Analysis and control of a non-linear parabolic unstable system

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This paper is concerned with a non-linear evolutive system of diffusion-reaction type. This system may, for finite values of the control, blow up in a finite time; consequently, classical methods based on a priori estimates on the solution do not seem well suited. We restrict the study to the strong solutions, and show that the implicit function theorem can be applied. If, in addition, the solution has to belong to some L^p-space, the problem can be treated in a similar manner by choosing some new spaces which are maximal in some sense. Previous results allow us to express the optimality conditions of control problems associated with the system. If the criterion includes a state cost in an L^p norm, this implies the use of abstract duality products; these may be viewed as an extension by continuity of integrals.

1. Setting of the problem

Let Ω be a bounded open set of $\mathbb{R}^3$, with C^∞ boundary Γ. Let T be a strictly positive real number and denote:

$$Q = \Omega \times]0, T[\,; \qquad \Sigma = \Gamma \times]0, T[\,.$$

Consider the system:

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y - y^3 = f \text{ in } Q\,, \\ &\frac{\partial y}{\partial n} = u \text{ on } \Sigma\,, \\ &y(x, 0) = h(x)\,, \quad \text{a.e. } x \in \Omega\,. \end{aligned} \tag{1.1}$$

Because we are interested in strong solutions of (1.1) we impose that y belongs to $H^{2,1}(Q)$ (for the definition of such spaces see [10]). We shall see that this implies that (f, u, h) belongs to $U = L^2(Q) \times H^{1/2,1/4}(\Sigma) \times H^1(\Omega)$. Then the existence of solutions for time t near 0 can easily be deduced from results of Ishii [5]. On the other hand, it is known that the non-linear term may cause a blowing up of the solution of (1.1) in a finite time [2, 8]: this means that there exists some $\bar{t} > 0$ such that (1.1) has a solution for $t \in [0, \bar{t}[$ and that $y(t)$ is unbounded as t approaches $\bar{t}$. Consequently, the usual method of studying parabolic equations, based on a priori estimates on the solution, does not work. We show how the use of the implicit function theorem allows us to obtain the following results:

(1) If (1.1) has a solution for $(f_0, u_0, h_0) \in U$, this solution is unique and, in a neighbourhood of (f_0, u_0, h_0) in U, (1.1) has a unique solution which depends in a smooth way on (f, u, h).

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(2) If we impose the regularity condition $y \in L^\alpha(Q)$, with $\alpha \in [2, +\infty]$, the preceding result still holds if, when $\alpha > 10$, we choose some new function spaces for y and (f, u, h).

(3) With these results it is possible to express the gradient of some criteria and to deduce the optimality conditions of control problems associated with equation (1.1).

2. Analysis of the state equation

We analyse equation (1.1) with the a priori restriction that $y \in H^{2,1}(Q)$. The trace theorems (Lions and Magenes [10, vol. 2, p. 10]) and (1.1) imply that $u \in H^{1/2,1/4}(\Sigma)$ and $h \in H^1(\Omega)$. In addition (Lions [9]), if the space dimension n is such that $n \leq 3$, for any $\lambda > 1$ the following relation holds:

$$W^{2,1;\lambda}(Q) \overset{n\leq 3}{\subset} L^\mu(Q),$$

with

$$\frac{1}{\mu} \geq \frac{1}{\lambda} - \frac{2}{5}, \quad \text{if } \frac{1}{\lambda} - \frac{2}{5} > 0, \qquad \mu = +\infty, \quad \text{if not}. \tag{2.1}$$

For $\lambda = 2$, noticing that $W^{2,1;2}(Q) = H^{2,1}(Q)$, we get:

$$H^{2,1}(Q) \overset{n\leq 3}{\subset} L^{10}(Q). \tag{2.2}$$

Consequently, if $y \in H^{2,1}(Q)$, $\mathrm{d}y/\mathrm{d}t - \Delta y - y^3$ is in $L^2(Q)$, and so by (1.1) f is in $L^2(Q)$. To sum up, (f, u, h) is in $U = L^2(Q) \times H^{1/2,1/4}(\Sigma) \times H^1(\Omega)$. Endowed with the norm

$$\|(f, u, h)\|_U = (\|f\|^2_{L^2(Q)} + \|u\|^2_{H^{1/2,1/4}(\Sigma)} + \|h\|^2_{H^1(\Omega)})^{1/2},$$

U is an Hilbert space. Define:

$$\mathcal{O} = \{(f, u, h) \in U \text{ such that (1.1) has (at least) a solution in } H^{2,1}(Q)\}.$$

We recall the implicit function theorem, which will be used in the sequel (see, for instance, Cartan [3]).

Theorem 2.1. *Let A, B and C be three Banach spaces and ϕ a continuously differentiable mapping from $A \times B$ onto C. Let $(a_0, b_0) \in A \times B$ such that*

$$\phi(a_0, b_0) = 0 \text{ in } C.$$

Then, if $(\partial\phi/\partial a)(a_0, b_0)$ is an isomorphism from A onto C, there exists a neighbourhood $\mathcal{B}$ of b_0 in B and a continuously differentiable mapping g from $\mathcal{B}$ onto A such that $g(b_0) = a_0$ and

$$\phi(g(b), b) = 0, \quad \forall B \in \mathcal{B}. \quad \square$$

Then:

Theorem 2.2. *$\mathcal{O}$ is an open, convex, non-empty subset of U. The mapping from $\mathcal{O}$ onto $H^{2,1}(Q)$, associating to (f, u, h) the solution of (1.1) is univocal and C^1.* □

We first state and prove a lemma, then prove the theorem.

Lemma 2.1. *For any $q \in L^\infty(0, T, L^3(\Omega))$ and $(f, u, h) \in U$, the equation*

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z + qz = f \text{ in } Q, \\ &\frac{\partial z}{\partial n} = u \text{ on } \Sigma, \\ &z(x, 0) = h(x), \text{ a.e. } x \in \Omega, \end{aligned} \tag{2.3}$$

has a unique solution in $H^{2,1}(Q)$. □

Proof of Lemma 2.1. For the sake of simplicity we denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and scalar product of $L^2(\Omega)$, and C_i some positive constants. We multiply the first equation (2.3) by z and integrate on Ω at a given $t \in [0, T]$. After integration by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z(\cdot,t)|^2 + \sum_{i=1}^{3}\int_\Omega \left(\frac{\partial z}{\partial x_i}(x,t)\right)^2 \mathrm{d}x = -\int_\Omega q(x,t)(z(x,t))^2\,\mathrm{d}x + \int_\Gamma u(\gamma,t)z(\gamma,t)\,\mathrm{d}\gamma + \int_\Omega f(x,t)z(x,t)\,\mathrm{d}x.$$

We notice that $1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{2}$; then, using Hölder's inequality:

$$\begin{aligned} \left|\int_\Omega q(x,t)(z(x,t))^2\,\mathrm{d}x\right| &\leq \|q(\cdot,t)\|_{L^3(\Omega)}\|z(\cdot,t)\|_{L^6(\Omega)}\|z(\cdot,t)\|_{L^2(\Omega)}, \\ &\leq \|q\|_{L^\infty(0,T,L^3(\Omega))}\|z(\cdot,t)\|_{L^6(\Omega)}\|z(\cdot,t)\|_{L^2(\Omega)}. \end{aligned}$$

The Sobolev imbedding theorems [1] imply that $H^1(\Omega) \overset{n\leq 3}{\subset} L^6(\Omega)$, so that for some $C_1 > 0$ depending on $\|q\|_{L^\infty(0,T,L^3(\Omega))}$:

$$\left|\int_\Omega q(x,t)(z(x,t))^2\,\mathrm{d}x\right| \leq \tfrac{1}{4}\|z(\cdot,t)\|^2_{H^1(\Omega)} + \frac{C_1}{2}|z(\cdot,t)|^2.$$

We deduce that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z(\cdot,t)|^2 + \sum_{i=1}^{3}\int_\Omega \left(\frac{\partial z}{\partial x_i}(x,t)\right)^2 \mathrm{d}x &\leq \tfrac{1}{4}\|z(\cdot,t)\|^2_{H^1(\Omega)} \\ &+ \frac{C_1}{2}|z(\cdot,t)|^2 + C_2\|u(\cdot,t)\|^2_{L^2(\Gamma)}\|z(\cdot,t)\|_{H^1(\Omega)} + |z(\cdot,t)|\,|f(\cdot,t)|, \\ &\leq \tfrac{1}{2}\|z(\cdot,t)\|^2_{H^1(\Omega)} + C_3|z(\cdot,t)|^2 + C_4\|u(\cdot,t)\|^2_{L^2(\Gamma)} + \tfrac{1}{2}|f(\cdot,t)|^2, \end{aligned}$$

so that, with

$$\|w\|^2_{H^1(\Omega)} = \sum_{i=1}^{3} \int_\Omega \left(\frac{\partial w}{\partial x_i}(x,t)\right)^2 \mathrm{d}x + |w|^2 ,$$

and by multiplying the preceding inequality by 2, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}|z(\cdot,t)|^2 + \sum_{i=1}^{3} \int_\Omega \left(\frac{\partial z}{\partial x_i}(x,t)\right)^2 \mathrm{d}x \leq C_5|z(\cdot,t)|^2 + C_6\|u(\cdot,t)\|^2_{L^2(\Omega)} + |f(\cdot,t)|^2 .$$

From Gronwall's inequality we deduce the unicity of the solution of (2.3) and get a priori estimates of z in $L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$. To deduce an estimate of qz in $L^2(Q)$, we notice that

$$\|qz\|^2_{L^2(Q)} = \int_0^T \|q(\cdot,t)z(\cdot,t)\|^2_{L^2(\Omega)} \, \mathrm{d}t .$$

Using Hölder's inequality with $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$, we get:

$$\begin{aligned} \|qz\|^2_{L^2(Q)} &\leq \int_0^T \|q(\cdot,t)\|^2_{L^3(\Omega)} \|z(\cdot,t)\|^2_{L^6(\Omega)} \, \mathrm{d}t , \\ &\leq \|q\|^2_{L^\infty(0,T,L^3(\Omega))} \|z\|^2_{L^2(0,T,L^6(\Omega))} , \\ &\leq C . \end{aligned}$$

Then we write (2.3) as:

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z = f - qz \text{ in } Q , \\ &\frac{\partial z}{\partial n} = u \text{ on } \Sigma , \\ &z(x,0) = h(x) , \quad \text{a.e. } x \in \Omega . \end{aligned} \tag{2.4}$$

Since we obtained an estimate of qz in $L^2(Q)$, the right-hand side of (2.4) is estimated in U. Then, considering z as a solution of (2.4) with a given right-hand side, we deduce from [10] an a priori estimate of z in $H^{2,1}(Q)$.

We now prove the existence of a solution by considering

$$q_N(x,t) = \inf(N, \sup(-N, q(x,t))) .$$

Obviously, $q_N \to q$ in $L^\infty(0, T, L^3(\Omega))$ as $N \to +\infty$. Since $q_N \in L^\infty(Q)$, it is known that equation (2.3), with q_N instead of q, has a solution z. But we estimated z_N in $H^{2,1}(Q)$ independently of N. So z_N has at least one weak limit point z in $H^{2,1}(Q)$. Passing to the limit in the equation we deduce that z is the solution of (2.3). □

Proof of Theorem 2.2. The set $\mathcal{O}$ is the range of $H^{2,1}(Q)$ through the mapping:

$$H^{2,1}(Q) \to U,$$
$$y \to \left(\frac{\partial y}{\partial t} - \Delta y - y^3, \frac{\partial y}{\partial n}, y(\cdot, 0)\right).$$

Because of (2.2) and the trace theorems [9], this mapping is continuous. Hence, $\mathcal{O}$ is connex and non-empty. Let us prove the unicity of the solution of (1.1). Let y and z be two solutions. Their difference, $w = y - z$, is the solution of:

$$\frac{\partial w}{\partial t} - \Delta w - (y^2 + yz + z^2)w = 0 \quad \text{in } Q,$$
$$\frac{\partial w}{\partial n} = 0 \text{ on } \Sigma, \tag{2.5}$$
$$w(x, 0) = 0, \quad \text{a.e. } x \in \Omega.$$

Since $H^{2,1}(Q) \overset{n \leq 3}{\subset} L^\infty(0, T, L^6(\Omega))$ (see [10]) the function $q = y^2 + yz + z^2$ is in $L^\infty(0, T, L^3(\Omega))$. Then Lemma 2.1 implies that the unique solution of (2.5) is $w = 0$. This means that the solution of (2.1) is unique.

To prove that $\mathcal{O}$ is an open set and that y depends in a smooth way on (f, u, h) we apply the implicit function theorem to

$$F: H^{2,1}(Q) \times U \to U,$$
$$(y, f, u, h) \to \left(\frac{\partial y}{\partial t} - \Delta y - y^3 - f, \frac{\partial y}{\partial n} - u, y(\cdot, 0) - h\right).$$

It is clear that F is C^1. The operator $\partial F/\partial y$ is defined by:

$$\frac{\partial F}{\partial y}(y): H^{2,1}(Q) \to U,$$
$$z \to \left(\frac{\partial z}{\partial t} - \Delta z - 3y^2 z, \frac{\partial z}{\partial n}, z(\cdot, 0)\right).$$

As before, we see that $y^2 \in L^\infty(0, T, L^3(\Omega))$ so that Lemma 2.1 implies that $(\partial F/\partial y)(y)$ is an isomorphism from $H^{2,1}(Q)$ onto U. This allows us to use the implicit function theorem which gives the result. □

We now take into account a constraint on the state of type $y \in L^\alpha(Q)$, $\alpha \in [2, +\infty]$. Because of (2.2) the constraint is automatically satisfied if $\alpha \leq 10$. To deal with the case $\alpha > 10$, we use some new spaces. For y, a natural space is $Y_\alpha = H^{2,1}(Q) \cap L^\alpha(Q)$. When endowed with the norm

$$\|y\|_{Y_\alpha} = \|y\|_{H^{2,1}(Q)} + \|y\|_{L^\alpha(Q)},$$

it is easily checked that Y_α is a Banach space: a Cauchy sequence $\{y_n\}$ in Y_α converges toward some y in $H^{2,1}(Q)$ and some z in $L^\alpha(Q)$. But the convergence in $L^\alpha(Q)$ implies the convergence a.e. on Q, hence $y = z$ and $y_n \to y$ in Y_α.

In order to define a new space for (f, u, h), let us call $z = z(f, u, h)$ the solution of the linear equation

$$\frac{\partial z}{\partial t} - \Delta z = f \text{ in } Q,$$
$$\frac{\partial z}{\partial n} = u \text{ on } \Sigma, \tag{2.6}$$
$$z(x,0) = h(x) \text{ p.p. } x \in \Omega.$$

If $(f, u, h) \in U$, then (2.6) has a unique solution z in $H^{2,1}(Q)$. Define:

$$U_\alpha = \{(f, u, h) \in U \text{ such that } z(f, u, h) \in L^\alpha(Q)\}.$$

Because (2.6) is linear, U_α is a vector space; we endow it with the norm:

$$\|(f, u, h)\|_{U_\alpha} = \|(f, u, h)\|_U + \|z(f, u, h)\|_{L^\alpha(Q)}.$$

Then, as for Y_α, it can be checked that U_α is a Banach space. An interesting particular case is when $f \in L^\lambda(Q)$, $2 \leq \lambda < +\infty$, $u \equiv 0$ and $h \equiv 0$. Then, from the results in [10], $z(f, 0, 0)$ is in $W^{2,1,\lambda}(Q)$. With (2.1) we see that z is in Y_α if $1/\alpha \geq (1/\lambda) - (2/5)$, so that

$$f \in L^{5\alpha/(5+2\alpha)}(Q) \Rightarrow (f, 0, 0) \in U_\alpha, \quad \forall \alpha \in \,]10, +\infty]. \tag{2.7}$$

Relation (2.7) allows us to prove the following lemma:

Lemma 2.2. *Let (f, u, h) be in $\mathcal{O}$, i.e. (1.1) has a solution y in $H^{2,1}(Q)$. Then for any $\alpha \in \,]10, +\infty]$:*

$$y \in L^\alpha(Q) \Leftrightarrow z(f, u, h) \in L^\alpha(Q). \quad \square$$

Proof. The function $w = y - z$ is the solution of:

$$\frac{\partial w}{\partial t} - \Delta w = y^3 \text{ in } Q,$$
$$\frac{\partial w}{\partial n} = 0 \text{ on } \Sigma,$$
$$w(x, 0) = 0, \quad \text{a.e. } x \in \Omega.$$

By (2.2) y^3 is in $L^{10/3}(Q)$. Because of (2.7) this implies that w is in $L^\infty(Q)$. This proves the lemma. $\square$

Remark 2.1. Lemma 2.2 shows that U_α is a 'good structure' for (f, u, h). The idea to consider only the linear part of the system equation to define a convenient space is from Lions [8], where it is applied to some other systems. Note that once we state that $y \in H^{2,1}(Q)$ and $(f, u, h) \in U$, the space U_α is the largest space for (f, u, h), and hence is optimal in this way. $\square$

Define:

$$\mathcal{O}_\alpha = \{(f, u, h) \in U_\alpha;\ (1.1) \text{ has a solution in } Y_\alpha\}.$$

Here is the analogue of Theorem 2.2.

Theorem 2.3. *$\mathcal{O}_\alpha$ is an open, connex, non-empty subset of U_α. The application from $\mathcal{O}_\alpha$ onto Y_α, associating to (f, u, h) the solution of (1.1), is univalued and C^1.* $\square$

Proof. The set $\mathcal{O}_\alpha$ is convex and non-empty because it is the range in U_α of the continuous application:

$$Y_\alpha \to U,$$
$$y \to \left(\frac{\partial y}{\partial t} - \Delta y - y^3, \frac{\partial y}{\partial n}, y(\cdot, 0)\right).$$

Let us check that the range of this mapping is in U_α. Since y is in Y_α, the definition of U_α implies that $(\partial y/\partial t - \Delta y, \partial y/\partial n, y(\cdot, 0))$ is in U_α. On the other hand, $Y_\alpha \subset H^{2,1}(Q)$, so by (2.2) $y^3 \in L^{10/3}(Q)$ and so, by (2.7), $(-y^3, 0, 0)$ is in U_α. So the sum of these two terms is also in U_α. The continuity of the application can be checked by similar arguments.

The unicity of y is a consequence of Theorem 2.2. We prove that $\mathcal{O}_\alpha$ is open and that $(f, u, h) \to y$ is C^1 with the implicit function theorem applied to

$$F: Y_\alpha \times U_\alpha \to U_\alpha,$$
$$(y, f, u, h) \to \left(\frac{\partial y}{\partial t} - \Delta y - y^3 - f, \frac{\partial y}{\partial n} - u, y(\cdot, 0) - h\right).$$

It is easily checked that F is C^1. Lemma 2.1 says that the linear equation $(\partial F/\partial y)(y)z = (f, u, h)$ has a unique solution z in $H^{2,1}(Q)$. But we can write z as $z_1 + z_2$, solution of:

$$\frac{\partial z_1}{\partial t} - \Delta z_1 = f \text{ in } Q,$$
$$\frac{\partial z_1}{\partial n} = u \text{ on } \Sigma,$$
$$z_1(x, 0) = h(x), \quad \text{a.e. } x \in \Omega,$$

and

$$\frac{\partial z_2}{\partial t} - \Delta z_2 = 3y^2 z \text{ in } Q,$$
$$\frac{\partial z_2}{\partial n} = 0 \text{ on } \Sigma,$$
$$z_2(x, 0) = 0, \quad \text{a.e. } x \in \Omega.$$

Since (f, u, h) and, by (2.7), $(3y^2 z, 0, 0)$ are in U_α, z_1 and z_2 are in Y_α and so is z. The result follows. □

3. Application to optimal control problems

We now consider (f, u, h) as control parameters and apply the preceding results to the study of some open-loop optimal control problems. Let α be in $[2, +\infty[$ and

$$J(f, u, h) = \begin{cases} \dfrac{1}{\alpha} \displaystyle\int_Q |y(f, u, h) - y_d|^\alpha \, dx \, dt, & \text{if } (f, u, h) \in \mathcal{O}_\alpha, \\ +\infty, & \text{if not;} \end{cases}$$

in this expression $y(f, u, h)$ is the solution of (1.1), y_d is an element of $L^\alpha(Q)$ and $\mathbb{C}_\alpha = \mathbb{C}$ if $\alpha \in [2, 10]$. We now compute the gradient of J, and first study the case $\alpha \leq 10$.

Proposition 3.1. *If $\alpha \in [2, 10]$ the mapping $(f, u, h) \to J(f, u, h)$ is of class C^1 from $\mathbb{C}$ onto $\mathbb{R}$. There exists an adjoint state $p \in W^{2,1;\alpha/\alpha-1}(Q)$, solution of:*

$$\begin{aligned} &-\frac{\partial p}{\partial t} - \Delta p - 3y^2 p = |y - y_d|^{\alpha-2}(y - y_d) \text{ in } Q, \\ &\frac{\partial p}{\partial n} = 0 \text{ on } \Sigma, \\ &p(x, T) = 0, \quad \text{a.e. } x \in \Omega, \end{aligned} \tag{3.1}$$

such that

$$\begin{aligned} \langle J'(f, u, h), (e, v, g)\rangle_{U'U} = &\int_Q p(x, t)e(x, t)\,\mathrm{d}x\,\mathrm{d}t + (p_{|\Sigma}, v)_\Sigma \\ &- \int_\Omega p(x, 0)g(x)\,\mathrm{d}x, \end{aligned} \tag{3.2}$$

where $p_{|\Sigma}$ is the trace of p on Σ and $(p_{|\Sigma}, \cdot)_\Sigma$ is the extension by continuity in $H^{1/2,1/4}(\Sigma)$ of the linear mapping $v \to \int_\Sigma p_{|\Sigma} v\,\mathrm{d}\Sigma$, defined on a dense subset of $H^{1/2,1/4}(\Sigma)$. □

Proof. J is C^1 as being the composition of two mappings of class C^1:

$$(f, u, h) \to y \to \frac{1}{\alpha}\int_Q |y - y_d|^\alpha\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\langle J'(f, u, h), (e, v, g)\rangle_{U'U} = \int_Q |y - y_d|^{\alpha-2}(y - y_d)z\,\mathrm{d}x\,\mathrm{d}t,$$

where $z \in H^{2,1}(Q)$ is the solution of the linearized state equation:

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z - 3y^2 z = e \text{ in } Q, \\ &\frac{\partial z}{\partial n} = v \text{ on } \Sigma, \\ &z(x, 0) = g(x), \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{3.3}$$

Because the space $L^2(Q)$ is being identified to its dual, U' is identical to $L^2(Q) \times H^{1/2,1/4}(\Sigma)' \times H^1(\Omega)'$, and hence $J'(f, u, h) \equiv (p, q, r)$, elements of the preceding spaces. With (3.3) we get:

$$\begin{aligned} &\int_Q p\left(\frac{\partial z}{\partial t} - \Delta z - 3y^2 z\right)\mathrm{d}x\,\mathrm{d}t + \left(q, \frac{\partial z}{\partial n}\right)_\Sigma + \langle r, z(\cdot, z)\rangle_{H^1(\Omega)'H^1(\Omega)} \\ &= \int_Q |y - y_d|^{\alpha-2}(y - y_d)z\,\mathrm{d}x\,\mathrm{d}t, \quad \forall z \in H^{2,1}(Q). \end{aligned} \tag{3.4}$$

In (3.4), $(\cdot,\cdot)_\Sigma$ means the duality product between $H^{1/2,1/4}(\Sigma)$ and its dual. We now interpret (3.4). Put

$$\bar{f} = |y - y_d|^{\alpha-2}(y - y_d) + 3y^2 p .$$

We easily check that $\bar{f}$ is, at least, in $L^{10/9}(Q)$ (p being in $L^2(Q)$) and

$$\int_Q p\left(\frac{\partial z}{\partial t} - \Delta z\right) \mathrm{d}x\, \mathrm{d}t = \int_Q \bar{f} z \, \mathrm{d}x\, \mathrm{d}t , \tag{3.5}$$

for any z in

$$Z = \left\{ z \in H^{2,1}(Q);\ \frac{\partial z}{\partial n} \equiv 0 \text{ on } \Sigma \text{ and } z(\cdot, 0) \equiv 0 \right\}.$$

Consider $\bar{f}$ as given. Then (3.5) defines p in a unique way. This is because for any $e \in L^2(Q)$, there exists $z = z(e) \in H^{2,1}(Q)$, the solution of:

$$\frac{\partial z}{\partial t} - \Delta z = e \text{ in } Q ,$$

$$\frac{\partial z}{\partial n} = 0 \text{ on } \Sigma ,$$

$$z(\cdot, 0) = 0 , \quad \text{a.e. } x \in \Omega .$$

As $z \in H^{2,1}(Q) \subset L^{10}(Q)$, the application $L: e \to \int_Q \bar{f} z(e) \, \mathrm{d}x\, \mathrm{d}t$ is linear continuous from $L^2(Q)$ onto $\mathbb{R}$. Then (3.5) is equivalent to:

$$\int_Q p(x, t) e(x, t) \, \mathrm{d}x\, \mathrm{d}t = L(e) , \quad \forall e \in L^2(Q) .$$

$L^2(Q)$ being identified to its dual, this equation admits a unique solution in $L^2(Q)$. Now consider $\bar{p}$, the solution of:

$$\frac{\partial \bar{p}}{\partial t} - \Delta \bar{p} = \bar{f} \text{ in } Q ,$$

$$\frac{\partial \bar{p}}{\partial n} = 0 \text{ on } \Sigma , \tag{3.6}$$

$$\bar{p}(x, T) = 0 , \quad \text{a.e. } x \in \Omega .$$

As $\bar{f} \in L^{10/9}(Q)$, equation (3.6) has a unique solution $\bar{p}$ in $W^{2,1;10/9}(Q)$ (see [11]); hence, by (2.1), $\bar{p}$ is also in $L^2(Q)$. Multiplying the first equation (3.6) by $z \in Y$ and integrating by parts, we check that $\bar{p}$ is the solution of (3.5) and so $p = \bar{p}$. From (3.6) and the definition of $\bar{f}$ we deduce that p is the solution of (3.1).

Let us show that p is in $W^{2,1;\alpha/\alpha-1}(Q)$. Since y is in $L^{10}(Q)$ and p is in $L^2(Q)$, y^2p is in $L^{10/7}(Q)$. On the other hand, $|y - y_d|^{\alpha-2}(y - y_d)$ is in $L^{\alpha/\alpha-1}(Q)$; hence, f is in $L^{\beta}(Q)$ with $\beta = \inf(\alpha/\alpha - 1, 10/7)$. Since p is the solution of (3.6), p is in $W^{2,1;\beta}(Q)$. We get the result if $\alpha/\alpha - 1 \leq 10/7$. Since $\alpha \in [2, 10]$, $\alpha/\alpha - 1$ is in $[10/9, 2]$, so the case $\alpha/\alpha - 1 \in [10/7, 2]$ remains open. In that case the preceding analysis shows that p is in $W^{2,1;10/7}(Q)$; hence, by (2.1) in $L^{10/3}(Q)$. Then y^2p is in $L^2(Q)$, and so $\bar{f}$ is in $L^{\alpha/\alpha-1}(Q)$. Since p is the solution of (3.6), it is in $W^{2,1;\alpha/\alpha-1}(Q)$.

We now clarify the relations between p, q and r, to obtain (3.2). Put $\beta = \alpha/(\alpha - 1)$. Since $p \in W^{2,1;\beta}(Q)$, we know [4] that

$$p(\cdot, 0) \in B^{2-2/\beta,\beta}(\Omega)\,; \qquad p_{|\Sigma} \in B^{2-1/\beta,1-1/2\beta;\beta}(Q)\,,$$

where $B^{\cdot,\cdot}(\Omega)$ is a Besov space (see [1]) and

$$B^{2s,s;\beta}(Q) = B^{s,\beta}(0, T; L^{\beta}(\Omega)) \cap L^{\beta}(0, T; B^{2s,\beta}(\Omega))\,.$$

Since $\alpha \in [2; 10]$, β is in $[10/9, 2]$. Since $\frac{1}{5} = 2 - 2 \times \frac{9}{10}$, $p(\cdot, 0)$ is at least in $B^{1/5,10/9}(\Omega)$. Because the first index of this Besov space is not an integer, it is equal to $W^{1/5,10/9}(\Omega)$ ([1]). We know [1] that:

$$W^{1/5,10/9}(\Omega) \overset{n\leq 3}{\subset} L^{\lambda}(\Omega)\,, \qquad \frac{1}{\lambda} = \frac{9}{10} - \frac{1/5}{3}\,.$$

i.e. $\lambda = 6/5$. So $p(\cdot, 0)$ is in $L^{6/5}(\Omega)$. Since $g \in H^1(\Omega)$ and $n = 3$, g is in $L^6(\Omega)$ too and so $\int_\Omega p(x, 0)g(x)\,dx$ is meaningful. Concerning $p_{|\Sigma}$, we know that it is in $L^{\beta}(\Sigma)$ and so in $L^{10/9}(\Sigma)$. Now suppose that $v \in H^{1/2,1/4}(\Sigma) \cap L^{10}(\Sigma)$. From (3.1) and (3.4), integrating by parts, we obtain:

$$-\int_\Omega p(x, 0)g(x)\,dx + \int_\Sigma p_{|\Sigma} v \, d\Sigma = \langle r, g \rangle_{H^1(\Omega)' H^1(\Omega)} + (q, v)_\Sigma\,.$$

Because this is true for any $g \in H^1(\Omega)$, it follows that:

$$-\int_\Omega p(x, 0)g(x)\,dx = \langle r, g \rangle_{H^1(\Omega)' H^1(\Omega)}\,,$$

and so

$$\int_\Sigma p_{|\Sigma} v \, d\Sigma = (q, v)_\Sigma\,, \quad \forall v \in H^{1/2,1/4}(\Sigma) \cap L^{10}(\Sigma)\,.$$

This is true in particular if $v \in \mathcal{D}(\bar{\Sigma})$ which is a dense subset of $H^{1/2,1/4}(\Sigma)$. So the continuous mapping

$$H^{1/2,1/4}(\Sigma) \to \mathbb{R}\,, \qquad v \to (q, v)_\Sigma\,,$$

is the extension by continuity to $H^{1/2,1/4}(\Sigma)$ of the mapping $v \to \int_\Sigma pv\,d\Sigma$. This proves the proposition. □

Remark 3.1. If $\alpha = 2$, p belongs to $H^{2,1}(Q)$ and $p_{|\Sigma} \in H^{3/2,3/4}(\Sigma)$ so that $(p_{|\Sigma}, v)_\Sigma$ is actually, for any $v \in H^{1/2,1/4}(\Sigma)$, equal to $\int_\Sigma pv\,d\Sigma$. This remains probably true for some values of α greater than 2; to prove it we would need an extension of (2.1) to spaces $W^{2s,s;\lambda}(Q)$ with $0 < s < 1$. □

We now extend the results to the case $\alpha > 10$. Since U_α is no longer, like U, a product space, we can no longer split the gradient into three terms.

Proposition 3.2. *If $\alpha \in]10, +\infty[$, the application $(f, u, h) \to J(f, u, h)$ is of class C^1 from $\mathcal{C}_\alpha$ onto $\mathbb{R}$. There* *exists an adjoint state* $p \in W^{2,1;\alpha/\alpha-1}(Q)$, *the solution of*:

$$-\frac{\partial p}{\partial t} - \Delta p - 3y^2 p = |y - y_d|^{\alpha-2}(y - y_d) \text{ in } Q,$$
$$\frac{\partial p}{\partial n} = 0 \text{ on } \Sigma, \tag{3.7}$$
$$p(x, T) = 0, \quad \text{a.e. } x \in \Omega,$$

and $\langle J'(f, u, h), (e, v, g)\rangle_{U_\alpha^* U_\alpha}$ *is the extension by continuity to* U_α *of*

$$\int_Q p(x, t)e(x, t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma p(\gamma, t)v(\gamma, t)\,\mathrm{d}\gamma\,\mathrm{d}t - \int_\Omega p(x, 0)g(x)\,\mathrm{d}x, \tag{3.8}$$

which is defined on a dense subset of U_α. □

Proof. J is still C^1 as the composition of two C^1 mappings. To get an explicit expression of the gradient, let us show that (3.7) has a solution for any $\alpha \in]10, +\infty[$. Let $q \in W^{2,1;\alpha/\alpha-1}(Q)$ be the solution of:

$$-\frac{\partial q}{\partial t} - \Delta q = |y - y_d|^{\alpha-2}(y - y_d) \text{ in } Q,$$
$$\frac{\partial q}{\partial n} = 0 \text{ on } \Sigma,$$
$$q(x, T) = 0, \quad \text{a.e. } x \in \Omega.$$

If p exists, $w = p - q$ is the solution of:

$$-\frac{\partial w}{\partial t} - \Delta w - 3y^2 w = 3y^2 q \text{ in } Q,$$
$$\frac{\partial w}{\partial n} = 0 \text{ on } \Sigma,$$
$$w(x, T) = 0, \quad \text{a.e. } x \in \Omega.$$

Because of (2.1), y^2 belongs to $L^5(Q)$ and q is at least in $L^{5/3}(Q)$, so that $3y^2q$ is in $L^{5/4}(Q)$. Define:

$$\hat{y}_d = y - |3y^2 q|^{1/4} s(q),$$

where $s(\cdot)$ is the sign function $\mathbb{R} \to \mathbb{R}$ defined by

$$s(a) = \begin{cases} -1, & \text{if } a < 0, \\ 0, & \text{if } a = 0, \\ +1, & \text{if } a > 0. \end{cases}$$

It is easily checked that $\hat{y}_d$ is in $L^5(\Omega)$ and that

$$3y^2 q = |y - \hat{y}_d|^3 (y - \hat{y}_d), \quad \text{a.e. in } Q.$$

Consequently, the equation on w appears as the co-state equation associated with the criterion $\frac{1}{5}\int_Q |y-\hat{y}_d|^5 \,dx\,dt$.

By Proposition 3.1, it admits a solution in $W^{2,1;5/4}(Q)$. This proves the existence of $p=q+w$, the solution of (3.7) in $W^{2,1;\alpha/\alpha-1}(Q)$. Expressing J as a product of derivatives and using (3.7) we get:

$$\langle J'(f,u,h),(e,v,g)\rangle_{U_\alpha' U_\alpha} = \int_Q \left(-\frac{\partial p}{\partial t} - \Delta p - 3y^2 p\right) z \,dx\,dt\,,$$

z being the solution in Y_α of the linearized state equation (3.3). We can choose arbitrarily z in Y_α, (e, v, g) being the functions of z through (3.3). We suppose that $z \in \mathscr{D}(\bar{Q})$. From the trace theorems used in the proof of Proposition 3.1, we deduce that p has a trace on Σ (resp. $\Omega\times\{0\}$) which is at least in $L^1(\Sigma)$ (resp. $L^1(\Omega)$). Integrating by parts, this allows us to write:

$$\langle J'(f,u,h),(e,v,g)\rangle_{U_\alpha' U_\alpha} = \int_Q pe \,dx\,dt + \int_\Sigma p_{|\Sigma} v \,dx\,dt - \int_\Omega p(x,0)g(x)\,dx\,,$$

and this is true for any $(e,v,g)\in U_\alpha$ becoming to the range of $\mathscr{D}(\bar{Q})$ by the application

$$z \to \left(\frac{\partial z}{\partial t} - \Delta z - 3y^2 z,\ \frac{\partial z}{\partial n},\ z(\cdot,0)\right).$$

Since this application is surjective from Y_α on U_α, and $\mathscr{D}(\bar{Q})$ is dense in Y_α, we deduce that the range of $\mathscr{D}(\bar{Q})$ is dense in U_α. This proves the proposition. □

Remark 3.2. If one of the elements (e, v, g) is regular enough to give a meaning to the corresponding integral in (3.8), then the gradient of J in the direction (e, v, g) splits into the sum of an integral and an abstract bilinear form. In the general case we cannot split the gradient because (e, v, g) are related by the condition $z(e,v,g)\in L^\alpha(Q)$. □

We now apply the preceding results to the study of an open-loop control problem. We consider $\alpha>0$, three positive constants N_1, N_2 and N_3, and K, a closed convex set in U. Define:

$$I(f,u,h) = \begin{cases} \frac{1}{\alpha}\int_Q |y-y_d|^\alpha \,dx\,dt + \frac{N_1}{2}\|f\|^2_{L^2(Q)} + \frac{N_2}{2}\|u\|^2_{H^{1/2,1/4}(\Sigma)} + \frac{N_3}{2}\|h\|^2_{H^1(\Omega)}\,, \\ \qquad \text{if } (f,u,h)\in \mathscr{O}_\alpha\,, \\ +\infty, \text{ if not}\,. \end{cases}$$

The control problem is:

$$\begin{aligned} &\text{minimize} \quad I(f,u,h)\,, \\ &(f,u,h)\in K\,. \end{aligned} \tag{3.9}$$

Our result is:

Theorem 3.1. *We suppose* $\alpha > 10$ *and*

(i) $\mathscr{O}_\alpha \cap K \neq \emptyset$,

(ii) $N_i > 0$, $i = 1, 2, 3$, *or K is bounded in U.*

Then (3.9) *has at least one solution. Any solution of* (3.9) *checks the necessary optimality conditions*

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y - y^3 = f \\ &-\frac{\partial p}{\partial t} - \Delta p - 3y^2 p = |y - y_d|^{\alpha-2}(y - y_d) \end{aligned}\right\} \text{ in } Q, \tag{3.10}$$

$$\frac{\partial y}{\partial n} = u\,; \qquad \frac{\partial p}{\partial n} = 0 \text{ on } \Sigma,$$

$$y(x,0) = h(x)\,; \qquad p(x,T) = 0, \text{ a.e. } x \in \Omega,$$

and

$$\langle J'(f,u,h), (e-f, v-u, g-h)\rangle_{U_a' U_a} + N_1 \int_Q f(e-f)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ N_2(u, v-u)_{H^{1/2,1/4}(\Sigma)} + N_3(h, g-h)_{H^1(\Omega)} \geq 0\,, \quad \forall (e,v,g) \in K, \tag{3.11}$$

$J'(f,u,h)$ being related to p through Proposition 3.2. □

Remark 3.3. Here is a formulation equivalent to (3.11) using no abstract linear form: $\forall (e,v,g) \in K$, for any sequence $(e_n, v_n, g_n) \to (e, v, g)$ in U_α in such a way that $(e_n - e, v_n - v, g_n - g)$ is 'smooth', we get:

$$\lim_{n\to\infty} \Bigg[\int_Q (p + N_1 f)(e_n - f)\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma (p_{|\Sigma} + N_2 u)(v_n - u)\,\mathrm{d}\Sigma$$

$$- \int_\Omega (p(x,0) + N_3 h(x))(g_n(x) - h(x))\,\mathrm{d}x \Bigg] \geq 0\,. \quad \square$$

Proof of Theorem 3.1. The infimum of I on $\mathcal{C}_\alpha \cap K$ is bounded because of (i). Let (f_n, u_n, h_n) be a minimizing sequence of I on $\mathcal{C}_\alpha \cap K$ and y_n the associated state. Because of (ii) and the definition of I, we get:

f_n is bounded in $L^2(Q)$,
u_n is bounded in $H^{1/2,1/4}(\Sigma)$,
h_n is bounded in $H^1(\Omega)$,
y_n is bounded in $L^\alpha(Q)$.

Consequently,

$$\frac{\partial y_n}{\partial t} - \Delta y_n = f_n + (y_n)^3$$

is bounded in $L^2(Q)$. We deduce of that an estimate of $\{y_n\}$ in $H^{2,1}(Q)$ hence in Y_α. So there exists (f,u,h,y) in $Z = L^2(Q) \times H^{1/2,1/4}(\Sigma) \times H^1(\Omega) \times Y_\alpha$ such that:

$$(f_n, u_n, h_n, y_n) \rightharpoonup (f, u, h, y) \text{ in } Z.$$

Since the inclusion of $H^{2,1}(Q)$ into $L^2(Q)$ is compact, $y_n \to y$ in $L^2(Q)$, and hence a.e. in Q. From a lemma of Lions [7, p. 12] we deduce that $(y_n)^3 \rightharpoonup y_n$ in $L^3(Q)$. This allows us to pass to the limit in the state equation. On the other hand, we can consider I as a convex function of (f, u, h, y) in $U \times Y_\alpha$, and hence weakly l.s.c., so that

$$I(f, u, h) \leq \liminf_{n\to\infty} I(f_n, u_n, h_n).$$

This implies that (f, u, h) is a solution of (3.9).

The necessary optimality conditions, (3.10) and (3.11), are an easy consequence of Proposition 3.2. □

Remark 3.4. A problem similar to (3.9) has been studied by Lions [9] who obtained the expression for the necessary optimality conditions in the case $\alpha \leq 10$; he uses a penalization-type method and so avoids analysis of the state equation. □

Remark 3.5. One can find in [2] an application of the same type of methods to other examples of parabolic systems, and in particular to a (1.1)-type system with a non-linearity in $y^{5/3}$ only, associated with boundary Neumann conditions in $L^2(\Sigma)$ and an initial condition in $L^2(\Omega)$. Also considered are a problem of control by coefficients and a problem of control of a second-order hyperbolic system. □

4. Conclusion

The analysis of an unstable parabolic equation of diffusion-reaction type, apt to explode in a finite time, led to the following conclusions: if the system equation admits a solution y on $[0, T]$ for a given value of the parameters, in some neighbourhood of these parameters, the system equation has a unique solution depending in a smooth way on the parameters. If y is imposed to be in some L^p space, the results are still true if, for $p > 10$, we choose new spaces for y and the parameters, depending on p. These spaces are related to the linear part of the equation. The preceding results allow us to study some control problems associated with the system. □

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