To $(P)_M$ we associate the exact penalty function

$$J_r(u) := J(y_u, u) + r\|(g(.,y_u) - M)^+\|_\infty$$

and the associated optimization problem

$$\min J_r(u) \ ; \ u \in U. \tag*{$(Q)_r$}$$

We endow $U$ with the metric

$$d(u,v) := mes\,\{x \in \Omega \ ; \ u_d(x) \neq v_d(x)\} + mes\,\{\sigma \in \Gamma \ ; \ u_b(\sigma) \neq v_b(\sigma)\}.$$

The following can be checked as in [4].

**Proposition 2** (i) *The mapping $u \to y_u$ is compact $(U,d) \to Y$.*

(ii) *If $(P)_M$ is strongly stable then there exists $r > 0$ such that any solution of $(P)_M$ is a local solution of $(Q)_r$ in the space $(U,d)$.*

We now fix that for some $r > 0$, $d > 0$, $\bar{u}$ is solution of

$$\min J_r(u) \ ; \ u \in U, \ d(u,\bar{u}) \leq \delta. \tag*{$(Q)_{r,\delta}$}$$

This problem has no more state constraints. However its cost is non-differentiable, and we will regularize it. The first idea is to approximate the $L^\infty$-norm by the $L^q$-norm, $q$ being a "large" number. The $L^q$ norm being not differentiable at 0 we will define a regularized cost as follows :

$$J_{r,q}(u) := J(y_u, u) + r[q^{-q} + \int_\Omega [(g(x, y_u(x)) - M)^+]^q dx]^{1/q},$$

and we consider the regularized problem

$$\min J_{r,q}(u) \ ; \ u \in U, \ d(u,\bar{u}) \leq \delta. \tag*{$(Q)_{r,\delta,q}$}$$

**Proposition 3** *The following identity holds*

$$\inf(Q)_{r,\delta} = \lim_{q \to \infty} \inf(Q)_{r,\delta,q}.$$

**Proof.** Define $U_\delta := \{u \in U \,;\, d(u,\bar{u}) \le \delta\}$. Let $u \in U_\delta$ be given. From the relations

$$\begin{aligned}\|(g(.,y_u)-M)^+\|_q &\le [q^{-q} + \int_\Omega [(g(x,y_u(x))-M)^+]^q dx]^{1/q}\\ &\le \frac{1}{q} + \|(g,.,y_u)-M)^+\|_q,\end{aligned}$$

and the convergence of $\|z\|_q \to \|z\|_\infty$, for all $z \in L^\infty(\Omega)$, we deduce that $J_{r,q}(u) \to J_r(u)$, hence

(*) $$\overline{\lim}\{\inf(Q)_{r,\delta,q}\} \le \min(Q)_{r,\delta}.$$

Let us prove the converse inequality. let $\{u_q\}$ be a sequence in $U_\delta$ (and $y_q$ the associated states) such that $J_{r,q}(u_q) \le \inf(Q)_{r,\delta,q} + 1/q$. Then, using (*) :

$$\overline{\lim} J_{r,q}(u_q) = \overline{\lim}\inf(Q)_{r,\delta,q} \le \min(Q)_{r,\delta} \le \underline{\lim} J_r(u_q).$$

We end the proof by checking that $\underline{\lim} J_{r,q}(u_q) = \overline{\lim} J_r(u_q)$. The sequence $\{y_q\}$ is indeed compact in $C(\bar{\Omega})$. Let $\hat{y}$ be a limit-point in $C(\bar{\Omega})$. Assume that $\hat{z} := (g(.,\hat{y}) - M)^+$ is such that $|\hat{y}|$ attains its maximum at $x_0 \in \bar{\Omega}$. For all $\varepsilon > 0$ there exists $\eta > 0$ and $k_0$ such that for a given subsequence

$$|(g(x,y_q(x))-M)^+| \ge \|\hat{z}\|_\infty - \varepsilon, \quad \forall x \in \bar{\Omega}, \quad \|x - x_0\| \le \eta, \quad \forall q > k_0.$$

From this we deduce that $\|(g(.,y_q)-M)^+\|_q \to \|(g(.,\hat{y})-M)^+\|_\infty$ and the result follows. □

## 3 Hamiltonian formulation of the variation of the cost and spike perturbations

We first consider a problem with differentiable cost and without state constraints, that generalizes $(Q)_{r,\delta,q}$. We show how to write the difference of costs as integral of difference of Hamiltonians. Then we will make use of spike perturbations. We consider the criterion

$$c(u) := J(y_u,u) + \Phi[\int_\Omega b(x,y_u(x))dx].$$

Here $\Phi : \mathbb{R} \to \mathbb{R}$ is $C^1$ and $b : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and has a continuous derivative w.r.t. $y$. Given two controls $u, v$ and the associated states $y_u, y_v$ we define, using the mean value theorem, some interpolated states $y^i$, $i = 1$ to 5, such that

$$y^i(x) \in [y_u(x), y_v(x)], \quad \forall x \in \Omega,$$

and

$$\begin{aligned}F(y_v,v) &= F(y_u,v) + F'_y(y^1,v)(y_v-y_u), \text{ a.e. in } \Omega,\\ L(y_v,v) &= L(y_u,v) + L'_y(y^2,v)(y_v-y_u), \text{ a.e. in } \Omega,\\ f(y_v,v) &= f(y_u,v) + f'_y(y^3,v)(y_v-y_u), \text{ a.e. on } \Gamma,\\ \ell(y_v,v) &= \ell(y_u,v) + \ell'_y(y^4,v)(y_v-y_u), \text{ a.e. on } \Gamma,\\ \Phi[\int_\Omega b(.,y_v)] &= \Phi[\int_\Omega b(.,y_u)] + \Phi'[\int_\Omega b(.,y^5)]\int_\Omega b'_y(.,y^5)(y_v-y_u),\end{aligned}$$



and the interpolated costate $p_{u,v}$ solution of

$$\left\{\begin{aligned}&-\Delta p_{u,v} = F'_y(y^1,v)p_{u,v} + L'_y(y^2,v) + \Phi'[\textstyle\int_\Omega b(.,y^5)]b'_y(.,y^5) \text{ in } \Omega,\\ &\alpha p_{u,v} + \frac{\partial p_{u,v}}{\partial n} = f'_y(y^3,v)p_{u,v} + \ell'_y(y^4,v) \text{ on } \Gamma.\end{aligned}\right. \tag{8}$$

Indeed the above equation has a unique solution in $Y$.

**Proposition 4** (Hamiltonian formulation of the variation of the cost). *For any two control $u, v$ in $U$, $p_{u,v}$ being solution of (8), the following holds :*

$$c(v) = c(u) + \int_\Omega [H(y_u,v,p_{u,v}) - H(y_u,u,p_{u,v})] + \int_\Gamma [h(y_u,v,p_{u,v}) - h(y_u,u,p_{u,v})]. \tag{9}$$

**Proof.** Put

$$\theta := \int_\Omega [L(y_v,v) - L(y_u,v)] + \int_\Gamma [\ell(y_v,v) - \ell(y_u,v)] + \Phi[\int_\Omega b(.,y_v)] - \Phi[\int_\Omega b(.,y_u)].$$

Then it is easily checked that

$$c(v) = c(u) + \int_\Omega [L(y_u,v) - L(y_u,u)] + \int_\Gamma [\ell(y_u,v) - \ell(y_u,u)] + \theta. \tag{10}$$

Using $\{y^i\}$, and (8) we find

$$\begin{aligned}\theta &= \int_\Omega L'_y(y^2,v)(y_v-y_u) + \int_\Gamma [\ell'(y^4,v)(y_v-y_u)\\ &\quad + \Phi'[\int_\Omega b(.,y^5)]\int_\Omega b'(.,y^5)(y_v-y_u)\\ &= \int_\Omega [-\Delta p_{u,v} - F'_y(y^1,v)p_{u,v}](y_v-y_u)dx\\ &\quad + \int_\Gamma [\alpha p_{u,v} + \frac{\partial p_{u,v}}{\partial n} - f'_y(y^3,v)p_{u,v}](y_v-y_u)d\sigma.\end{aligned}$$

Now by Green's formula

$$\int_\Omega -\Delta p_{u,v}(y_v-y_u) = -\int_\Gamma \frac{\partial p_{u,v}}{\partial n}(y_v-y_u) + \int_\Gamma p_{u,v}\frac{\partial(y_v-y_u)}{\partial n} - \int_\Omega p_{u,v}\Delta(y_v-y_u)$$

hence, after simplification

$$\begin{aligned}\theta &= \int_\Omega [-\Delta(y_v-y_u) - F'_y(y^1,v)(y_v-y_u)]p_{u,v}\\ &\quad + \int_\Gamma [\alpha(y_v-y_u) + \frac{\partial(y_v-y_u)}{\partial n} - f'_y(y^3,v)]p_{u,v}.\end{aligned}$$

Using the state equations for $y_u$ and $y_v$ we find

$$\theta = \int_\Omega [F(y_u,v) - F(y_u,u)]p_{u,v} + \int_\Gamma [f(y_u,v) - f(y_u,u)]p_{u,v}.$$

This and (10) give the conclusion. □

We define

$$\begin{aligned}\omega_k(x_0) &:= \{x \in \Omega\ ;\ \|x - x_0\| \leq 1/k\},\\ \gamma_k(\sigma_0) &:= \{\sigma \in \Gamma\ ;\ \|\sigma - \sigma_0\| \leq 1/k\}.\end{aligned}$$

We say that a sequence $v^k = (v_d^k, v_b^k)$ in $U$ is a distributed (resp. boundary) spike perturbation of $u \in U$ around $x_0 \in \Omega$ (resp. $\sigma_0 \in \Gamma$) if for some $v \in K_d$ (resp. $v \in K_b$)

$$v_b^k = u_b \text{ and } v_d^k(x) = \begin{cases} v \text{ if } x \in \omega_k(\sigma_0),\\ u_d(x) \text{ if not;}\end{cases}$$

respectively :

$$v_d^k = u_b \text{ and } v_b^k(\sigma) = \begin{cases} v \text{ if } \sigma \in \gamma_k(\sigma_0),\\ u_b(\sigma) \text{ if not.}\end{cases}$$

By $p_u$ we mean $p_{u,u}$.

**Proposition 5** *Let $u \in U$ be given. Then*

(i) *For almost all $x_0 \in \Omega$ the following holds : if $v^k$ is a distributed spike perturbation of $u$ around $x_0$, and $y_k$ is the associated state,*

$$\lim_{k\to\infty} \frac{1}{mes(\omega_k(x_0))}[c(v^k) - c(u)] = H(y_u(x_0), v, p_u(x_0)) - H(y_u(x_0), u(x_0), p_u(x_0)).$$

(ii) *For almost $\sigma_0 \in \Gamma$ the following holds : if $v^k$ is a boundary spike perturbation of $u$ around $\sigma_0$, and $y_k$ is the associated state, then*

$$\lim_{k\to\infty} \frac{c(v^k) - c(u)}{mes(\gamma_k(\sigma_0))} = h(\sigma_0, y_u(\sigma_0), v, p_u(\sigma_0)) - h(\sigma_0, y_u(\sigma_0), u(\sigma_0), p_u(\sigma_0)).$$

**Proof.** We give only the proof of case (ii) ; case (i) can be dealt with in a similar way. We have from Proposition 4

$$c(v^k) - c(u) = \int_{\gamma_k(\sigma_0)} [h(y_u, v, p_k) - h(y_u, u, p_k)],$$

where $p_k$ is the interpolated costate $p_{v^k,u}$. From the (easily established) uniform convergence of $p_k \to p_u$ in $C(\bar{\Omega})$, and the boundedness of the data we deduce that (if the limit exists)

$$\lim_{k\to\infty} \frac{c(v^k) - c(u)}{mes(\gamma_k(\sigma_0))} = \lim_{k\to\infty} mes(\gamma_k(\sigma_0))^{-1} \int_{\gamma_k(\sigma_0)} [h(y_u, v, p_u) - h(y_u, u, p_u)].$$

Now by continuity we always have

$$mes(\gamma_k(\sigma_0))^{-1} \int_{\gamma_k(\sigma_0)} h(y_u, v, p_u) \to h(y_u(\sigma_0), v, p_u(\sigma_0)),$$



hence the formula holds if $\sigma_0$ is a Lebesgue point of $\sigma \to h(y_u(\sigma), u(\sigma), p_u(\sigma))$ ; the set of Lebesgue point being of full measure, we get the result. □

From the above results we deduce an optimality system for the minimization of $c(u)$ over $U$.

**Theorem 2** *If $\hat{u}$ minimizes $c(u)$ over $U$ then, denoting $\hat{y} := y_{\hat{u}}$ and $\hat{p} := p_{\hat{u}}$, the following holds :*

$$\begin{aligned} H(\hat{y}(x), \hat{u}_d(x), \hat{p}(x)) &= \min_{v\in K_d} H(\hat{y}(x), v, \hat{p}(x)), \quad a.e. \text{ in } \Omega,\\ h(\hat{y}(x), \hat{u}_b(x), \hat{p}(x)) &= \min_{v\in K_b} h(\hat{y}(x), v, \hat{p}(x)), \quad a.e. \text{ on } \Gamma.\end{aligned}$$

## 4 Back to state constrained problems

We now end the proof of the main result, as follows. Assuming $(P)_M$ strongly stable it follows that $\bar{u}$ (solution of $(P)_M$) is also solution of $(Q)_{r,\delta}$ for $r$ large enough and $\delta$ small enough (Prop. 2) ; by Prop. 3, $\bar{u}$ is an $\varepsilon(q)$-solution of $Q_{r,\delta,q}$, with

$$\varepsilon(q) = J_{r,q}(\bar{u}) - \inf(Q)_{r,\delta,q}$$

and $\varepsilon(q) \searrow 0$ as $q \nearrow \infty$. Now $u \to J_{r,q}(\bar{u})$ is continuous from $(U_\delta, d)$ into $\mathbb{R}$ and $(U_\delta, d)$ is a complete metric space. Applying Ekeland's principle [10], we deduce that there exists $u_q \in U_\delta$ such that

$$d(\bar{u}, u_q) \leq \sqrt{\varepsilon(q)}, \tag{11}$$

$$J_{r,q}(u_q) \leq J_{r,q}(u) + \sqrt{\varepsilon(q)}\, d(u, u_q), \quad \forall u \in U. \tag{12}$$

Now, if $v^k$ is, say, a boundary spike perturbation of $u_q$ around $\sigma_0$, if follows that

$$d(v^k, u_q) \leq mes\gamma_k(\sigma_0),$$

hence with (12)

$$0 \leq \underline{\lim}_{k\to\infty} \frac{J_{r,q}(v^k) - J_{r,q}(u_q)}{mes(\gamma_k(\sigma_0))} + \sqrt{\varepsilon(q)}.$$

We may now apply Prop. 5 ; we obtain the following

**Theorem 3** *For any $q > 1$ there exists $u_q \in U$ satisfying (11) and such that denoting $y_q := y_{u_q}$, there exists $p_q \in Y$, $\lambda_q \in L^1(\Omega)$ such that*

$$\begin{cases} -\Delta p_q = F'_y(y_q, u_q)p_q + L'_y(y_q, u_q) + g'_y(., y)\lambda_q & \text{in } \Omega,\\ \alpha p_q + \dfrac{\partial p_q}{\partial n} = f'_y(y_q, u_q)p_q + \ell'_y(y_q, u_q) & \text{on } \Gamma,\end{cases} \tag{13}$$

$$\lambda_q = r[q^{-q} + \int_\Omega [(g(x, y_q(x)) - M)^+]^q dx]^{1/q-1}[(g(., y_q) - M)^+]^{q-1} \tag{14}$$

*and*

$$H(y_q,u_q,p_q) \leq \min_{v\in K_d} H(y_q,v,p_q)+\sqrt{\varepsilon(q)} \quad a.e.\ in\ \Omega, \qquad (15)$$

$$h(y_q,u_q,p_q) \leq \min_{v\in K_b} h(y_q,v,p_q)+\sqrt{\varepsilon(q)} \quad a.e.\ on\ \Gamma. \qquad (16)$$

## 5 From the approximate to the original control problem

We end the proof of Thm. 1 by passing to the limit in the optimality system stated in Theorem 3. Note that, as $d(u_q,\bar u)\leq\sqrt{\varepsilon(q)}$, and $u\to y_u$ is continuous in $(U,d)$, we have that $y_q\to\bar y$ in $Y$. It remains to pass to the limit in (13)-(16). First let us note that an estimate of $\lambda_q$ in $L^1(\Omega)$ can be obtained as in [1], hence a subsequence of $\lambda_q$ $*$-converges to $\bar\lambda\in M(\bar\Omega)$. That $\lambda_q\geq 0$ implies $\bar\lambda\geq 0$. Obviously $\bar\lambda$ has support where $g(x,\bar y(x))=M$. As $(g(x,\bar y(x))-M)^+=0$ it follows that $\int_{\bar\Omega} g(x,\bar y(x))-M)^+d\bar\lambda(x)=0$ ; (5) follows. Now let us pass to the limit in the costate equation (13). From a given subsequence, from $\lambda_q \overset{*}{\rightharpoonup} \bar\lambda$ in $M(\bar\Omega)$, it follows that $g'_y(.,y_q)\lambda_q \overset{*}{\rightharpoonup} g'_y(.,\bar y)\bar\lambda$ in $M(\bar\Omega)$. We see that (4) will follows from the study of the abstract problem

$$\left\{\begin{array}{ll} -\Delta p+ap=\lambda_d & in\quad \Omega,\\ \alpha p+\dfrac{\partial p}{\partial n}=\lambda_b & on\quad \Gamma,\end{array}\right.$$

with $a\in L^\infty(\Omega), a\geq 0$, $\lambda_d$, $\lambda_b$ traces on $\Omega$ and $\Gamma$ of $\lambda\in M(\bar\Omega)$. We study this equation by the method of transposition. To $(f,g)\in L^s(\Omega)\times W^{1,s}(\Gamma)$ we associate $z$ solution of

$$\left\{\begin{array}{ll} -\Delta z+az=f & in\quad \Omega,\\ \alpha z+\dfrac{\partial z}{\partial n}=g & on\quad \Gamma.\end{array}\right.$$

It is known that $z\in C(\bar\Omega)$ and that the mapping $(f,g)\to z$ is dense in $M(\bar\Omega)$. Integrating (formally) by parts we obtain, $q$ being the trace of $p$ :

$$\int_\Omega fp+\int_\Gamma gq=\int_{\bar\Omega} zd\lambda.$$

This equation in $(p,q)$ has a unique solution in the dual space $L^{s'}(\Omega)\times W^{-1,s'}(\Gamma)$ with $1/s'+1/s=1$. Now it can be proved as in Casas [9] that $p\in W^{1,\sigma}(\Omega)$ for all $\sigma<n/(n-1)$ and the $q$ can be interpreted as the trace of $p$. Also the estimate of $p$ in $W^{1,\sigma}(\Omega)$ can be obtained independently of $a\geq 0$. Coming back to our problem, it follows that $p_q$ is bounded in $W^{1,\sigma}(\Omega)$. We can extract a subsequence such that $\lambda_q \overset{*}{\rightharpoonup} \bar\lambda$ in $M(\bar\Omega)$ and $p_q\rightharpoonup\bar p$ in $W^{1,\sigma}(\Omega)$ for all $\sigma<n/(n-1)$. From this follows the strong convergence of $p_q$ in $L^1(\Omega)$. Passing to the limit we obtain (4). This allows to pass to the limit in (14),(15),(16) ; we deduce that (5), (6), (7) hold. □

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