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ON THE CHOICE OF THE FUNCTION SPACES FOR SOME STATE-CONSTRAINED CONTROL PROBLEMS

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ABSTRACT

We study a quadratic parabolic control problem with pointwise final state constraints. As the set of admissible states has an empty interior, the existence of Lagrange multipliers cannot be proved directly. We obtain, however some optimality conditions by expressing the fact that among a space of regular perturbations of the optimal control, the null perturbation is optimal. We show that the qualification hypothesis can be effectively checked in some examples and that the information given by the optimality conditions is useful because it allows to get some regularity results for the optimal control.

1. INTRODUCTION.

Let V and W be two Banach spaces. We consider the following abstract convex state constrained control problem

$$(1.1)\qquad \begin{cases} \min J(u) , \\ u \in K,\ y_u \in Z_{ad}, \end{cases}$$

where $J(.)$ is a smooth convex functional from V into $\mathbb{R}$, $u \to y_u$ is a linear continuous mapping from V into W, and K, Z_{ad} are two closed convex subsets of V and W. In order to apply the rules on subdifferential calculus of convex analysis [5], it is necessary

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to choose the function spaces such that Z_{ad} has a non-empty interior. In the control problems studied by the authors in [2], the following hypothesis holds :

$$(1.2)\quad \begin{cases} \text{there exists a Banach space } Y \subset W \text{ such that :} \\ \text{(i) } y_{\bar{u}} \in Y,\ \bar{u} \text{ being the solution of (1.1.) ,} \\ \text{(ii) the restriction of } Z_{ad} \text{ to } Y \text{ has a non-empty interior} \\ \qquad \text{in } Y. \end{cases}$$

Then a natural choice for the control space is :

$$(1.3)\qquad U = \{u \in V \ ;\ y_u \in Y\} \ ;$$

i.e. the regularity of the state is taken into account in the definition of the control space. An extension of the results to nonlinear systems is also made in [2].

This paper is concerned with the case when there exists no such space Y such that (1.2) holds. Let $\bar{u}$ be a solution of (1.1) and $\bar{y} = y_{\bar{u}}$ the associated state. Our idea is to consider a space $Y \subset W$, that does not contain $\bar{y}$ in general, for which the closed convex set :

$$Z_{\bar{u}} = (Z_{ad} - \bar{y}) \cap Y$$

has a non-empty interior in Y. Then we define U by (1.3) and consider the new problem :

$$(1.4)\quad \begin{cases} \min J(\bar{u} + v), \\ v \in (K - \bar{u}) \cap U \ ;\ y_v \in Z_{\bar{u}}. \end{cases}$$

We know that $v = 0$ is a solution of (1.4) and that $Z_{\bar{u}}$ has a non empty interior in Y. Then, if the following hypothesis holds :

(1.5) $\exists v_0 \in (K - \bar{u}) \cap U \; ; \; y_{v_0} \in \overset{\circ}{Z}_{\bar{u}}$,

there exists some Lagrange multiplier associated to the constraint $y_v \in Z_{\bar{u}}$. Then, coming back to problem (1.1), we deduce the existence of a Lagrange multiplier associated to the state constraint $y_{\bar{u}} \in Z_{ad}$.

This method is simple and its spirit general. However, some doubts can raise on its interest, for two reasons. The first is the possibility of checking the qualification hypothesis (1.5) in some particular examples. The second is that, because of the abstract nature of the space U, it is not clear whether the existence of a multiplier gives any useful information.

In this paper, the method is applied to the control of some parabolic system and we show that the two difficulties stated aboved are overcomed. Specifically, we prove that the qualification hypothesis holds and we use the optimality conditions to deduce some regularity results on the optimal control, which are not obvious.

The paper is concerned only with some convex control problems of linear systems. We mention that the ideas of this paper can be combined with the results obtained by the authors in [2], in order to deal with the case of nonlinear systems. However, this would increase too much the length of this paper.

2. SETTING OF THE MODEL PROBLEM.

Let Ω be an open bounded subset of $\mathbb{R}^n$ $(n \leq 3)$ with smooth (C^∞) boundary Γ. Let $T > 0$ be given. We denote :

$$Q = \Omega \times]0,T[, \; \Sigma = \Gamma \times]0,T[.$$

We consider a system governed by the parabolic equation :

$$(2.1)\quad \begin{cases} \frac{\partial y}{\partial t}(x,t) - \Delta y(x,t) = u(x,t) & \text{a.e. in } Q, \\ \frac{\partial y}{\partial n}(\gamma,t) \equiv 0 & \text{on } \Sigma, \\ y(x,0) \equiv 0 & \text{in } \Omega, \end{cases}$$

where u is a function in $L^2(Q)$. This equation admits (J.L. Lions, E. Magenes [8]) a unique solution y_u in the space :

$$H^{2,1}(Q) = \{y \in H^1(Q) \; ; \; y \in L^2(0,T,H^2(\Omega))\}.$$

As $H^{2,1}(Q)$ is included in $\mathscr{C}([0,T],H^1(\Omega))$ the trace of y_u at time T is well defined. Let $N \geq 0$, $y_d \in L^2(\Omega)$, and K, a non-empty closed convex subset of $L^2(Q)$, be given ; we consider the problem

$$(2.2)\quad \begin{cases} \min J(u) = \frac{N}{2} \int_Q (u(x,t))^2 \, dx\, dt + \frac{1}{2} \int_\Omega (y_u(x,T) - y_d(x))^2 dx, \\ u \in K, \; y_u(x,T) \geq 0, \text{ a.e. } x \in \Omega. \end{cases}$$

A control u_0 is said admissible if $u_0 \in K$ and $y_{u_0}(.,T)$ is positive. We give a result of existence of solutions for problem (2.2) :

Theorem 2.1 :

If an admissible control exists and

(2.3) $N > 0$ *or* K *is bounded in* $L^2(Q)$,
problem (2.2) *has a solution ; if* $N > 0$ *this solution is unique.* □

This theorem is easily proved by considering a minimizing sequence. See for instance J.L. Lions [6].

3. FIRST ORDER OPTIMALITY CONDITIONS.

We first notice that, in general, a solution $\bar{u}$ of problem (2.2) has no reason to be in a space of functions smoother than $L^2(Q)$. In fact, let u_1 be a positive element of $L^2(Q)$. If $N = 0$, $K \supset \{u_1\}$ and $y_d = y_{u_1}(.,T)$, then u_1 is a solution of (2.2).

The linear mapping $u \to y_u(.,T)$ is continuous from $L^2(Q)$ into $H^1(\Omega)$. However, the set :

$$\{z \in H^1(\Omega) \ ; \ z(x) \geq 0, \text{ a.e. } x \in \Omega\} \ ,$$

has an empty interior in $H^1(\Omega)$ iff $n > 1$. Hence, for $n = 1$, a qualification hypothesis associated to the usual spaces for $\bar{u}$ and $\bar{y}$ allow to express the optimality conditions. From now on, we restrict our attention to the case $n = 2,3$. Then we have no direct mean to apply the classical rules on subdifferential calculus (I. Ekeland, R. Temam [5]).

We consider the spaces :

$$Y = \{y \in H^{2,1}(Q) \ ; \ y(.,0) \equiv 0 \ ; \ \frac{\partial y}{\partial n}(\sigma) \equiv 0 \text{ on } \Sigma \ ;$$

$$y(.,T) \in L^\infty(\Omega)\}.$$

$$U = \{u \in L^2(Q) \ ; \ y_u \in Y\}.$$

Endowed with the norm of the graph, Y and U are Banach spaces. We prove that the space U is rather large.

<u>Lemma 3.1</u> :

(i) The space $L^\beta(Q)$ *is imbedded in* U *for* $\beta > (n+2)/2$.

(ii) The space U *is dense in* $L^2(Q)$. □

Proof :

Assertion (ii) is a consequence of (i) and of the density of the imbedding $L^\beta(Q) \subset L^2(Q)$ for $\beta > 2$. Hence is it sufficient to prove (i). We use some results on the heat equation in $L^\beta(Q)$, collected in J.L. Lions [7], ch. 1. If $u \in L^\beta(Q)$, for $\beta \in]1,+\infty[$, equation (2.1) admits a solution y_u in :

$$W^{2,1;\beta}(Q) = \{y \in W^{1,\beta}(Q) \ ; \ y \in L^\beta(0,T,W^{2,\beta}(\Omega))\}.$$

An element of $W^{2,1;\beta}(Q)$ has a trace at time T in the Besov space $B^{2(1-1/\beta),\beta}(\Omega)$. The Sobolev imbedding theorem (R.A. Adams [1]) implies that $B^{2(1-1/\beta),\beta}(\Omega) \subset C(\bar{\Omega})$ if $\beta > (n+2)/2$, where n is the space dimension. This proves the lemma. □

The lemma implies that if $K \subset L^\beta(Q)$, for some $\beta > (n+2)/2$, then $L^\beta(Q)$ is a convenient space for the control. This includes, for instance, the case

$$K = \{u \in L^2(Q) \ ; \ |u(x,t)| \leq 1 \ \text{ a.e. on } Q\}.$$

In order to deal with the general case, we define, given a solution $\bar{u}$ of (2.2) :

$$Z_{\bar{u}} = \{z \in L^\infty(\Omega) \ ; \ z(x) \geq - y_{\bar{u}}(x,T) \ \text{ a.e. on } \Omega\} .$$

We notice that $Z_{\bar{u}}$ is a closed convex subset of $L^\infty(\Omega)$ with a non-empty interior. Now consider the problem :

$$(3.1) \qquad \begin{cases} \min J(\bar{u} + v), \\ v \in (K - \bar{u}) \cap \mathcal{U}, \ y_v(\cdot,T) \in Z_{\bar{u}}. \end{cases}$$

Obviously, $\bar{u} \equiv 0$ is a solution of (3.1). The key point is that the function space involved in (3.1) allow us to express the optimality conditions. We define $Z = L^\infty(\Omega)$.

Theorem 3.1 :

We do the following qualification hypothesis :

(3.2) $\exists v_0 \in (K - \bar{u}) \cap \mathcal{U}\ ;\ \ y_{v_0} \in \mathring{Z}_{\bar{u}}$.

Then there exists $\bar{p} \in \mathcal{U}'$, $\bar{q} \in L^2(0,T,H^1(\Omega))$, $\bar{\mu} \in Z'$ *such that*

$$(3.3)\quad \begin{cases} -\dfrac{\partial \bar{q}}{\partial t} - \Delta \bar{q} \equiv 0 \quad \text{in } Q, \\ \dfrac{\partial \bar{q}}{\partial n} \equiv 0 \text{ on } \Sigma\ ,\ \bar{q}(.,T) = y_{\bar{u}}(.,T) - y_d \text{ in } \Omega . \end{cases}$$

$$(3.4)\quad \langle \bar{\mu}, z \rangle_{Z'Z} \leq 0, \qquad \forall z \in Z_{\bar{u}},$$

$$(3.5)\quad \langle \bar{p}, \frac{\partial y}{\partial t} - \Delta y \rangle_{\mathcal{U}'\mathcal{U}} = \langle \bar{\mu}, y(.,T) \rangle_{Z'Z}, \quad \forall y \in Y,$$

and :

$$(3.6)\quad \int_Q (N\bar{u} + \bar{q})(v - \bar{u})dx\,dt + \langle \bar{p}, v-\bar{u} \rangle_{\mathcal{U}'\mathcal{U}} \geq 0, \quad \forall v \in K, \ \forall v-\bar{u} \in \mathcal{U}. \quad \square$$

Remark 3.1 :

As in [2], if $\bar{u}$ is internal to $K \cap \mathcal{U}$ in $\mathcal{U}$, the density of the imbedding $\mathcal{U} \subset L^2(Q)$ and (3.6) allow to identify $\bar{p}$ with an element of $L^2(Q)$. $\square$

Proof of Theorem 3.1 :

We know that (3.3) has a unique solution $\bar{q}$ in $L^2(0,T,H^1(\Omega))$. It is not difficult to see that the linear mapping :

$$\mathcal{U} \to \mathbb{R},$$

$$v \to \int_Q (N\bar{u} + \bar{q})v\ dx\,dt,$$

is the G-derivative of the function $v \to J(\bar{u} + v)$ in $\mathcal{U}$ at $v \equiv 0$. Define $L \in \mathcal{L}(\mathcal{U},Z)$ by $Lv = y_v(.,T)$. The qualification hypothesis (3.2) and the rules on sub-differential calculus (I. Ekeland, R. Temam [5]) imply the existence of $\bar{\mu} \in Z'$ such that (3.4) holds and, as $\bar{v} \equiv 0$ is a solution of problem (3.1) :

$$(3.7) \qquad \int_Q (N\bar{u} + \bar{q})v \, dx \, dt + \langle L^*\bar{\mu}, v\rangle_{\mathbb{U}'\mathbb{U}} \geq 0, \ \forall v \in (K - \bar{u}) \cap \mathbb{U}.$$

Let us call $\bar{p} = L^*\bar{\mu}$. It is easily seen that (3.7) is equivalent to (3.6). Now, the definition of $\bar{p}$ implies :

$$\langle\bar{p}, v\rangle_{\mathbb{U}'\mathbb{U}} = \langle\bar{\mu}, y_v(\cdot, T)\rangle, \ \forall v \in \mathbb{U}.$$

As system (2.1) makes an isomorphism between $\mathbb{U}$ and Y, this is equivalent to (3.5). This proves the theorem. □

The preceding result may seem difficult to handle because of the abstract nature of the space $\mathbb{U}$. However, we will restrict the information given by (3.3)-(3.6) to some perturbations of the optimal control in $L^\beta(Q)$, and we will see later that this allows to get some usable result. We define :

$$\hat{Z}_{\bar{u}} = Z_{\bar{u}} \cap C(\bar{\Omega}),$$

$$Y_\beta = \{y \in W^{2,1;\beta}(Q) \ ; \ y(\cdot, 0) \equiv 0 \ ; \ \frac{\partial y}{\partial n} \equiv 0 \text{ on } \Sigma\}.$$

We call $M(\bar{\Omega})$ the space of regular bounded additive measures on $\bar{\Omega}$, which is the dual of $C(\bar{\Omega})$. For all $\beta > 1$, we denote $\beta' = \beta/(\beta - 1)$.

Theorem 3.2 :

Let $\bar{q}$, $\bar{p}$, $\bar{\mu}$ *be such that* (3.3)-(3.6) *hold. Then there exists* $\bar{p}_0$ *in* $L^{\beta'}(Q)$ *, for all* $\beta > (n+2)/2$, *which coincides with the restriction of* $\bar{p}$ *to* $L^\beta(Q)$ *, and such that the restriction* $\bar{\mu}_0$ *of* $\bar{\mu}$ *to* $C(\bar{\Omega})$ *satisfies*

$$(3.8) \qquad \int_\Omega z(x) d\bar{\mu}_0 \leq 0, \quad \forall z \in \hat{Z}_{\bar{u}},$$

$$(3.9) \qquad \int_Q \bar{p}_0 \left(\frac{\partial y}{\partial t} - \Delta y\right) dx \, dt = \int_\Omega y(x,T) d\bar{\mu}_0, \ \forall y \in Y_\beta,$$

and

$$(3.10) \qquad \int_Q (N\bar{u} + \bar{q} + \bar{p}_0)(v - \bar{u})dx\,dt \geq 0, \quad \forall v \in K, \quad v - \bar{u} \in L^\beta(Q). \qquad \square$$

Proof :

As $\bar{\mu}_0$ is the restriction of $\bar{u}$ to $C(\bar{\Omega})$, (3.8) is a direct consequence of (3.4).

Take $\beta > (n+2)/2$. As $L^\beta(Q)$ is continuously embedded in $\mathcal{U}$, the mapping

$$L^\beta(Q) \to R \quad ,$$

$$v \to \langle\bar{p},v\rangle \quad ,$$

is a continuous linear form on $L^\beta(Q)$. As the dual of $L^\beta(Q)$ is $L^{\beta'}(Q)$, this means that there exists $\bar{p}_\beta \in L^{\beta'}(Q)$ such that, with (3.5)(3.6) :

$$\int_Q \bar{p}_\beta(\frac{\partial y}{\partial t} - \Delta y)dx\,dt = \langle\bar{\mu}_0, y(\cdot,T)\rangle_{M(\bar{\Omega}),C(\bar{\Omega})}, \ \forall y \in Y_\beta,$$

$$\int_Q (N\bar{u} + \bar{q} + \bar{p}_\beta)(v - \bar{u})dx\,dt \geq 0, \ \forall v \in K, \ v-\bar{u} \in L^\beta(Q).$$

To get the conclusion, we have to show that all the restrictions $\bar{p}_\beta$ of $\bar{p}$ are equal to some $\bar{p}_0$ a.e. on Q. But as $L^{\beta_2}(Q) \subset L^{\beta_1}(Q)$ if $\beta_2 > \beta_1$, we have

$$\langle p,v\rangle_{\mathcal{U}'\mathcal{U}} = \int_Q \bar{p}_{\beta_2} v = \int_Q \bar{p}_{\beta_1} v, \qquad \forall v \in L^{\beta_2}(Q),$$

hence :

$$\int_Q (\bar{p}_{\beta_2} - \bar{p}_{\beta_1})v = 0, \qquad \forall v \in L^{\beta_2}(Q),$$

which implies that $\bar{p}_{\beta_2} \equiv \bar{p}_{\beta_1}$. $\square$

We will now show that the above analysis can be effectively applied ; i.e., for some special choices of the convex K, the qualification hypothesis can be checked and the analysis of the optimality conditions gives some effective results.

4. THE CASE OF LOCAL CONSTRAINTS ON THE CONTROL.

We suppose that for some $\alpha \in \mathbb{R}$, we have :

$$K = \{u \in L^2(Q) \; ; \; u \geq \alpha \quad \text{a.e. on } Q\}. \tag{4.1}$$

Obviously, if α is non negative, the state constraint plays no role, but it is not so if $\alpha < 0$. In order to apply Theorem 3.2, we check the qualification hypothesis (3.2). Let $\bar{u}$ be a solution of problem (2.2).

Lemma 4.1 :

If K *is given by* (4.1), *hypothesis* (3.2) *holds.*

Proof :

We take $v \equiv 1$ on Q. Obviously v_o is in $(K - \bar{u}) \cap U$. We have $y_v(\cdot,T) \equiv T$. This implies that $y_v \in \overset{o}{Z}_{\bar{u}}$ and proves the lemma. □

Lemma 4.1 and Theorem 3.2 imply that (3.8)-(3.10) hold in this case. From (3.10) we deduce the usual complementarity conditions :

Proposition 4.1 :

If (4.1) *holds, the following complementarity conditions are checked* :

$$\left.\begin{cases} \bar{u} \geq \alpha \quad , \; N\bar{u} + \bar{q} + \bar{p}_o \geq 0 \\ (N\bar{u} + \bar{q} + \bar{p}_o)(\bar{u} - \alpha) = 0 \end{cases}\right\} \text{a.e on } Q. \tag{4.2}$$

□

We then deduce, if $N > 0$, a regularity result on $\bar{u}$ which is not a priori obvious. Given $\varepsilon \in]0,T[$, we define :

$$Q_\varepsilon = \Omega \times]0,T-\varepsilon[.$$

Theorem 4.1 :

If $N > 0$ *and* (4.1) *holds, problem* (2.2) *has a unique solution* $\bar{u}$ *whose restriction to* $\bar{Q}_\varepsilon$ *is in* $C(\bar{Q}_\varepsilon) \cap H^1(Q_\varepsilon)$. □

Proof :

The existence and unicity of $\bar{u}$ is a consequence of Theorem 2.1. It is a classical trick in optimal control to prove that if $N > 0$, then (4.2) is equivalent to :

$$\bar{u} = \max [\alpha, -\frac{1}{N}(\bar{q} + \bar{p}_0)] \quad \text{a.e. on } Q.$$

From (3.9) we deduce that :

$$(4.3) \quad \begin{cases} -\dfrac{\partial \bar{p}_0}{\partial t} - \Delta \bar{p}_0 \equiv 0 \text{ in } Q, \\ \dfrac{\partial \bar{p}_0}{\partial n} \equiv 0 \quad \text{on } \Sigma . \end{cases}$$

Hence, by the regularizing properties of the heat equation applied to $\bar{q}$ and $\bar{p}_0$, we deduce that $-\frac{1}{N}(\bar{q} + \bar{p}_0)$ is in $C(\bar{Q}_\varepsilon) \cap H^1(Q_\varepsilon)$, and so is $\bar{u}$ (for the fact that the maximum of two functions in $H^1(Q_\varepsilon)$ is in $H^1(Q_\varepsilon)$, see e.g. G. Stampacchia [10]).

5. CONSTRAINT ON THE NORM OF THE CONTROL IN $L^2(Q)$.

We define :

$$B = \{u \in L^2(Q) \ ; \ \|u\|_{L^2(Q)} \leq 1\} ,$$

and suppose that

(5.1) $\quad K = B.$

Lemma 5.1 :

Hypothesis (5.1) *imply that the qualification hypothesis* (3.2) *holds.*

Proof :

If $\bar{u} \equiv 0$, the result is obvious. If not, we separate two cases :

a) If $\bar{u}$ is non negative, there exists $M > 0$ such that the set

$$\theta_M = \{(x,t) \in Q \ ; \ \bar{u}(x,t) \geq M\}$$

has a strictly positive measure. For $\alpha \geq 0$, define

$$v_\alpha(x,t) = \begin{cases} \alpha - M \text{ if } (x,t) \in \theta_M, \\ \alpha \quad \text{if not.} \end{cases}$$

Obviously, $\bar{u} + v_\alpha$ is positive for all $\alpha \geq 0$, and $\|\bar{u} + v_\alpha\|_{L^2(Q)} < 1$ if α is less than some $\alpha_0 > 0$. But :

$$y_{\bar{u}+v_\alpha}(.,T) = \alpha T + y_{\bar{u}+v_0}(.,T) \geq \alpha T,$$

by the maximum principle. Hence for $\alpha < \alpha_0$, $v_\alpha = (\bar{u} + v_\alpha) - \bar{u}$ is in $(K - \bar{u}) \cap U$ and y_{v_α} is in $\mathring{Z}_{\bar{u}}$, and so v_α checks (3.2).

b) If $\bar{u}$ is not non negative, there exists $M > 0$ such that :

$$\hat{\theta}_M = \{(x,t) \in Q \ ; \ \bar{u}(x,t) \leq -M\}$$

has a strictly positive measure. For $\alpha \geq 0$ define :

$$v_\alpha(x,t) = \begin{cases} \alpha + M \quad \text{if } (x,t) \in \hat{\theta}_M, \\ \alpha \qquad \text{if not.} \end{cases}$$

For α small enough, v_α is in $(K - \bar{u}) \cap U$ and :

$$y_{\bar{u}+v_\alpha}(.,T) \geq y_{v_\alpha}(.,T) \geq \alpha T.$$

Hence, v_α checks (3.2) for α small enough. □

We now prove some regularity results :

Theorem 5.1 :

The multiplier $\bar{p}_0$ *defined in Theorem 3.2 is in* $L^2(Q)$. □

Theorem 5.1 is a consequence of (3.10) and of the following lemma :

Lemma 5.2 :

Let u *be in* B *and* p *in* $L^1(Q)$ *be such that :*

$$\int_Q p(v - u)dx\, dt \leq 0, \forall v \in B, \quad v - u \in L^\infty(Q). \tag{5.2}$$

Then $p \in L^2(Q)$ *and* $p \equiv \alpha u$ *for some* $\alpha \geq 0$. □

Proof :

Let $M > 0$ be given and define :

$$Q_M = \{x \in Q \; ; \; |p(x)| \leq M\}.$$

Let $v \in L^2(Q)$ be such that :

$$v \equiv u \quad \text{on } Q - Q_M, \tag{5.3}$$

$$\|v\|_{L^2(Q)} < 1. \tag{5.4}$$

Let $\alpha > 0$ be given and define :

$$v_\alpha = \begin{cases} u \text{ in } Q - Q_M, \\ u + \max(-\alpha, \min(\alpha, v-u)) \text{ in } Q_M. \end{cases}$$

Then, $v_\alpha - u$ is in $L^\infty(Q)$. When $\alpha \to +\infty$, $v_\alpha \to v$ in $L^2(Q)$, hence $\|v_\alpha\|_{L^2(Q)} < 1$ for α superior to some α_0. Hence

$$\int_Q p(v_\alpha - u) = \int_{Q_M} p(v_\alpha - u) \le 0, \quad \alpha > \alpha_0.$$

But as the integral is in Q_M and $p_{|Q_M}$ is in $L^\infty(Q_M)$ we can pass to the limit when $\alpha \to +\infty$; hence

$$\int_{Q_M} p(v - u) \le 0,$$

for any v such that (5.3) (5.4) hold. This remains true when (5.3) holds and $\|v\|_{L^2(Q)} = 1$. Hence $p_{|Q_M}$ is an out ward normal at $u_{|Q_M}$ to the convex set :

$$K_M = \{v \in L^2(Q_M) \; ; \|v\|^2_{L^2(Q_M)} \le 1 - \|u\|^2_{L^2(Q-Q_M)}\}.$$

This implies that for some $\alpha_M \ge 0$ we have :

$$p(x) = \alpha_M u(x) \quad \text{a.e. on } Q_M.$$

Hence, if $u \equiv 0$ on Q, $p \equiv 0$ on Q and if $u \not\equiv 0$ on Q, α_M is equal to some $\alpha \ge 0$ independent of M, for M great enough, As $\text{mes}\,(Q - \bigcup_{M>0} Q_M) = 0$, this proves the lemma. □

We deduce from the optimality conditions a regularity result for $\bar{u}$ if $N > 0$.

Theorem 5.1 :

If $N > 0$ *and* (5.1) *holds, problem* (2.2) *has a unique solution* $\bar{u}$ *whose restriction to* $\bar{\Omega} \times [0,T[$ *is of class* C^∞. □

Proof :

The existence and unicity of $\bar{u}$ is a consequence of Theorem 2.1.

We have for $\beta>(n+2)/2$:

$$\int_Q (N\bar{u} + \bar{q} + \bar{p}_o)(v-\bar{u})dx\, dt \geq 0, \forall v \in B, v-\bar{u} \in L^\beta(Q),$$

hence there exists $\alpha \geq 0$ such that :

$$N\bar{u} + \bar{q} + \bar{p}_o = -\alpha\bar{u} \quad \text{on } Q,$$

hence if $N > 0$:

$$\bar{u} = \frac{-1}{N+\alpha}(\bar{q} + \bar{p}_o) \text{ on } Q.$$

From (3.3), (4.3) and the regularizing properties of the heat equation, we deduce the desired result. □

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