ON THE CHOICE OF THE FUNCTION SPACES
FOR SOME STATE-CONSTRAINED CONTROL PROBLEMS

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ABSTRACT

We study a quadratic parabolic control problem with pointwise
final state constraints. As the set of admissible states has an
empty interior, the existence of Lagrange multipliers cannot be
proved directly. We obtain, however some optimality conditions by
expressing the fact that among a space of regular perturbations of
the optimal control, the null perturbation is optimal. We show that
the qualification hypothesis can be effectively checked in some
examples and that the information given by the optimality condi-
tions is useful because it allows to get some regularity results for the
optimal control.

1. INTRODUCTION.

Let $V$ and $W$ be two Banach spaces. We consider the following
abstract convex state constrained control problem

$$
\begin{align*}
\min & \ J(u), \\
u \in K, \ y_u \in Z_{ad},
\end{align*}
$$

(1.1)

where $J(.)$ is a smooth convex functional from $V$ into $\mathbb{R}$, $u \mapsto y_u$ is
a linear continuous mapping from $V$ into $W$, and $K, Z_{ad}$ are two
closed convex subsets of $V$ and $W$. In order to apply the rules on
subdifferential calculus of convex analysis [5], it is necessary
to choose the function spaces such that $Z_{ad}$ has a non-empty interior. In the control problems studied by the authors in [2], the following hypothesis holds:

\[
\begin{align*}
\text{(1.2)} \\
\begin{cases}
\text{there exists a Banach space } Y \subset W \text{ such that :} \\
\quad (i) \quad y_u \in Y, \; \tilde{u} \text{ being the solution of (1.1),} \\
\quad (ii) \quad \text{the restriction of } Z_{ad} \text{ to } Y \text{ has a non-empty interior in } Y.
\end{cases}
\end{align*}
\]

Then a natural choice for the control space is:

\[
\text{(1.3)} \quad U = \{ u \in Y \; ; \; y_u \in Y \} ;
\]

i.e. the regularity of the state is taken into account in the definition of the control space. An extension of the results to nonlinear systems is also made in [2].

This paper is concerned with the case when there exists no such space $Y$ such that (1.2) holds. Let $\tilde{u}$ be a solution of (1.1) and $\tilde{y} = y_{\tilde{u}}$ the associated state. Our idea is to consider a space $Y \subset W$, that does not contain $\tilde{y}$ in general, for which the closed convex set:

\[
Z_u = (Z_{ad} - \tilde{y}) \cap Y
\]

has a non-empty interior in $Y$. Then we define $U$ by (1.3) and consider the new problem:

\[
\begin{align*}
\text{(1.4)} \\
\begin{cases}
\min J(\tilde{u} + v), \\
\quad v \in (K - \tilde{u}) \cap U \; ; \; y_v \in Z_u
\end{cases}
\end{align*}
\]

We know that $v = 0$ is a solution of (1.4) and that $Z_u$ has a non-empty interior in $Y$. Then, if the following hypothesis holds:
at $Z_{ad}$ has a non-empty
den by the authors in [2],

$Y \subseteq \mathcal{W}$ such that:

The solution of (1.1)

to $Y$ has a non-empty interior

\begin{equation}
\psi^* (K - \bar{u}) \cap \mathcal{U} ; y_{v_0} \in \mathcal{Z}'_u,
\end{equation}

there exists some Lagrange multiplier associated to the constraint

$y_v \in \mathcal{Z}'_u$. Then, coming back to problem (1.1), we deduce the
existence of a Lagrange multiplier associated to the state
constraint $y_u \in \mathcal{Z}_{ad}$.

This method is simple and its spirit general. However, some
doubts can raise on its interest, for two reasons. The first is
the possibility of checking the qualification hypothesis (1.5) in
some particular examples. The second is that, because of the
abstract nature of the space $U$, it is not clear whether the
existence of a multiplier gives any useful information.

In this paper, the method is applied to the control of some
parabolic system and we show that the two difficulties stated
above are overcome. Specifically, we prove that the qualification
hypothesis holds and we use the optimality conditions to deduce
some regularity results on the optimal control, which are not
obvious.

The paper is concerned only with some convex control problems
of linear systems. We mention that the ideas of this paper can be
combined with the results obtained by the authors in [2.1], in
order to deal with the case of nonlinear systems. However, this
would increase too much the length of this paper.

2. SETTING OF THE MODEL PROBLEM.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ ($n \leq 3$) with smooth ($C^\omega$)
boundary $\Gamma$. Let $T > 0$ be given. We denote:

\begin{align*}
Q &= \Omega \times [0,T], \\
\Sigma &= \Gamma \times [0,T].
\end{align*}
We consider a system governed by the parabolic equation:

\[
\begin{align*}
&\frac{\partial y}{\partial t}(x,t) - \Delta y(x,t) = u(x,t) \quad \text{a.e. in } Q, \\
&\frac{\partial y}{\partial n}(y,t) = 0 \quad \text{on } \Sigma, \\
&y(x,0) = 0 \quad \text{in } \Omega,
\end{align*}
\]

(2.1)

where \( u \) is a function in \( L^2(Q) \). This equation admits (J.L. Lions, E. Magenes [8]) a unique solution \( y_u \) in the space:

\[ H^{2,1}(Q) = \{ y \in H^1(Q) : y \in L^2(0,T,H^2(\Omega)) \}. \]

As \( H^{2,1}(Q) \) is included in \( C([0,T],H^1(\Omega)) \) the trace of \( y_u \) at time \( T \) is well defined. Let \( N \geq 0, y_d \in L^2(Q), \) and \( K \), a non-empty closed convex subset of \( L^2(Q) \), be given; we consider the problem

\[
\begin{align*}
\min J(u) &= \frac{N}{2} \int_0^T (u(x,t))^2 \, dt + \frac{1}{2} \int_\Omega (y_u(x,T) - y_d(x))^2 \, dx, \\
u &\in K, \ y_u(x,T) \geq 0, \ \text{a.e. } x \in \Omega.
\end{align*}
\]

(2.2)

A control \( u_0 \) is said admissible if \( u_0 \in K \) and \( y_{u_0}(.,T) \) is positive. We give a result of existence of solutions for problem (2.2):

**Theorem 2.1:**

If an admissible control exists and

\[ N > 0 \quad \text{or } K \text{ is bounded in } L^2(Q), \]

problem (2.2) has a solution; if \( N > 0 \) this solution is unique. \( \square \)

This theorem is easily proved by considering a minimizing sequence. See for instance J.L. Lions [6].
the parabolic equation:

\[ a.e. \ in \ Q, \]

the parabolic equation: equation admits (J.L. Lions, in the space:

\[ L^2(0,T;H^2(\Omega)) \]

the trace of \( y_u \) at \( t_1 \in L^2(\Omega) \), and \( K \), a non-empty nen; we consider the problem

\[ \int_{\Omega} dx \ dt + \frac{1}{2} \int_{\Omega} \left( y_{u_\sigma}(x,t) - \gamma_d(x) \right)^2 dx, \]

\( \sigma \in \Omega. \)

if \( u_0 \in K \) and \( y_{u_0}(.,t) \) is one of solutions for problem

\[ 2(q), \]

is the solution is unique. \[ \]

\textit{Iterating a minimizing sequence.}

3. \textsc{First Order Optimality Conditions.}

We first notice that, in general, a solution \( \bar{u} \) of problem (2.2) has no reason to be in a space of functions smoother than \( L^2(Q) \). In fact, let \( u_1 \) be a positive element of \( L^2(Q) \). If \( N = 0, K = \{ u_1 \} \) and \( \gamma_d = y_{u_1}(.,T) \), then \( u_1 \) is a solution of (2.2).

The linear mapping \( u \to y_u(.,T) \) is continuous from \( L^2(\Omega) \) into \( H^1(\Omega) \). However, the set:

\[ \{ z \in H^1(\Omega); z(x) = 0, a.e. \ x \in \Omega \}, \]

has an empty interior in \( H^1(\Omega) \) iff \( n > 1 \). Hence, for \( n = 1 \), a qualification hypothesis associated to the usual spaces for \( \bar{u} \) and \( \gamma \) allow to express the optimality conditions. From now on, we restrict our attention to the case \( n = 2, 3 \). Then we have no direct mean to apply the classical rules on subdifferential calculus (I. Ekeland, R. Temam [5]).

We consider the spaces:

\[ Y = \{ y \in H_{\mathcal{S}}^{2,1}(\Omega); y(.,0) = 0; \frac{\partial y}{\partial n}(\sigma) = 0 \ on \ \Sigma; \]

\[ y(.,T) \in L^\infty(\Omega) \} \]

\[ U = \{ u \in L^2(\Omega); y_u \in Y \} \]

Endowed with the norm of the graph, \( Y \) and \( U \) are Banach spaces. We prove that the space \( U \) is rather large.

\textbf{Lemma 3.1:}

(i) The space \( L^\beta(Q) \) is imbedded in \( U \) for \( \beta > (n+2)/2 \).

(ii) The space \( U \) is dense in \( L^2(Q) \). \[ \]
Proof:

Assertion (ii) is a consequence of (i) and of the density of the imbedding $L^2(Q) \subset L^2(Q)$ for $p > 2$. Hence it is sufficient to prove (i). We use some results on the heat equation in $L^p(\Omega)$, collected in J.L. Lions [7], ch. 1. If $u \in L^p(\Omega)$, for $p \in ]1, +\infty[$, equation (2.1) admits a solution $\gamma_{t}$ in:

$$W^{2,1,1} \cap \{ y \in W^{1,1,1}(\Omega) \cap L^{2}(0,T,W^{2,1} \cap \{ y \in L^{2}(0,T,W^{2,q}(\Omega)) \}.$$ 

An element of $W^{1,1,1}(\Omega)$ has a trace at time $T$ in the Besov space $\mathcal{B}^{2(1-1/p)} \cap \mathcal{B}(\Omega)$. The Sobolev imbedding theorem (R.A. Adams [1]) implies that $\mathcal{B}^{2(1-1/p)} \cap \mathcal{B}(\Omega) \subset C(\Omega)$ if $p > (n+2)/2$, where $n$ is the space dimension. This proves the lemma.

The lemma implies that if $K \subset L^p(\Omega)$, for some $p > (n+2)/2$, then $L^p(\Omega)$ is a convenient space for the control. This includes, for instance, the case

$$X = \{ u \in L^2(\Omega) \ ; \ |u(x,t)| \leq 1 \ a.e. \mbox{ on } \Omega \}.$$ 

In order to deal with the general case, we define, given a solution $\tilde{u}$ of (2.2) :

$$Z_{\tilde{u}} = \{ z \in L^m(\Omega) \ ; \ z(x) = -\gamma_{t}(x,T) \ a.e. \mbox{ on } \Omega \}.$$ 

We notice that $Z_{\tilde{u}}$ is a closed convex subset of $L^m(\Omega)$ with a non-empty interior. Now consider the problem:

$$\begin{align*}
\text{min} \ J(\tilde{u} + v), \\
v \in (K - \tilde{u}) \cap U, \gamma_{t}(\cdot,T) \subset Z_{\tilde{u}}.
\end{align*}$$

(3.1)

Obviously, $\tilde{u} \equiv 0$ is a solution of (3.1). The key point is that the function space involved in (3.1) allow us to express the optimality conditions. We define $Z = L^\infty(\Omega)$. 

of (1) and of the density of the heat equation in $L^B(\Omega)$, Hence is it sufficient to solve $u \in L^B(\Omega)$, for a solution $y_u$ in:

$$y \in L^B(0,T;W^{2,B}(\Omega)).$$

\text{ace at time } T \text{ in the Besov embedding theorem (R.A. Adams)} \hspace{1cm} \square

\text{1}(\Omega), \text{ for some } B > (n+2)/2, \text{ where } n \text{ is the control. This includes,}

\hspace{1cm} \text{: 1 a.e. on } \Omega.

\text{1 case, we define, given a}

\hspace{1cm} \gamma_\bar{u}(x,T) \text{ a.e. on } \Omega).

\text{in some subset of } L^\omega(\Omega) \text{ with a problem :}

\hspace{1cm} Z_\bar{u},

i.1. The key point is that allow us to express the $L^N(\Omega)$.

---

**Theorem 3.1:**

We do the following qualification hypothesis:

$$\exists \bar{v}_0 \in (K \ominus \bar{u}) \cap \bar{U}; \quad y_{\bar{v}_0} \in Z_\bar{u}.$$

Then there exists $\bar{v} \in U$, $\bar{q} \in L^2(0,T;H^1(\Omega))$, $\bar{u} \in Z$ such that

$$\left\{\begin{array}{ll}
-\bar{\partial}_t \bar{q} - \Delta \bar{q} \equiv 0 & \text{in } \Omega, \\
\bar{\partial}_n \bar{q} \equiv 0 & \text{on } \Sigma, \\
\bar{q}(\cdot, T) = y_{\bar{v}_0}(\cdot, T) - y_d & \text{in } \Omega.
\end{array}\right.$$ (3.3)

$$\bar{u}, Z \in Z_\bar{u}, \quad \forall z \in Z_\bar{u},$$ (3.4)

$$\gamma_\bar{u} \phi \gamma_{\bar{u}} \geq 0, \quad \forall \phi \in Y,$$ (3.5)

and:

$$\left\{\int_Q (\bar{u} \bar{u} + \bar{q})(v - \bar{u}) dx \, dt + \int_\Sigma \bar{p} \cdot \nu \bar{u} \bar{v}_0 \gamma_{\bar{u}} v \geq 0, \quad \forall v \in K, \quad \forall \bar{v}_0 \in U. \hspace{1cm} \square\right.$$ (3.6)

**Remark 3.1:**

As in [2], if $\bar{u}$ is internal to $K \cap \bar{U}$ in $U$, the density of the embedding $U \subset L^2(\Omega)$ and (3.6) allow to identify $\bar{p}$ with an element of $L^2(\Omega)$.

Proof of Theorem 3.1:

We know that (3.3) has a unique solution $\bar{q}$ in $L^2(0,T;H^1(\Omega))$. It is not difficult to see that the linear mapping:

$$\bar{u} \rightarrow R,$$

$$v \rightarrow \int_Q (\bar{u} + \bar{q})v \, dx \, dt,$$

\text{is the G-derivative of the function } v \rightarrow J(\bar{u} + v) \text{ in } U \text{ at } v = 0.

\text{Define } L = \mathcal{L}(U, Z) \text{ by } L v = y_v(\cdot, T). \text{ The qualification hypothesis (3.2) and the rules on sub-differential calculus (I. Ekeland, R. Temam [5]) imply the existence of } \bar{u} \in Z \text{ such that (3.4) holds}$$
and, as \( \tilde{\nu} = 0 \) is a solution of problem (3.1):

\[
(3.7) \quad \int_Q (\tilde{\mu} + \tilde{\eta})_t \, dx \, dt + \langle L^* \tilde{\mu}, \nu \rangle_{\mathcal{U}'} \leq 0, \quad \forall \nu \in (K - \tilde{\mu}) \cap \mathcal{U}.
\]

Let us call \( \tilde{\rho} = L^* \tilde{\mu} \). It is easily seen that (3.7) is equivalent to (3.6). Now, the definition of \( \tilde{\rho} \) implies:

\[
\langle \tilde{\rho}, \nu \rangle_{\mathcal{U}'} = \langle \tilde{\mu}, \nu \rangle_{\mathcal{U}'}(.,T), \quad \forall \nu \in \mathcal{U}.
\]

As system (2.1) makes an isomorphism between \( \mathcal{U} \) and \( Y \), this is equivalent to (3.5). This proves the theorem.

The preceding result may seem difficult to handle because of the abstract nature of the space \( \mathcal{U} \). However, we will restrict the information given by (3.3)-(3.6) to some perturbations of the optimal control in \( L^0(\Omega) \), and we will see later that this allows to get some usable result. We define:

\[
\tilde{z}_U = \tilde{z}_U \cap C(\tilde{\Omega}),
\]

\[
Y_\beta = \{ y \in W^{2,1;\beta}_0(\Omega) ; y(.0) = 0 ; \frac{\partial y}{\partial n} = 0 \text{ on } \Sigma \}.
\]

We call \( M(\tilde{\Omega}) \) the space of regular bounded additive measures on \( \tilde{\Omega} \), which is the dual of \( C(\tilde{\Omega}) \). For all \( \beta > 1 \), we denote \( \beta' = \beta/(\beta - 1) \).

**Theorem 3.2:**

Let \( \tilde{\eta}, \tilde{\rho}, \tilde{\mu} \) be such that (3.3)-(3.6) hold. Then there exists \( \tilde{\nu}_0 \in L^\beta(\Omega) \), for all \( \beta > (n+2)/2 \), which coincides with the restriction of \( \tilde{\rho} \) to \( L^\beta(\Omega) \), and such that the restriction \( \tilde{\nu}_0 \) of \( \tilde{\nu} \) to \( C(\tilde{\Omega}) \) satisfies

\[
(3.8) \quad \int_\tilde{\Omega} z(x) d\tilde{\nu}_0 \leq 0, \quad \forall z \in \tilde{z}_U,
\]

\[
(3.9) \quad \int_Q \tilde{\rho}_0 \frac{\partial y}{\partial t} - \Delta y dx \, dt = \int_\Omega y(x, T) d\tilde{\nu}_0, \quad \forall y \in Y_\beta.
\]
\[ m (3.1) : \]
\[ \mu \geq 0, \forall \nu \in (K - \tilde{u}) \cap U. \]

y seen that (3.7) is equivalent plies:

\[ \epsilon U. \]

hism between \( Y \) and \( Y \), this is theorem. \( \square \)

fficult to handle because of otherwise, we will restrict the one perturbations of the optimal er that this allows to get some

\[ \{ \beta > 0 \}; \beta \epsilon \{ H(\alpha) \} \}

.r bounded additive measures all \( \beta > 1 \), we denote

\[ (3.10) \]
\[ \int_{Q} (H\tilde{u} + \tilde{q} + \tilde{p}_{0})(\nu - \tilde{u})dx \ dt \geq 0, \forall \nu \in K, \quad \nu - \tilde{u} \in L_{0}^{\beta}(Q). \]

\[ \text{Proof:} \]

As \( \tilde{u}_{0} \) is the restriction of \( \tilde{u} \) to \( C(\tilde{u}) \), (3.8) is a direct consequence of (3.4).

Take \( \beta > (n+2)/2 \). As \( L_{0}^{\beta}(Q) \) is continuously embedded in \( U \), the mapping

\[ L_{0}^{\beta}(Q) \rightarrow \mathbb{R}, \]
\[ \nu \rightarrow \langle \tilde{p}, \nu \rangle, \]

is a continuous linear form on \( L_{0}^{\beta}(Q) \). As the dual of \( L_{0}^{\beta}(Q) \) is \( L_{0}^{\beta}(Q) \), this means that there exists \( \tilde{p}_{0} \in L_{0}^{\beta}(Q) \) such that, with (3.5) (3.6):

\[ \int_{Q} \tilde{p}_{0}(\Delta y - \gamma y)dx \ dt = \langle \tilde{p}_{0}, y(T) \rangle_{H(\alpha), C(\alpha)} , \quad \forall \nu \in H, \]
\[ \int_{Q} (H\tilde{u} + \tilde{q} + \tilde{p}_{0})(\nu - \tilde{u})dx \ dt \geq 0, \forall \nu \in K, \nu \epsilon L_{0}^{\beta}(Q). \]

To get the conclusion, we have to show that all the restrictions \( \tilde{p}_{0} \) of \( \tilde{p} \) are equal to some \( \tilde{p}_{0} \), a.e. on \( Q \). But as \( L_{0}^{\beta_{2}}(Q) \subset L_{0}^{\beta_{1}}(Q) \) if \( \beta_{2} > \beta_{1} \), we have

\[ \langle \tilde{p}, \nu \rangle_{H(\alpha)} = \int_{Q} \tilde{p}_{0} \nu = \int_{Q} \tilde{p}_{1} \nu, \quad \forall \nu \in L_{0}^{\beta_{1}}(Q), \]

hence:

\[ \int_{Q} (\tilde{p}_{2} - \tilde{p}_{1}) \nu = 0, \quad \forall \nu \in L_{0}^{\beta_{2}}(Q), \]

which implies that \( \tilde{p}_{2} \equiv \tilde{p}_{1} \). \( \square \)
We will now show that the above analysis can be effectively applied; i.e., for some special choices of the convex K, the qualification hypothesis can be checked and the analysis of the optimality conditions gives some effective results.

4. THE CASE OF LOCAL CONSTRAINTS ON THE CONTROL

We suppose that for some \( \alpha \in \mathbb{R} \), we have:

\[
(4.1) \quad K = \{ u \in L^2(Q) \; ; \; u \geq \alpha \; \text{a.e. on } Q \}.
\]

Obviously, if \( \alpha \) is non-negative, the state constraint plays no role, but it is not so if \( \alpha < 0 \). In order to apply Theorem 3.2, we check the qualification hypothesis (3.2). Let \( \tilde{u} \) be a solution of problem (2.2).

Lemma 4.1:

If \( K \) is given by (4.1), hypothesis (3.2) holds.

Proof:

We take \( v \equiv 1 \) on \( Q \). Obviously \( v \) is in \( (K - \tilde{u}) \cap U \). We have \( y_v(T, T) \equiv T \). This implies that \( y_v \in \mathbb{Z} \tilde{u} \) and proves the lemma. \( \square \)

Lemma 4.1 and Theorem 3.2 imply that (3.8)-(3.10) hold in this case. From (3.10) we deduce the usual complementarity conditions:

Proposition 4.1:

If (4.1) holds, the following complementarity conditions are checked:

\[
(4.2) \quad \begin{cases}
\tilde{u} \geq \alpha, \quad N\tilde{u} + \tilde{q} + \tilde{p}_o \geq 0 \\
(N\tilde{u} + \tilde{q} + \tilde{p}_o)(\tilde{u} - \alpha) = 0 
\end{cases} \text{ a.e. on } Q. \]

\( \square \)
analysis can be effectively conducted and the analysis of the active results.

THE CONTROL

We have:

\[ \Omega = \Omega \times [0,T] \cap \{ x \mid x = \epsilon \} . \]

The state constraint plays a role in order to apply Theorem 3.2, (3.2). Let \( \tilde{u} \) be a solution of (3.2) holds.

is in \( (K - \tilde{u}) \cap U \). We have \( \tilde{u} \) and prove the lemma. □

that (3.8)-(3.10) hold. In this complementarity conditions:

\[ \begin{align*}
& c_1 \partial^2 \tilde{p} - \Delta \tilde{p}_0 = 0 \quad \text{in } Q, \\
& \frac{\partial \tilde{p}_0}{\partial n} = 0 \quad \text{on } \Sigma .
\end{align*} \]

Hence, by the regularizing properties of the heat equation applied to \( \bar{q} \) and \( \bar{p}_0 \), we deduce that \(-\frac{1}{N} (\bar{q} + \bar{p}_0)\) is in \( C(\overline{Q}_\epsilon) \cap H^1(Q_\epsilon) \), and so is \( \tilde{u} \) (for the fact that the maximum of two functions in \( H^1(Q_\epsilon) \) is in \( H^1(Q_\epsilon) \), see e.g. G. Stampacchia [10]).

5. CONSTRAINT ON THE NORM OF THE CONTROL IN \( L^2(Q) \).

We define:

\[ B = \{ u \in L^2(Q) ; \| u \|_{L^2(Q)} \leq 1 \} , \]

and suppose that
(5.1) \( K = B. \)

**Lemma 5.1:**

Hypothesis (5.1) imply that the qualification hypothesis (3.2) holds.

**Proof:**

If \( \tilde{\nu} \equiv 0 \), the result is obvious. If not, we separate two cases:

a) If \( \tilde{\nu} \) is non negative, there exists \( M > 0 \) such that the set

\[
\hat{O}_M = \{(x,t) \in Q; \tilde{\nu}(x,t) \geq M\}
\]

has a strictly positive measure. For \( \alpha \geq 0 \), define

\[
v_\alpha(x,t) = \begin{cases} \alpha - M & \text{if } (x,t) \in \hat{O}_M, \\ \alpha & \text{if not.} \end{cases}
\]

Obviously, \( \tilde{\nu} + v_\alpha \) is positive for all \( \alpha \geq 0 \), and

\[\|\tilde{\nu} + v_\alpha\|_{L^2(Q)} < 1 \text{ if } \alpha \text{ is less than some } \alpha_0 > 0. \]

But:

\[
v_\alpha(x,t) = \alpha \tilde{T} + v_{\tilde{\nu} + v_\alpha}(x,t) \geq \alpha \tilde{T},
\]

by the maximum principle. Hence for \( \alpha < \alpha_0 \), \( v_\alpha = (\tilde{\nu} + v_\alpha) - \tilde{\nu} \) is in \((K - \tilde{\nu}) \cap U\) and \( v_\alpha \) is in \( \hat{O}_M \), and so \( v_\alpha \) checks (3.2).

b) If \( \tilde{\nu} \) is not non negative, there exists \( M > 0 \) such that:

\[
\hat{O}_M = \{(x,t) \in Q; \tilde{\nu}(x,t) \leq -M\}
\]

has a strictly positive measure. For \( \alpha \geq 0 \) define:

\[
v_\alpha(x,t) = \begin{cases} \alpha + M & \text{if } (x,t) \in \hat{O}_M, \\ \alpha & \text{if not.} \end{cases}
\]
For a small enough, \( v_\alpha \) is in \((K - \bar{u}) \cap U\) and:

\[
y_{\bar{u} + v_\alpha}(., T) \geq y_{v_\alpha}(., T) \geq \alpha T.
\]

Hence, \( v_\alpha \) checks (3.2) for a small enough. \( \square \)

We now prove some regularity results:

**Theorem 5.1**: The multiplier \( \bar{p}_0 \) defined in Theorem 3.2 is in \( L^2(Q) \). \( \square \)

Theorem 5.1 is a consequence of (3.10) and of the following lemma:

**Lemma 5.2**: Let \( u \) be in \( B \) and \( p \) in \( L^1(Q) \) be such that:

\[
(5.2) \quad \int_Q p(v - u)dxdt \leq 0, \quad \forall v \in B, \quad v - u \in L^\infty(Q).
\]

Then \( p \in L^2(Q) \) and \( p = au \) for some \( a \geq 0 \). \( \square \)

**Proof**:

Let \( M > 0 \) be given and define:

\[
Q_M = \{ x \in Q ; |p(x)| \leq M \}.
\]

Let \( v \in L^2(Q) \) be such that:

\[
(5.3) \quad v \equiv u \text{ on } Q - Q_M,
\]

\[
(5.4) \quad \|v\|_{L^2(Q)} < 1
\]

Let \( \alpha > 0 \) be given and define:

\[
v_\alpha = \begin{cases} u \text{ in } Q - Q_M, \\ u + \max (-\alpha, \min(\alpha, v - u)) \text{ in } Q_M. \end{cases}
\]
Then, $v_\alpha - u$ is in $L^\omega(Q)$. When $\alpha \to +\infty$, $v_\alpha \to v$ in $L^2(Q)$, hence $\|v_\alpha\|_{L^2(Q)} < 1$ for $\alpha$ superior to some $a_0$. Hence

$$\int_Q p(v_\alpha - u) = \int_{Q_M} p(v_\alpha - u) < 0, \quad \alpha > a_0.$$ 

But as the integral is in $Q_M$ and $p|_{Q_M}$ is in $L^\omega(Q_M)$ we can pass to the limit when $\alpha \to +\infty$; hence

$$\int_{Q_M} p(v - u) < 0,$$

for any $v$ such that (5.3) (5.4) hold. This remains true when (5.3) holds and $\|v\|_{L^2(Q)} = 1$. Hence $p|_{Q_M}$ is an outward normal at $u|_{Q_M}$ to the convex set:

$$K_M = \{v \in L^2(Q_M) ; \frac{\|v\|_{L^2(Q_M)}}{L^2(Q-Q_M)} < 1 - \frac{\|u\|_{L^2(Q-Q_M)}}{L^2(Q-Q_M)} \}.$$ 

This implies that for some $\alpha_M > 0$ we have:

$$p(x) = \alpha_M u(x) \text{ a.e. on } Q_M.$$ 

Hence, if $u \equiv 0$ on $Q$, $p \equiv 0$ on $Q$ and if $u \not\equiv 0$ on $Q$, $\alpha_M$ is equal to some $\alpha \geq 0$ independent of $M$, for $M$ great enough. As $\operatorname{mes}(Q - u, Q_M) = 0$, this proves the lemma. \qed

We deduce from the optimality conditions a regularity result for $\bar{u}$ if $N > 0$.

**Theorem 5.1:**

If $N > 0$ and (5.1) holds, problem $(2.2)$ has a unique solution $\bar{u}$ whose restriction to $\bar{u} \times 10, \Pi_{is}$ of class $C^\infty$. \qed

**Proof:**

The existence and unicity of $\bar{u}$ is a consequence of Theorem 2.1.
We have for $\beta(n+2)/2$:

$$
\int_Q (N\tilde{u} + \tilde{q} + \tilde{p}_o)(v-u) dx dt \geq 0, \forall v \in B, v - \tilde{u} \in L^p(Q),
$$

hence there exists $\alpha \geq 0$ such that:

$$
N\tilde{u} + \tilde{q} + \tilde{p}_o = -\alpha \tilde{u} \text{ on } Q,
$$

hence if $N > 0$:

$$
\tilde{u} = -\frac{1}{N+\alpha} (\tilde{q} + \tilde{p}_o) \text{ on } Q.
$$

From (3.3), (4.3) and the regularizing properties of the heat equation, we deduce the desired result. □

REFERENCES.


