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PONTRYAGIN'S PRINCIPLE FOR THE OPTIMAL CONTROL OF SEMILINEAR ELLIPTIC SYSTEMS WITH STATE CONSTRAINTS

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Abstract

We consider a nonconvex optimal control problem for a semilinear elliptic system with distributed control and rather general nonlinearity in the cost and state equation. The problem includes punctual constraints in the control and the cost. We derive for "almost all" problems of this type necessary optimality conditions in qualified form involving the minimization of some Hamiltonian. These conditions appear as the natural extension of Pontryagin's principle for the optimal control of O.D.E.'s

1 Introduction

In this paper we will state necessary optimality conditions, analogous to Pontryagin's principle for the control of O.D.E.'s, for the following problem : the state equation is

$$\begin{cases} -\Delta y = f(y(x), u(x)) & \text{a.e. } x \text{ in } \Omega, \\ y = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, with smooth boundary $\partial\Omega$. The control $u(x)$ is in some bounded (not necessarily closed) subset K of $\mathbb{R}$, a.e. $x \in \Omega$, i.e. $u \in U$ where

$$U := \{u : \Omega \to K, \text{ measurable}\}.$$

We also assume that the mapping $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, f'_y exists, is non positive and is continuous : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Under this hypotheses it is not difficult to check that the state equation has a unique solution $y \in C_0(\Omega)$ (space of continuous functions on Ω, null on $\partial\Omega$), that we will denote y_u. Now given a continuous mapping $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, differentiable with respect to the first variable with $L'_y(y, u)$ continuous : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we define

$$J(u) := \int_\Omega L(y_u(x), u(x))dx.$$

For $M \in \mathbb{R}^+$ we consider the optimal control problem

$$\min J(u) \; ; \; u \in U \; ; \; g(y_u(x)) \leq M \text{ on } \Omega. \tag{P_M}$$

where g is a C^1 mapping.

Such a problem, but without state constraints, has been considered in Bonnans and Casas (1991) who stated necessary optimality conditions involving a maximum principle, as in Pontryagin's principle. In a recent paper the author [3] obtained an extension of this maximum principle including the state constraints, for almost all values of the parameter M. However the optimality conditions there are obtained in an unqualified form. This is due to the technique of quadratic penalization, in which it is difficult to give an estimate of the multiplier associated to the state constraint. Hence a normalization procedure is used in [3] when passing to the limit, and this gives results that are valid for almost all M, but in unqualified form. We will obtain qualified results by using exact penalization (this being valid for almost all M) and approximating the exact penalized problem by a smooth approximating.

Our motivation is purely theoretical and not related to any real-world problem. The interest of the result presented here is that it gives a different view on Pontryagin's principle. Indeed in the control system there is no idea of causality and the dimension is more than one ; however, our way of proving the maximum principle appears to be at the same time general and simple. Indeed (although some regularization processes have to be dealt with, checking at each time the stability of the infimal cost, in order to be able to apply Ekeland's principle ; also one has to obtain some estimates on the data of the optimality system of the regularized problem in order to be able to pass to the limit) there is no complicated argument in the proof and the main tools are exact penalization, Ekeland's principle, and some elementary estimates on L^s norms. Hence, even in the case of the control of ordinary differential equations our proof may be an alternative to more complicated arguments.

We briefly compare our results to the existing litterature. First order optimality conditions for a nonlinear state equation are obtained in Bonnans and Casas [1]. The case of ill-posed systems (f'_y nonmonotone) but with a control entering linearly in the state equation and quadratically in the cost in dealt with in Bonnans and Casas [2]. A problem with a finite number of state constraints is considered in Raitum [8]. For optimal control of a variational inequality without state constraints, a maximum principle is obtained in Bonnans and Tiba [5] (see also Tiba [9]).

2 Statement of the main result

We start by some preliminaries. Let $M(\Omega)$ be the dual of $C_0(\Omega)$ (space of bounded measures).

The boundary condition implies $\inf(P_M) = +\infty$ when $M < 0$. As $\inf(P_M) \searrow$ when $M \nearrow$, let $\bar{M}$ be the smallest number such that $\inf P_{\bar{M}} < \infty$, i.e. $P_{\bar{M}}$ is feasible. Our hypotheses imply $\inf(P_M) > -\infty$ for all $M \geq \bar{M}$.

We will say that (P_M) has a stable cost at $M = M_0$, if for some $r > 0$

$$\inf P_{M'} \geq \inf P_M - r|M' - M| + o(M' - M_0).$$

As $\inf P_M$ is monotonous, $M \to \inf P_M$ is differentiable a.e.

$M > \bar{M}$, hence (P_M) has a stable cost a.e. $M \geq \bar{M}$.

By $H(y, u, p)$ we denote the Hamiltonian associated to the control problem :

$$H(y, u, p) := L(y, u) + pf(y, u).$$

Theorem 1 *If (P_M) has a stable cost for some $M \geq \bar{M}$, for all solution $\bar{u}$ of (P_M), denoting $\bar{y} = y_{\bar{u}}$, there exist $\bar{\lambda} \in M(\Omega)$, $\bar{p} \in W_0^{1,s}(\Omega)$ for all $s < \frac{n}{n-1}$, such that*

$$\int_\Omega (z - g(\bar{y}))d\bar{\lambda} \leq 0 \text{ for all } z \in C_0(\Omega)\ ;\ z(x) \leq M, \quad (2)$$

$$\begin{cases} -\Delta\bar{p} = f'_y(\bar{y}(x), \bar{u}(x))\bar{p} + L'_y(\bar{y}(x), \bar{u}(x)) + \bar{\lambda}g'(y_u(x)) \\ \quad \text{in } M(\Omega), \\ \\ \bar{p} = 0 \text{ on } \partial\Omega, \end{cases} \quad (3)$$

$$H(\bar{y}(x), \bar{u}(x), \bar{p}(x)) = \min_{v \in K} H(\bar{y}(x), v, \bar{p}(x)), \quad \text{a.e. on } \Omega. \quad (4)$$

Remark 1 Actually equation (4) uniquely defines $\bar{p}$ in $W_0^{1,s}(\Omega)$ for $s < \frac{n}{n-1}$.

3 Exact penalization and regularization

We establish the link between stability of the cost and exact penalization. This kind of argument, due to Clarke [6], essentially derives from the perturbation theory presented in connection with duality theory.

Proposition 1 *If (P_M) is stable, i.e. for some $r > 0$ and*
$|M' - M| \leq \varepsilon$ we have

$$\inf(P_{M'}) \geq \inf P_M - r|M' - M|, \quad (5)$$

then the so-called exact penalty function

$$J_r(u) := J(u) + r\|(g(y_u) - M)^+\|_\infty$$

is such that $\bar{u}$ is a solution of

$$\min J_r(u)\ ;\ u \in U,\ g(y_u) \leq M + \varepsilon. \quad (6)$$

Proof Relation (5) says that $M' \to \inf P_{M'} + r|M' - M|$ has a minimum at M, in $[M - \varepsilon, M + \varepsilon]$ i.e.

$$\begin{aligned} \inf P_M &= \inf\{\inf P_{M'} + r|M' - M|,\ M' \in [M - \varepsilon, M + \varepsilon]\}, \\ &= \inf\{J(u) + r|M' - M|\ ;\ u \in \mathcal{U}\ ;\ g(y_u) \leq M'\ ; \\ &\quad M' \in [M - \varepsilon, M + \varepsilon]\}. \end{aligned}$$

Minimizing first with respect to M' for fixed $u \in U$ we find

$$\begin{aligned} \inf P_M &= \inf\{J(u) + r\|(g(y_u) - M)^+\|_\infty\ ;\ u \in U, \\ &\quad g(y_u) \leq M + \varepsilon\}, \\ &= \inf\{J_r(u)\ ;\ u \in U,\ g(y_u) \leq M + \varepsilon\}. \square \end{aligned}$$

Proposition 1 reduces the proof of Theorem 1 to the derivation of optimality conditions for the problem

$$\min J_r(u)\ ;\ u \in U,\ g(y_u) \leq M + \varepsilon. \quad (P_r)$$

As the state constraint above is not binding at $\bar{u}$ is essentially reduces to the study of a problem without state constraint. Pick $\bar{u}$ solution of P_r and define the distance for u, v in U as

$$d(u, v) := mes\{x \in \Omega\ ;\ u(x) \neq v(x)\}.$$

As K is bounded, (U, d) is a complete metric space. Define now for $\alpha > 0$

$$U_\alpha := \{v \in U\ ;\ d(\bar{u}, v) \leq \alpha\}.$$

For $\alpha > 0$ small enough we have $g(y_u) \leq M + \varepsilon$ for all $u \in U_\alpha$ hence $\bar{u}$ is a local solution of

$$\min J_r(u)\ ;\ u \in U_\alpha. \quad (Q_r)$$

We now define for $q > 1$ and $\varepsilon > 0$ the following costs and control problems :

$$\begin{aligned} J_{r,q}(u) &:= J(u) + r\|(g(y_u(x)) - M)^+\|_q, \\ J_{r,q,\varepsilon}(u) &:= J(u) + r[\varepsilon + \int_\Omega [(g(y_u(x)) - M)^+]^q]^{1/q}, \end{aligned}$$

$$\min J_{r,q}(u)\ ;\ u \in U_\alpha \quad (Q_{r,q})$$

$$\min J_{r,q,\varepsilon}(u)\ ;\ u \in U_\alpha. \quad (Q_{r,q,\varepsilon})$$

Note that $Q_{r,q,\varepsilon}$ has a differentiable cost. We check the continuity of the infimal cost through this regularization procedure.

Lemma 1 *For all $r > 0$ and $q > 1$ one has*

$$\lim_{\varepsilon \searrow 0}\{\inf Q_{r,q,\varepsilon}\} = \inf Q_{r,q}.$$

Proof As $J_{r,q,\varepsilon}(u) \geq J_{r,q}(u)$ we have $\liminf_{\varepsilon \searrow 0} Q_{r,q,\varepsilon} \geq \inf Q_{r,q}$. But from $\inf Q_{r,q} \leq J_{r,q}(u) = \lim_{\varepsilon \searrow 0} J_{r,q,\varepsilon}(u)$, for all $u \in U_\alpha$, we deduce the converse inequality. $\square$

Lemma 2 *For all $r > 0$ one has*

$$\inf Q_r = \lim_{q\to\infty} \inf Q_{r,q}.$$

Proof From $\|z\|_q \to \|z\|_\infty$ for a given z in $L^\infty(\Omega)$we deduce that $J_r(u) = \lim_{q\to\infty} J_{r,q}(u)$, for all u in U_α, hence

$$\inf Q_r \geq \lim_{q\to\infty} \inf\{\inf Q_{r,q}\}.$$

Pick u in U_α. The hypotheses made on f give a uniform bound of Δy_u in $L^\infty(\Omega)$, hence $\{g(y_u)\}$ is uniformly Lipschitz with constant C. Let x_0 be a point where y_u attains its maximum. Then

$$\eta := (g(y_u(x)) - M)^+ = \|(g(y_u(x)) - M)^+\|_\infty.$$

For $\varepsilon > 0$, define $B_\varepsilon := \{x \in \Omega \ ; \ \|x - x_0\| \leq \varepsilon\}$. One has

$$\begin{aligned} \|(g(y_u(x)) - M)^+\|_q &\geq \left[\int_{B_\varepsilon} [(g(y_u(x)) - M)^+]^q\right]^{1/q}, \\ &\geq mes(B_\varepsilon)^{1/q}(\eta - C\varepsilon)^+, \\ &\geq mes(B_\varepsilon)^{1/q}\eta - mes(B_\varepsilon)^{1/q}C\varepsilon. \end{aligned}$$

The set Ω being smooth, there exists $\alpha(\varepsilon) > 0$ such that $mes(B_\varepsilon) > \alpha(\varepsilon)$ with $\alpha(\varepsilon)$ not depending on u. Also we may assume that $mes(B_\varepsilon) \leq 1$. For q large enough, $\alpha(\varepsilon)^{1/q} \geq 1 - \varepsilon$ hence

$$\begin{aligned} \|(g(y_u) - M)^+\|_q &\geq (1-\varepsilon)\eta - C\varepsilon, \\ &= (1-\varepsilon)\|(g(y_u(x)) - M)^+\|_\infty - C\varepsilon. \end{aligned}$$

Let $\gamma = \sup\{\|(g(y_u(x)) - M)^+\|_\infty \ ; \ u \in U\}$; we obtain for $q > q_0$ and all u in U

$$J_{r,q}(u) \geq J_r(u) - r(C+\gamma)\varepsilon,$$

hence $\inf Q_{r,q} \geq \inf Q_r - (C+\gamma)\varepsilon$. This proves the converse inequality. □

4 Approximate optimality conditions for the regularized problem

Let $\theta > 0$ be given. From Lemma 1 and 2, if $\bar{u}$ is solution of (Q_r), then for $q > 0$ large enough it is a $\theta/2$-solution of $Q_{r,q}$. Pick such a q ; for $\varepsilon > 0$ small enough, $\bar{u}$ is a θ-solution of $Q_{r,q,\varepsilon}$. This is a smooth problem without state constraints. As it is not of the form considered in Bonnans and Casas [4] we have to extend their result to our case in order to obtain the optimality system. We define the costate equation as

$$\left\{ \begin{array}{l} -\Delta p = f'_y(y,u)p + L'_y(y,u) \\ \qquad +r[\varepsilon + \int_\Omega [(g(y_u) - M)^+]^q]^{1/q-1} \\ \qquad [(g(y_u) - M)^+]^{q-1} g'(y_u) \\ \text{in } \Omega, \\ \\ p = 0 \text{ on } \partial\Omega. \end{array} \right.$$

Let u, v be in U_α and y_u, y_v their associated state. We need to define an interpolated costate as in [4]. Using the mean value theorem we get $\tilde{y}$, $\approx{y}$ with $\tilde{y}(x), \approx{y}(x)$ in $[y_u(x), y_v(x)]$ for all $x \in \Omega$, such that

$$\begin{aligned} f(y_v, v) &= f(y_u, v) + f'_y(\tilde{y}, v)(y_v - y_u), \\ L(y_v, v) &= L(y_u, v) + L'_y(\approx{y}, v)(y_v - y_u). \end{aligned}$$

Similarly, as

$$\Phi : L^q(\Omega) \to \mathbb{R}, \quad y \to [\varepsilon + \int_\Omega [(g(y_u) - M)^+]^q]^{1/q},$$

is C^1 with derivative

$$\Phi'(y) = [\varepsilon + \int_\Omega [(g(y_u) - M)^+]^q]^{1/q-1}[(g(y_u) - M)^+]^{q-1} g'(y_u)$$

we get $\hat{y}$ in $\{\alpha y_u + (1-\alpha)y_v, \quad \alpha \in [0,1]\}$ solution of

$$\Phi(y_v) = \Phi(y_u) + \Phi'(\hat{y})(y_v - y_u).$$

Define now $p_{u,v}$ as the solution of

$$\left\{ \begin{array}{l} -\Delta p_{u,v} = f'_y(\tilde{y}, v)p_{u,v} + L'_y(\approx{y}, v) + \Phi'(\hat{y}) \text{ in } \Omega, \\ p_{u,v} = 0 \text{ on } \partial\Omega. \end{array} \right.$$

Simple computation as in [4] give the following result.

Lemma 3 *(Hamiltonian formulation of the variation of the cost) For all u, v in U_α one has*

$$J_{r,q,\varepsilon}(v) = J_{r,q,\varepsilon}(u) + \int_\Omega [H(y_u, v, p_{u,v}) - H(y_u, u, p_{u,v})].$$

Now $u \to J_{r,q,\varepsilon}(u)$ is continuous for the distance d define above and (U_α, d) is a complete metric space. Applying Ekeland's principle to the θ-solution $\bar{u}$ we deduce

Proposition 2 *There exists $u_{q,\varepsilon} \in U_\alpha$ with $d(\bar{u}, u_{q,\varepsilon}) \leq \sqrt{\theta}$ and*

$$J_{r,q,\varepsilon}(u_{q,\varepsilon}) \leq J_{r,q,\varepsilon}(u) + \sqrt{\theta} d(u_{q,\varepsilon}, u), \quad \forall u \in U_\alpha.$$

For $\theta < \alpha^2$ the constraint $d(\bar{u}, v) \leq \alpha$ is not binding at $u_{q,\varepsilon}$. Combining this result with Lemma 3, and considering spike perturbations we obtain (with $y_{q,\varepsilon} := y_{u_{q,\varepsilon}}$) :

Theorem 2 *One has, when $\theta < \alpha^2$:*

$$H(y_{q,\varepsilon}, u_{q,\varepsilon}, p_{q,\varepsilon}) \leq \inf_{v\in K} H(y_{q,\varepsilon}, v, p_{q,\varepsilon}) + \sqrt{\theta}, \quad \textit{a.e. on } \Omega,$$

with $p_{q,\varepsilon}$ solution of

$$\left\{ \begin{array}{l} -\Delta p_{q,\varepsilon} = f'_y(y_{q,\varepsilon}, u_{q,\varepsilon})p_{q,\varepsilon} + L'_y(y_{q,\varepsilon}, u_{q,\varepsilon}) \\ \quad +\lambda_{q,\varepsilon} g'(y_{q,\varepsilon}) \text{ in } \Omega, \\ p_{q,\varepsilon} = 0 \text{ on } \partial\Omega, \end{array} \right.$$

where

$$\lambda_{q,\varepsilon} := r[\varepsilon + \int_\Omega [(g(y_{q,\varepsilon}) - M)^+]^q]^{1/q-1}[(g(y_{q,\varepsilon}) - M)^+]^{q-1}.$$

5 Proof of the main result

We have to pass to the limit in the optimality conditions of Theorem 2. We easily obtain when $\theta \searrow 0$, $q \nearrow \infty$, $\varepsilon \searrow 0$, that $d(u_{q,\varepsilon}, \bar{u}) \to 0$, $y_{q,\varepsilon} \to \bar{y}$ in $C_0(\Omega)$ and, assuming $\{\lambda_{q,\varepsilon}\}$ bounded in $M(\Omega)$ that $p_{q,\varepsilon}$ is bounded in $W_0^{1,s}(\Omega)$ for $s < n/(n-1)$ and has $\bar{p}$ solution of (4) as weak limit. Also if $\bar{\lambda}$ is a weak-∞ limit of $\{\lambda_{q,\varepsilon}\}$ in $M(\Omega)$, relation (2) is a simple consequence of the definition of $\lambda_{q,\varepsilon}$. Finally (4) can be deduced from the corresponding relation in Theorem 2.

Hence the only delicate point is to obtain an estimate of $\|\lambda_{q,\varepsilon}\|_{M(\Omega)} = \|\lambda_{q,\varepsilon}\|_1$. It is sufficient to deal with the case $\varepsilon = 0$. Indeed if $z := (y_{q,\varepsilon} - a)^+$ then $\|\lambda_{q,\varepsilon}\|_1 \leq \|\lambda\|_1$ with $\lambda := (\int_\Omega z^q)^{1/q-1} z^{q-1}$. Now $\|\lambda\|_1 = \|z\|_q^{1-q}\|z\|_{q-1}^{q-1} = (\|z\|_{q-1}/\|z\|_q)^{q-1}$, hence we look for a relation like

$$\|z\|_{q-1} \leq C^{\frac{1}{q-1}} \|z\|_q.$$

Apply Hölder's inequality : $\|fg\|_r \leq \|f\|_p \|g\|_s$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$ and here $f = z, g = 1, p = q, r = q-1$ hence $\frac{1}{s} = \frac{1}{r} - \frac{1}{p} = \frac{1}{q(q-1)}$. We obtain

$$\|z\|_{q-1} \leq mes(\Omega)^{\frac{1}{q(q-1)}} \|z\|_q = C_q^{\frac{1}{q-1}} \|z\|_q$$

with $C_q := mes(\Omega)^{1/q}$. For q large enough, $C_q \leq 2$, hence the desired inequality is obtained. □

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