SOME STABILITY CONCEPTS AND THEIR APPLICATIONS IN OPTIMAL CONTROL PROBLEMS

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Some Stability Concepts and Their Applications in Optimal Control Problems*

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Abstract

In this work we are concerned with state-constrained optimal control problems. Our aim is to derive the optimality conditions and to prove the convergence of the numerical approximations. To deal with these questions, whose difficulty is motivated by the presence of the state constraints, we consider some concepts of stability of the optimal cost functional with respect to small perturbations of the set of feasible states. While weak and strong stability on the right allow us to derive optimality conditions in a non-qualified and qualified form respectively, the weak stability on the left is the key to prove the convergence of the numerical approximations.

1 Introduction

In this paper we consider an optimal control problem with pointwise state constraints governed by a semilinear elliptic partial differential equation. We study the influence of some stability properties in the derivation of the optimality conditions satisfied by the optimal control, obtained through the penalization of the state constraints, and in the convergence of the numerical approximations of the optimal control problem. Since the ideas are quite general, this study can be

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extended to other state-constrained control problems, for instance to the control of evolution partial differential equations or ordinary differential equations.

It is well known that optimal control problems become more difficult to be studied when some state constraints are present. Under the hypothesis of non-empty interior of the feasible state set, it is possible to derive some optimality conditions in a non-qualified form (or also called Fritz John type conditions). The Slater condition has been widely used to obtain these conditions in a qualified form; see [2], [3], [4], [5], [12]. In these papers the calmness notion introduced by Rockafellar and Clarke, see [16], was considered as an alternative to Slater condition. This condition has been also used by the second author to prove the convergence of the numerical approximation of optimal control problems; see [11], [14].

While the Slater condition is a stability hypothesis that looks at the optimal control in connection with the state constraints, here we will consider some stability conditions that point out the dependence of the optimal cost functional with respect to small perturbations of the set of feasible states. We will distinguish between weak and strong stability and stability on the left and on the right. The strong stability on the right is equivalent, in a local sense, to the existence of a exact penalty function, which can be used to derive the optimality conditions in a qualified form. This concept has been used under the name of calmness, Burke [9]. The weak stability on the right allows to utilize the classical differentiable non-exact penalty function to derive these optimality conditions in a non-qualified form; see [1], [8], [7] for the application of these ideas to the derivation of the Pontryagin's principle for state-constrained control problems. The stability on the left has shown to be useful in the analysis of the convergence of the numerical approximation of state-constrained optimal control problems; see [13].

The plan of the paper is as follows: in the next section we formulate the optimal control problem and prove the existence of a solution; in Section 3 the definitions of the stability concepts are given and the stability of the control problem is investigated; in Section 4 the optimality conditions are derived and finally the convergence of the numerical approximations is stated in Section 5.

## 2 Setting of the control problem

Let \( \Omega \) be an open bounded convex subset of \( \mathbb{R}^n \) \( (n \leq 3) \) with boundary \( \Gamma \). Let us consider the following boundary value problem:

\[
Ay = - \sum_{i,j=1}^{n} \partial_{x_i} (a_{ij}(x) \partial_{x_j} y(x)) + a_0(x) y(x),
\]
with \( a_{ij} \in C^{0,1}(\overline{\Omega}) \) and \( a_0 \in L^\infty(\Omega) \) satisfying:

\[
\begin{cases}
\exists m > 0 \text{ such that } \sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \geq m|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \Omega, \\
a_0(x) \geq 0 \quad \text{a. e. } x \in \Omega.
\end{cases}
\]

Let \( \phi : R \to R \) be an increasing monotone function of class \( C^1 \), with \( \phi(0) = 0 \). Given \( u \in L^2(\Omega) \), then the Dirichlet problem

\[
\begin{cases}
Ay + \phi(y) = u & \text{in } \Omega \\
y = 0 & \text{on } \Gamma
\end{cases}
\]

has a unique solution \( y_u \in H^1_0(\Omega) \cap H^2(\Omega) \). The regularity of \( y_u \) can be obtained as follows: first we deduce the boundedness of \( y_u \) by adapting the techniques of Stampacchia [21] to the nonlinear case or some other method, see for example Bonnans and Casas [6], and then the \( H^2(\Omega) \)-regularity is consequence of Grisvard’s regularity results [20]. Furthermore the following estimate can be proved:

\[
||y_u||_{H^2(\Omega)} \leq C(1 + ||u||_{L^2(\Omega)}),
\]

where \( C \) is independent of \( u \).

Now the control problem is formulated

\[
(P_b) \begin{cases}
\text{minimize } J(u) = \int_\Omega L(x, y_u(x), u(x))dx \\
\text{subject to } u \in K \text{ and } g(y_u(x)) \leq \delta \quad \forall x \in \overline{\Omega},
\end{cases}
\]

where \( g : R \to R \) is continuous, \( L : \Omega \times R^2 \to R \) is a Carathéodory function, \( \delta \) is a fixed real number and \( K \) is a bounded convex weakly-\( * \) closed nonempty subset of \( L^\infty(\Omega) \). We will make the following additional assumptions about \( L \):

**H1** For every \((x, y) \in \Omega \times R \), \( L(x, y, \cdot) : R \to R \) is a convex function and for every \( x \in \Omega \) \( L(x, \cdot, \cdot) : R^2 \to R \) is a function of class \( C^1 \).

**H2** For each \( M > 0 \) there exists a function \( \psi_M \in L^1(\Omega) \) such that for all \( x \in \Omega, |y| \leq M \) and \( |u| \leq M \) the following inequality holds

\[
\left| \frac{\partial L}{\partial u}(x, y, u) \right| + \left| \frac{\partial L}{\partial y}(x, y, u) \right| + |L(x, y, u)| \leq \psi_M(x).
\]

Under these hypotheses we have the following theorem of existence of solution:

**Theorem 1** There exists a number \( \delta_0 \in R \) such that \( (P_b) \) has at least one solution for every \( \delta \geq \delta_0 \), whereas \( (P_b) \) has no feasible control for \( \delta < \delta_0 \).
Proof. Using the continuity of the inclusion $H^2(\Omega) \subset C(\overline{\Omega})$ and the boundedness of $K$ in $L^\infty(\Omega)$, we deduce from (2) the existence of a constant $C_1 > 0$ such that

$$\|y_u\|_{L^\infty(\Omega)} \leq C_1 \quad \forall u \in K.$$ 

Let $M$ and $m$ be the supremum and infimum respectively of $g$ on $[-C_1, +C_1]$. Then it is obvious that $(P_\delta)$ has no feasible control for $\delta < m$, and however all elements of $K$ are feasible for $\delta \geq M$. Let $\delta_0$ be the infimum of the values $\delta$ for which $(P_\delta)$ has feasible controls. Then $m \leq \delta_0 \leq M$ and $(P_\delta)$ has no feasible control for $\delta < \delta_0$. Let us prove that there exists at least one feasible control for $(P_{\delta_0})$. Let $\{\delta_j\}$ be a decreasing sequence converging to $\delta_0$ and $\{u_j\} \subset K$ a sequence of controls such that $u_j$ is feasible for $(P_{\delta_j})$. Since $K$ is bounded, we can extract a subsequence, denoted in the same way, converging weakly-$\star$ in $L^\infty(\Omega)$ towards an element $u_0 \in K$. Then the compactness of the inclusion $H^2(\Omega) \subset C(\overline{\Omega})$ implies the uniform convergence of $\{u_{\delta_j}\}$ to $u_0$ and therefore

$$g(y_{u_0}(x)) = \lim_{j \to \infty} g(y_{u_j}(x)) \leq \lim_{j \to \infty} \delta_j = \delta_0 \quad \forall x \in \overline{\Omega},$$

which proves that $u_0$ is a feasible control for $(P_{\delta_0})$.

To conclude the proof we must establish the existence of an optimal control of $(P_\delta)$ for all $\delta \geq \delta_0$. Let $\{u_k\} \subset K$ be a minimizing sequence of $(P_\delta)$. We can extract a subsequence, denoted again in the same manner, that converges weakly-$\star$ in $L^\infty(\Omega)$ to an element $u_\delta \in K$. Arguing as above, it can be proved that $g(y_{u_\delta}(x)) \leq \delta$ for every $x \in \overline{\Omega}$. Then $u_\delta$ is a feasible control for $(P_\delta)$ and moreover

$$J(u_\delta) \leq \liminf_{k \to \infty} J(u_k) = \inf(P_\delta),$$

where the first inequality is obtained with the help of hypotheses H1) and H2) (see Cesari [15] or Ekeland and Temam [18]) and the last equality follows from the definition of minimizing sequence. Consequently $u_\delta$ is a solution of $(P_\delta)$. ■

3 Some stability concepts associated to $(P_\delta)$

Now we give some stability definitions previously introduced by Bonnans and Casas in [8]:

Definition 1 We will say that problem $(P_\delta)$ is weakly stable on the right if

$$\liminf_{\delta \searrow \delta_0} \inf(P_{\delta'}) = \inf(P_{\delta}),$$

and weakly stable on the left if

$$\liminf_{\delta' \nearrow \delta} \inf(P_{\delta'}) = \inf(P_{\delta}).$$
\((P_\delta)\) is said strongly stable on the right (resp. left) if there exist \(\epsilon > 0\) and \(r > 0\) such that:

\[
\inf(P_\delta) - \inf(P_{\delta'}) \leq r(\delta' - \delta) \quad \forall \delta' \in [\delta, \delta + \epsilon],
\]

respectively

\[
\inf(P_{\delta'}) - \inf(P_\delta) \leq r(\delta - \delta') \quad \forall \delta' \in [\delta - \epsilon, \delta].
\]

If \((P_\delta)\) is weakly (resp. strongly) stable on the left and on the right, it will be called weakly (resp. strongly) stable.

The first result to state is that almost all problems are stable

**Proposition 1** Let \(\delta_0\) be as in Theorem 1. Then for every \(\delta > \delta_0\), except at most a countable number of them (resp. a set of zero measure), the problem \((P_\delta)\) is weakly (resp. strongly) stable.

**Proof.** If we define \(\varphi : [\delta_0, +\infty) \rightarrow R\) by \(\varphi(\delta) = \inf(P_\delta)\), then \(\varphi\) is a decreasing monotone function and therefore \(\varphi\) is continuous (resp. differentiable) at each point except at most a countable number of them (resp. a set of zero measure). Finally it is obvious that the continuity (resp. differentiability) of \(\varphi\) at \(\delta\) implies the weak (resp. strong) stability of \((P_\delta)\). \(\blacksquare\)

By using the maximum principle for elliptic partial differential equations, Casas proved in [13] the following result:

**Proposition 2** Let us assume that \(g : [a, b] \rightarrow R\) is a strictly increasing monotone function, where \(a\) and \(b\) are defined by the expressions

\[
a = \inf\{y_u(x) : x \in \overline{\Omega}, u \in K\} \quad \text{and} \quad b = \sup\{y_u(x) : x \in \overline{\Omega}, u \in K\}.
\]

If \(\delta > \delta_0\), \(u_\delta\) is a solution of \((P_\delta)\) and there exists \(u_0 \in K\), \(u_0 \neq u_\delta\), such that \(u_0(x) \leq u_\delta(x)\) for each \(x \in \overline{\Omega}\), then \((P_\delta)\) is weakly stable on the left.

Next we will study sufficient conditions for strong stability on the left of \((P_\delta)\). First we prove that the Slater condition implies strong stability on the left. Let us remark how the classical Slater condition is written for \((P_\delta)\) at the optimal control \(u_\delta\). If \(g\) is of class \(C^1\), then the Slater hypothesis at \(u_\delta\) is enounced as follows: there exists an element \(u_0 \in K\) such that if \(y_\delta\) is the state associated to \(u_\delta\) and \(z_0 \in H_0^1(\Omega) \cap H^2(\Omega)\) is the solution of

\[
\begin{cases}
Az_0 + \phi'(y_\delta)z_0 = u_0 - u_\delta \quad \text{in} \quad \Omega \\
z_0 = 0 \quad \text{on} \quad \Gamma,
\end{cases}
\]

then

\[
g(y_\delta(x)) + g'(y_\delta(x))z_0(x) < \delta \quad \forall x \in \overline{\Omega}.
\]

**Proposition 3** Let us suppose that \(g\) is of class \(C^1\). If the Slater condition (8) is satisfied at \(u_\delta\), then \((P_\delta)\) is strongly stable on the left.
Proof. Let \( u_\lambda = u_\delta + \lambda(u_0 - u_\delta) \). Then we deduce from (8)

\[
\lim_{\lambda \downarrow 0} \left[ g(y_{u_\lambda}) + \frac{g(y_{u_\lambda}) - g(y_{u_\delta})}{\lambda} \right] = g(y_{u_\delta}) + g'(y_{u_\delta})z_0 < \delta.
\]

Since the previous convergence is uniform, then there exist two numbers \( \lambda_0 > 0 \) and \( \theta \in (0, \delta) \) such that

\[
g(y_{u_\lambda}) + \frac{g(y_{u_\lambda}) - g(y_{u_\delta})}{\lambda} \leq \theta < \delta \quad \forall \lambda \leq \lambda_0.
\]

Let us take \( \epsilon > 0 \) such that

\[
\lambda = \frac{\delta - \delta'}{\delta - \theta} \leq \lambda_0 \quad \forall \delta' \in [\delta - \epsilon, \delta).
\] (9)

Then for every \( \delta' \in [\delta - \epsilon, \delta) \) we have

\[
\sup_{x \in \Omega} g(y_{u_\lambda}(x)) = \sup_{x \in \Omega} \left[ g(y_{u_\lambda}(x)) + \lambda \frac{g(y_{u_\lambda}(x)) - g(y_{u_\delta}(x))}{\lambda} \right] \leq
\]

\[
\lambda \sup_{x \in \Omega} \left[ g(y_{u_\lambda}(x)) + \frac{g(y_{u_\lambda}(x)) - g(y_{u_\delta}(x))}{\lambda} \right] + (1 - \lambda) \sup_{x \in \Omega} g(y_{u_\delta}(x)) \leq
\]

\[
\lambda \theta + (1 - \lambda) \delta = \delta',
\]

which implies that \( u_\lambda \) is feasible for \( (P_{\delta'}) \).

Finally, denoting by

\[
c = \sup_{\lambda \in (0, \lambda_0]} \frac{J(u_\lambda) - J(u_\delta)}{\lambda} < +\infty \quad \text{and} \quad r = \frac{c}{\delta - \theta},
\]

we get with (9)

\[
\inf(P_{\delta'}) \leq J(u_\lambda) = J(u_\delta) + \lambda \frac{J(u_\lambda) - J(u_\delta)}{\lambda} \leq
\]

\[
J(u_\delta) + c\lambda = \inf(P_\delta) + \frac{\delta - \delta'}{\delta - \theta} = \inf(P_\delta) + r(\delta - \delta') \quad \forall \delta' \in [\delta - \epsilon, \delta),
\]

which concludes the proof. ■

Now we consider the case of a convex nondifferentiable function \( g \), which corresponds to the classical case \( g(t) = |t| \). In this context the Slater condition is formulated as follows:

\[
g(y_\delta(x) + z_0(x)) < \delta \quad \forall x \in \overline{\Omega},
\] (10)

where \( z_0 \) is the solution of (7). Then we have the following result:
**Proposition 4** If \( g \) is convex and Slater condition (10) is satisfied at \( u_\delta \), then \( (P_\delta) \) is strongly stable on the left.

**Proof.** This proof follows the same steps than the previous one. We define \( u_\lambda \) in the same way and from (10) and the continuity of \( g \) we deduce the existence of \( \theta \in (0, \delta) \) and \( \lambda_0 > 0 \) satisfying

\[
g \left( y_{u_\delta} + \frac{y_{u_\lambda} - y_{u_\delta}}{\lambda} \right) \leq \theta < \delta \quad \forall \lambda \leq \lambda_0.
\]

Now taking \( \lambda \) as in (9) and using the convexity of \( g \) we obtain

\[
\sup_{x \in \Omega} g(y_{u_\delta}(x)) = \sup_{x \in \Omega} \left( y_{u_\delta}(x) + \lambda \frac{y_{u_\lambda}(x) - y_{u_\delta}(x)}{\lambda} \right) \leq
\]

\[
\frac{\lambda}{\delta} \sup_{x \in \Omega} \left( y_{u_\delta}(x) + \frac{y_{u_\lambda}(x) - y_{u_\delta}(x)}{\lambda} \right) + (1 - \lambda) \sup_{x \in \Omega} g(y_{u_\delta}(x)) \leq \delta'.
\]

The proof finishes as that of Proposition 3. 

**Corollary 1** If \( \phi \) is linear and \( g \) is convex, then \( (P_\delta) \) is strongly stable on the left for every \( \delta > \delta_0 \).

**Proof.** If \( \phi \) is linear, then the Slater condition (10) becomes: there exists \( u_0 \in K \) such that \( g(y_{u_0}(x)) < \delta \) for every \( x \in \Omega \). Therefore it is enough to take \( u_0 \) as any feasible control for \( (P_{\delta_0}) \) and apply the previous proposition.

So far we have studied the stability on the left. Concerning to the stability on the right of \( (P_\delta) \), we have the following result:

**Proposition 5** For every \( \delta \geq \delta_0 \) the problem \( (P_\delta) \) is weakly stable on the right.

**Proof.** Let \( u_\delta \) be a solution of \( (P_\delta) \). Since \( K \) is bounded, we deduce the existence of a sequence \( \{\delta_j\}_{j=1}^{\infty} \) such that \( \delta_j \searrow \delta \) when \( j \to \infty \) and

\[
\lim_{j \to \infty} u_{\delta_j} = \bar{u} \quad \text{weakly-* in } L^\infty(\Omega)
\]

for some \( \bar{u} \in K \). If \( y_j \) and \( \bar{y} \) are the states associated to \( u_{\delta_j} \) and \( \bar{u} \) respectively, we have that \( y_j \to \bar{y} \) uniformly in \( \Omega \). Therefore \( \bar{u} \) is a feasible control for \( (P_{\delta}) \). Now, using the convexity of \( L \) with respect to the third variable and the feasibility of \( u_\delta \) for every \( (P_{\delta'}) \), with \( \delta' > \delta \), we get

\[
\inf(P_\delta) \leq J(\bar{u}) \leq \liminf_{j \to \infty} J(u_{\delta_j}) = \liminf_{\delta \searrow \delta} \inf(P_{\delta'}) \leq J(u_\delta) = \inf(P_\delta),
\]

which proves (3).
4 Optimality conditions

The aim of this section is to prove the following theorem

**Theorem 2** Let \( \overline{u} \in K \) be a solution of \((P_\delta)\) and \( \overline{y} \in H^2(\Omega) \cap H^1_0(\Omega) \) the associated state. Then there exist a real number \( \overline{\alpha} \geq 0 \), a Borel measure \( \overline{\mu} \in M(\Omega) \) and an element \( \overline{p} \in W^{1,s}(\Omega) \), for all \( s \in [1,n/(n-1)] \), satisfying

\[
\overline{\alpha} + ||\overline{\mu}||_{M(\Omega)} > 0, \tag{11}
\]

\[
\begin{align*}
A^* \overline{p} + \phi'(\overline{y}) \overline{p} &= \overline{\alpha} \frac{\partial L}{\partial y}(x, \overline{y}(x), \overline{u}(x)) + g'(\overline{y}(x)) \overline{\mu} \text{ in } \Omega, \\
\overline{p} &= 0 \text{ on } \Gamma,
\end{align*} \tag{12}
\]

\[
\int_{\Omega} (z(x) - g(\overline{y}(x))) d\overline{\mu}(x) \leq 0 \quad \forall z \in C_0(\Omega) \text{ with } z(x) \leq \delta \quad \forall x \in \Omega, \tag{13}
\]

and

\[
\int_{\Omega} \left( \overline{p}(x) + \overline{\alpha} \frac{\partial L}{\partial u}(x, \overline{y}(x), \overline{u}(x)) \right) (u(x) - \overline{u}(x)) dx \geq 0 \quad \forall u \in K. \tag{14}
\]

Moreover, if \((P_\delta)\) is strongly stable on the right, then \((12)-(14)\) is satisfied with \( \overline{\alpha} = 1 \).

**Remark 1** From \((13)\) follows that \( \overline{\mu} \) is a positive measure whose support is included in the set

\[
F_\delta = \{ x \in \Omega : g(\overline{y}(x)) = \delta \}.
\]

Some qualitative properties of the optimal control can be deduced from the previous optimality system. In particular, it can be stated in many situations that the optimal control is a continuous function in \( \Omega \ F_\delta \), i.e. the singularities of \( \overline{u} \) are located at the points where the state constraint is active; see for example [10].

The proof of Theorem 2 is divided into several steps. First for every \( \gamma > 0 \) we introduce a penalized control problem:

\[
(Q_\gamma) \begin{cases}
\min J_\gamma(u) = \int_{\Omega} \left[ L(x, y_u(x), u(x)) + \frac{1}{2\gamma} ((g(\overline{y}(x)) - \delta)^+) \right] dx \\
u \in K.
\end{cases}
\]

The first issue to remark is the following

**Proposition 6** The following identity holds

\[
\lim \inf_{\gamma \downarrow 0} (Q_\gamma) = \inf (P_\delta).
\]

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Proof. Let \( \{u_\gamma\} \) be a family of \( \gamma \)-solutions of problems \((Q_\gamma)\) and \( \{y_\gamma\} \) the associated states:

\[
J(u_\gamma) \leq \inf(Q_\gamma) + \gamma.
\]

From the definition of \((Q_\gamma)\) it follows that \((g(y_\gamma(x)) - \delta)^+ \to 0\) in \(L^2(\Omega)\), which, together with (2) and the compactness of the inclusion \(H_0^1(\Omega) \cap H^2(\Omega) \subset C_0(\Omega)\), implies the convergence \((g(y_\gamma(x)) - \delta)^+ \to 0\) in \(C_0(\Omega)\). Therefore

\[
\delta_\gamma = \|(g(y_\gamma(x)) - \delta)^+\|_{L^\infty(\Omega)} + \delta \to 0 \text{ if } \gamma \searrow 0.
\]

As \(u_\gamma\) is feasible pair for \((P_{\delta_\gamma})\) we deduce that

\[
\inf(P_{\delta_\gamma}) \leq J(y_\gamma, u_\gamma) \leq \inf(Q_\gamma) + \gamma.
\]

Then, using the weak stability of \((P_\delta)\) on the right, Proposition 5, we obtain

\[
\inf(P_{\delta}) = \lim_{\gamma \searrow 0} \inf(P_{\delta_\gamma}) \leq \lim_{\gamma \searrow 0} \{\inf(Q_\gamma) + \gamma\} = \lim_{\gamma \searrow 0} \inf(Q_\gamma) \leq \inf(P_{\delta}),
\]

with the last inequality due to the fact that \((y_\delta, u)\) is feasible for \((Q_\gamma)\) whenever it is feasible for \((P_{\delta})\), with the same cost. \(\blacksquare\)

Let us remark that the convexity assumption stated in \(\textbf{H1}\) can be replaced by the weak stability of \((P_\delta)\) on the right and the proof of the previous proposition is still valid. In fact the same remark remains true for Theorem 2.

To prove the first part of Theorem 2 we will use the next result, called Ekeland's variational principle:

**Theorem 3 (Ekeland [17])** Let \((E, d)\) be a complete metric space, \(F : E \to R \cup \{+\infty\}\) a lower semicontinuous function and let \(e_\epsilon \in E\) satisfy

\[
F(e_\epsilon) \leq \inf_{\epsilon \in E} F(\epsilon) + \epsilon^2.
\]

Then there exists an element \(\overline{e}_\epsilon \in E\) such that

\[
F(\overline{e}_\epsilon) \leq F(e_\epsilon), \quad d(\overline{e}_\epsilon, e_\epsilon) \leq \epsilon
\]

and

\[
F(\overline{e}_\epsilon) \leq F(e) + \epsilon d(e, \overline{e}_\epsilon) \quad \forall e \in E.
\]

**Proof of Theorem 2: First Part.** Thanks to Proposition 6 we have that \(\overline{u}\) is a \(\epsilon_2^2\)-solution of \((Q_\gamma)\), i.e.

\[
J_\gamma(\overline{u}) = J(\overline{u}) = \inf(P_\delta) \leq \inf(Q_\gamma) + \epsilon_2^2,
\]

with \(\epsilon_\gamma \searrow 0\) when \(\gamma \to 0\). Applying Theorem 3, with \(E = K, F = J_\gamma\) and \(d(u, v) = \|u - v\|_{L^\infty(\Omega)}\), we obtain the existence of a control \(u_\gamma \in K\) satisfying

\[
\left\{
\begin{aligned}
J_\gamma(u_\lambda) &\leq J_\gamma(\overline{u}), \quad \|u_\gamma - \overline{u}\|_{L^\infty(\Omega)} \leq \epsilon_\gamma \quad \text{and} \\
J_\gamma(u_\lambda) &\leq J_\gamma(u) + \epsilon_\gamma \|u_\gamma - u\|_{L^\infty(\Omega)} \forall u \in K.
\end{aligned}
\right.
\]

(15)
Since $J_\gamma$ is Gâteaux differentiable and $\varphi(u) = \|u - u\|_{L^\infty(\Omega)}$ is a convex function we obtain (for instance, see [5, Lemma 2])

$$J'_\gamma(u_\gamma) \cdot (u - u_\gamma) + \epsilon_\gamma \|u_\gamma - u\|_{L^\infty(\Omega)} \geq 0 \quad \forall u \in K. \quad (16)$$

Now it is easy to verify that

$$J'_\gamma(u_\gamma) \cdot (u - u_\gamma) = \int_\Omega \left( p_\gamma(x) + \frac{\partial L}{\partial y}(x, y_\gamma(x), u_\gamma(x)) \right) dx,$$

where $y_\gamma$ is the state associated with $u_\gamma$ and $p_\gamma$ is the solution of

$$\begin{cases}
A^* p_\gamma + \phi'(y_\gamma)p_\gamma = \frac{\partial L}{\partial y} (\cdot, y_\gamma, u_\gamma) + \frac{1}{\gamma} (g(y_\gamma) - \delta)^+ g'(y_\gamma) \quad \text{in } \Omega, \\
p_\gamma = 0 \quad \text{on } \Gamma.
\end{cases} \quad (18)$$

Remark that $\frac{\partial L}{\partial y} (\cdot, y_\gamma, u_\gamma) \in L^1(\Omega)$ thanks to assumption H2). Therefore $p_\gamma \in W_0^{1,1}(\Omega)$ for all $s \in [1, n/(n - 1))$; see [21] and [10].

Defining

$$\overline{\alpha}_\gamma = \left( 1 + \frac{1}{\gamma} (g(y_\gamma) - \delta)^+ \|_{M(\Omega)} \right)^{-1}$$

$$\overline{\mu}_\gamma = \frac{\overline{\alpha}_\gamma}{\gamma} (g(y_\gamma) - \delta)^+, \quad \text{and} \quad \overline{p}_\gamma = \overline{\alpha}_\gamma p_\gamma,$$

we obtain

$$\overline{\alpha}_\gamma + \|\overline{\mu}_\gamma\|_{M(\Omega)} = 1, \quad (19)$$

$$\int_\Omega (z(x) - g(y_\gamma(x))) \overline{\mu}_\gamma(x) \leq 0 \quad \forall z \in C_0(\Omega) \text{ with } z(x) \leq \delta \quad \forall x \in \Omega, \quad (20)$$

with (18)

$$\begin{cases}
A^* \overline{p}_\gamma + \phi'(y_\gamma)p_\gamma = \overline{\alpha}_\gamma \frac{\partial L}{\partial y} (\cdot, y_\gamma, u_\gamma) + g'(y_\gamma) \overline{\mu}_\gamma \quad \text{in } \Omega, \\
\overline{p}_\gamma = 0 \quad \text{on } \Gamma,
\end{cases} \quad (21)$$

and with (16) and (17)

$$\int_\Omega \left( \overline{p}_\gamma(x) + \overline{\alpha}_\gamma \frac{\partial L}{\partial y}(x, y_\gamma(x), u_\gamma(x)) \right) (u(x) - u_\gamma(x)) dx +$$

$$\epsilon_\gamma \|u_\gamma - u\|_{L^\infty(\Omega)} \geq 0 \quad \forall u \in K. \quad (22)$$

Now we must pass to the limit in (19)–(21) to deduce (11)–(14). From (15) we have that $u_\gamma \to \overline{u}$ in $L^\infty(\Omega)$, hence $y_\gamma \to \overline{y}$ in $H_0^1(\Omega) \cap H^2(\Omega)$. On the other hand, from the definition of $\overline{\mu}_\gamma$ follows that $\{\overline{\mu}_\gamma\}_{\gamma > 0}$ is bounded in $M(\Omega)$, so we
extract a subsequence weakly-$\ast$ convergent to an element $\bar{\mu} \in M(\Omega)$. Using the assumption $H2$ we obtain that the right hand side of (21) converges in $M(\Omega)$ for a subsequence to the right hand side of (12). Then the corresponding subsequence of $\{\bar{\mu}_\gamma\}_{\gamma > 0}$ converges weakly in $W^{1,\gamma}_0(\Omega)$ to an element $\bar{\mu}$ satisfying (12). To obtain (11) it is enough to remember (19) and remark that

$$\bar{\alpha} + \|\bar{\mu}\|_{M(\Omega)} = \bar{\alpha} + \frac{1}{\delta} < \bar{\mu}, g(\bar{y}) > =$$

$$\lim_{\gamma \to 0} \left( \bar{\alpha}_\gamma + \frac{1}{\delta} < \bar{\mu}_\gamma, g(y_\gamma) > \right) = \lim_{\gamma \to 0} (\bar{\alpha}_\gamma + \|\bar{\mu}_\gamma\|_{M(\Omega)}) = 1.$$

Finally, it is easy to pass to the limit in (19), (21) and (22).

The rest of the section is devoted to the proof of the optimality conditions (12)–(14), with $\bar{\alpha} = 1$, under the assumption of strong stability of $(P_\delta)$ on the right. Instead of the classical differentiable penalization considered above, now we will analyze the exact penalization of the state constraints.

**Proposition 7** If $r > 0$ satisfies (5), then $\bar{u}$ is a local solution in $(K, d)$ of the penalized control problem

$$\begin{cases}
\min J_r(u) = \int_\Omega L(x, y_u(x), u(x))dx + r\|(g(y_u) - \delta)^+\|_{L\infty(\Omega)} \\
u \in K.
\end{cases}$$

**Proof.** From (5) it follows

$$\inf(P_\delta) = \inf \{ J(u) + r(\delta' - \delta) : u \in K, \ g(y_u(x)) \leq \delta', \ \delta' \in [\delta, \delta + \epsilon] \}.$$ 

Minimizing first with respect to $\delta'$ for fixed $u$ we find

$$\inf(P_\delta) = \inf \{ J(u) + r\|(g(y_u) - \delta)^+\|_{L\infty(\Omega)} : u \in K, \ g(y_u(x)) \leq \delta + \epsilon \}.$$ 

Since the mapping $u \in L\infty(\Omega) \rightarrow y_u \in C_0(\Omega)$ is continuous, we deduce the existence of a ball $B_\lambda(\bar{u}), \lambda > 0$, such that

$$\|g(y_u)\|_{L\infty(\Omega)} < \delta + \epsilon \ \ \forall u \in B_\lambda(\bar{u}),$$

which, together with the previous identity, proves that $\bar{u}$ is a local solution of the penalized control problem.

Let us take $\lambda > 0$ as in the proof of the previous proposition and $r > 0$ verifying (5). We introduce the problem

$$\begin{cases}
(Q_r) \min J_r(u) = \int_\Omega L(x, y_u(x), u(x))dx + r\|(g(y_u) - \delta)^+\|_{L\infty(\Omega)} \\
u \in B_\lambda(\bar{u}).
\end{cases}$$
Then $\overline{u}$ is a solution of this problem. We passed from a state-constrained control problem to another control problem without state constraints. The difficulty in this new problem is that the penalization term is not differentiable. To overcome this difficulty we define

$$J_{r,q}(u) = \int_\Omega L(x, y_u(x), u(x)) dx + r \left( q^{-q} + \int_\Omega [(g(y_u(x)) - \delta)^+]^q dx \right)^{1/q}$$

and

$$(Q_{r,q}) \left\{ \begin{array}{l}
\min_{u \in B_\Lambda(\overline{u})} J_{r,q}(u)
\end{array} \right.$$

with $q > 1$. Note that $(Q_{r,q})$ has a differentiable cost and moreover it represents an approximation of $(Q_r)$:

**Proposition 8** The following identity holds

$$\inf(Q_r) = \lim_{q \to \infty} \inf(Q_{r,q}).$$

**Proof.** From the convergence $\|z\|_{L^q(\Omega)} \to \|z\|_{L^\infty(\Omega)}$ for every $z \in L^\infty(\Omega)$ and the inequalities

$$\| (g(y_u) - \delta)^+ \|_{L^q(\Omega)} \leq \left( q^{-q} + \int_\Omega [(g(y_u(x)) - \delta)^+]^q dx \right)^{1/q} \leq$$

$$\frac{1}{q} + \| (g(y_u) - \delta)^+ \|_{L^q(\Omega)}$$

we deduce that $J_{r,q}(u) \to J_r(u)$ when $q \to +\infty$. Therefore for every $u \in B_\Lambda(\overline{u})$

$$\limsup_{q \to +\infty} \inf(Q_{r,q}) \leq \limsup_{q \to +\infty} J_{r,q}(u) = J_r(u),$$

hence

$$\limsup_{q \to +\infty} \inf(Q_{r,q}) \leq \inf(Q_r). \quad (23)$$

Now we prove the converse inequality. Let $\{u_q\}_{q \geq 1} \subset B_\Lambda(\overline{u})$ be a sequence such that

$$J_{r,q}(u_q) \leq \inf(Q_{r,q}) + \frac{1}{q}.$$

Then using (23) we obtain

$$\liminf_{q \to +\infty} J_{r,q}(u_q) \leq \limsup_{q \to +\infty} J_{r,q}(u_q) = \limsup_{q \to +\infty} \inf(Q_{r,q}) \leq \inf(Q_r) \leq \liminf_{q \to +\infty} J_r(u_q).$$

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We end the proof by checking that
\[
\liminf_{q \to \infty} J_r(u_q) \leq \liminf_{q \to \infty} J_{r,q}(u_q).
\]
Let \( \{q_k\}_{k=1}^{\infty} \) a sequence converging to \(+\infty\) such that
\[
\liminf_{q \to \infty} J_r(u_q) = \lim_{k \to \infty} J_r(u_{q_k}).
\]
Let \( \{y_{q_k}\}_{k=1}^{\infty} \) the states associated to \( \{y_{q_k}\}_{k=1}^{\infty} \). Since \( \{y_{q_k}\}_{k=1}^{\infty} \) is bounded in \( H^1_0(\Omega) \cap H^2(\Omega) \) we can extract a subsequence, denoted in the same way, such that \( y_{q_k} \to \tilde{y} \) in \( C_0(\Omega) \). Let \( z = (g(\tilde{y}) - \delta)^+ \) and let us assume that \( \tilde{y} \) attains its maximum in \( x_0 \in \overline{\Omega} \). Then for every \( \epsilon > 0 \) there exist \( \rho_\epsilon > 0 \) and \( k_\epsilon \in \mathbb{N} \) such that
\[
|(g(y_{q_k}(x) - \delta)^+) \geq ||z||_{L^\infty(\Omega)} - \epsilon, \quad \forall x \in \overline{\Omega}, \quad ||x - x_0|| \leq \rho_\epsilon \text{ and } \forall k \geq k_\epsilon.
\]
From this we deduce that
\[
(||z||_{L^\infty(\Omega)} - \epsilon) [m(B_{\rho_\epsilon}(x_0))]^{1/q_k} \leq ||(g(y_{q_k}(x)) - \delta)^+]_{L^{q_k}(\Omega)} \leq

||g(y_{q_k}(x)) - \delta^+]_{L^{q_k}(\Omega)} [m(\Omega)]^{1/q_k},
\]
hence
\[
||z||_{L^\infty(\Omega)} - \epsilon \leq \lim_{k \to \infty} ||(g(y_{q_k}(x)) - \delta)^+]_{L^{q_k}(\Omega)} \leq ||z||_{L^\infty(\Omega)}.
\]
Since \( \epsilon > 0 \) is arbitrary, we have
\[
\lim_{k \to \infty} ||(g(y_{q_k}(x)) - \delta)^+]_{L^{q_k}(\Omega)} = ||z||_{L^\infty(\Omega)}.
\]
Finally
\[
\liminf_{q \to \infty} J_r(u_q) = \liminf_{k \to \infty} J(u_{q_k}) + ||z||_{L^\infty(\Omega)} =

\liminf_{k \to \infty} J_{r,q_k}(u_{q_k}) \leq \liminf_{q \to \infty} J_{r,q}(u_q). \quad \blacksquare
\]

**Proof of Theorem 2: Second Part.** Thanks to propositions 7 and 8 we deduce that \( \overline{u} \) is a \( \epsilon_2^2 \)-solution of \((Q_{r,q})\), with \( \epsilon_2 \nearrow 0 \) when \( q \to \infty \). Then Theorem 3 claims the existence of a control \( u_q \in K \) satisfying
\[
\begin{cases}
J_{r,q}(u_q) \leq J_{r,q}(\overline{u}), & ||u_q - \overline{u}||_{L^\infty(\Omega)} \leq \epsilon_2 \\
J_{r,q}(u_q) \leq J_{r,q}(u) + \epsilon_2 ||u_q - u||_{L^\infty(\Omega)} & \forall u \in K.
\end{cases}
\]
Using again [5, Lemma 2], we obtain
\[
J'_{q,r}(u_q) \cdot (u - u_q) + \epsilon_2 ||u_q - u||_{L^\infty(\Omega)} \geq 0 \quad \forall u \in K,
\]

\[
J'_{q,r}(u_q) \cdot (u - u_q) + \epsilon_2 ||u_q - u||_{L^\infty(\Omega)} \geq 0 \quad \forall u \in K.
\]
with
\[ J_{\eta}(u_{\eta})(u - u_{\eta}) = \int_{\Omega} \left( \frac{\partial L}{\partial y}(x, y_{\eta}(x), u_{\eta}(x)) \right) dx, \tag{26} \]
where \( y_{\eta} \) is the state associated with \( u_{\eta} \) and \( p_{\eta} \) is the solution of
\[
\begin{align*}
A^*p_{\eta} + \phi'(y_{\eta})p_{\eta} &= \frac{\partial L}{\partial y}(\cdot, y_{\eta}, u_{\eta}) + \mu_{\eta} g'(y_{\eta}) \text{ in } \Omega, \\
p_{\eta} &= 0 \text{ on } \Gamma.
\end{align*}
\tag{27}
\]
and
\[
\mu_{\eta} = r \left( q^{-q} + \int_{\Omega} [(g(y_{\eta}(x)) - \delta)^+]^q dx \right)^{1/q - 1} \left[ (g(y_{\eta}) - \delta)^+ \right]^{q-1}.
\]
From here we deduce
\[
\|\mu_{\eta}\|_{M(\Omega)} \leq r \left( \int_{\Omega} [(g(y_{\eta}(x)))^+]^q dx \right)^{1/q - 1} \int_{\Omega} [(g(y_{\eta}(x)))^+]^{q-1} dx =
\]
\[
r \|g(y_{\eta}) - \delta\|^+ \|g(y_{\eta}) - \delta\|^+ \|L^q(\Omega)\|_{L^{q-1}(\Omega)}.
\]
Applying Hölder’s inequality with exponents \( q/(q - 1) \) and \( q \) it follows
\[
\|z\|_{L^{q-1}(\Omega)} \leq m(\Omega)^{1/q} \|z\|_{L^q(\Omega)} \quad \forall z \in L^q(\Omega),
\]
which together with the previous relation leads to
\[
\|\mu_{\eta}\|_{M(\Omega)} \leq m(\Omega)^{1/q} r \|g(y_{\eta}) - \delta\|^+ \|L^q(\Omega)\| \leq 
\]
\[
m(\Omega)^{2/q} r \|g(y_{\eta}) - \delta\|^+ \|_{\infty} \leq M < +\infty \quad \forall \eta > 1.
\]

Now we can pass to the limit as in the proof of the first part of the theorem, with the help of (24)–(27), and obtain (12)–(14) with \( \overline{\sigma} = 1. \]  

5 Finite element approximations of \((P_{\delta})\)

The aim of this section is to give a convergence result of the numerical approximations of \((P_{\delta})\). Here the convexity assumption stated in \( H1 \) is essential, however the differentiability hypothesis on \( L \) is not necessary. The presentation of this section follows the steps of [13]. Let \( \{T_{h}\}_{h>0} \) be a family of triangulations in \( \overline{\Omega} \). We associate two parameters with each element \( T \in T_h \): \( \rho(T) \) and \( \sigma(T), \rho(T) \) denotes the diameter of the set \( T \) and \( \sigma(T) \) is the diameter of the largest ball contained in \( T \) and it is supposed that \( h = \max_{T \in T_h} \rho(T) \). We make the following regularity hypotheses of the triangulation:

- There exist two positive numbers \( \rho \) and \( \sigma \) such that:

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\[
\frac{\rho(T)}{\sigma(T)} \leq \sigma \quad \text{and} \quad \frac{h}{\rho(T)} \leq \rho \quad \forall T \in \mathcal{T}_h \quad \text{and} \quad \forall h > 0.
\]

- Let us take \( \overline{\Omega}_h = \bigcup_{T \in \mathcal{T}_h} T \), \( \Omega_h \) its interior and \( \Gamma_h \) its boundary. Then we assume that \( \overline{\Omega}_h \) is convex and the vertices of \( \mathcal{T}_h \) placed on the boundary \( \Gamma_h \) are points of \( \Gamma \).

To every boundary triangle \( T \) of \( \mathcal{T}_h \) we associate another triangle \( \tilde{T} \subset \overline{\Omega} \) with two interior sides to \( \Omega \) coincident with two sides of \( T \) and the third side is the curvilinear arc of \( \Gamma \) limited by the other two sides. We denote by \( \mathcal{F}_h \) the family formed by these boundary triangles with a curvilinear side and the interior triangles to \( \Omega \) of \( \mathcal{T}_h \), so \( \overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \).

Now let us consider the spaces

\[
U_h = \{ u_h \in L^\infty(\Omega) : u_h|_T \text{ is constant } \forall T \in \mathcal{F}_h \}
\]

and

\[
V_h = \{ y_h \in C(\overline{\Omega}) : y_h|_T \in \mathcal{P}_1 \forall T \in \mathcal{T}_h \text{ and } y_h(x) = 0 \forall x \in \overline{\Omega}\setminus\Omega_h \},
\]

where \( \mathcal{P}_1 \) is the space of the polynomials of degree less than or equal to 1. It is obvious that \( V_h \subset H^1_0(\Omega) \). For each \( u_h \in U_h \) we denote by \( y_h(u_h) \) the unique element of \( V_h \) that satisfies:

\[
\sum_{i,j=1}^n \int_{\Omega_h} a_{ij}(x) \partial_{x_i} y_h(x) \partial_{x_j} z_h(x) dx + \int_{\Omega_h} a_0(x) y_h(x) z_h(x) dx + \int_{\Omega_h} \phi(y_h) z_h dx = \int_{\Omega_h} u_h z_h dx \quad \forall z_h \in V_h.
\]

For each \( h > 0 \) let \( K_h \) be a nonempty convex bounded closed subset of \( U_h \) in such a way that \( \{ K_h \}_{h>0} \) constitutes an internal approximation of \( K \) in the following sense:

1. \( K_h \subset K \) for each \( h > 0 \).
2. For every \( u \in K \) there exists a family \( \{ u_h \}_{h>0} \), with \( u_h \in K_h \), verifying that \( u_h(x) \rightarrow u(x) \) for almost every point \( x \in \Omega \).

See for example Glowinski et al. [19, Vol. 1] for these questions.

Finally we state the finite dimensional optimal control problem:

\[
\begin{cases}
\text{(P}_{\delta h}\text{)} \quad \text{minimize} \quad J_h(u_h) = \int_{\Omega_h} L(x, y_h(u_h), u_h) dx \\
\text{subject to } u_h \in K_h \text{ and } g(y_h(u_h)(x_j)) \leq \delta \forall j \in I_h,
\end{cases}
\]
where \( \{x_j\}_{j=1}^{n(h)} \) is the set of vertices of \( T_h \) and \( I_h \) is the index set corresponding to interior vertices. Note that \( y_h(u_h) \) is zero at the boundary vertices.

The following theorem states the existence of a solution for \((P_{bh})\).

**Theorem 4** For every \( \delta > \delta_0 \) there exists \( h_{\delta} > 0 \) such that \((P_{bh})\) has at least one solution \( \bar{u}_h \) for all \( h \leq h_{\delta} \).

**Proof.** Since \( K_h \) is a compact set and \( J_h \) is continuous, the existence of a solution of \((P_{bh})\) will be assured if we prove that the set of feasible controls is nonempty. Let \( u_0 \in K \) be a feasible control for \((P_{b})\) and let us take \( u_{0h} \in K_h \) such that \( u_{0h}(x) \rightarrow u_0(x) \) for almost every point \( x \in \Omega \) as \( h \rightarrow 0 \). Then \( y_h(u_{0h}) \rightarrow y_{u_0} \) uniformly in \( \bar{\Omega} \). Since \( g(y_{u_0}(x)) \leq \delta_0 \) for every \( x \in \bar{\Omega} \), from the uniform convergence and the inequality \( \delta > \delta_0 \) we deduce the existence of \( h_{\delta} > 0 \) such that the inequality \( g(y_h(u_{0h})(x)) \leq \delta \) holds for all \( x \in \bar{\Omega} \) and each \( h \leq h_{\delta} \). Therefore \( u_{0h} \) is a feasible control for \((P_{bh})\), which completes the proof.

**Remark 2** From the previous proof it follows easily that \( h_{\delta} \geq h_{\delta'} \) if \( \delta > \delta' \). On the other hand, we do not know if problems \((P_{bh})\) have feasible controls. Obviously the argument used in the proof is valid only if \( \delta > \delta_0 \).

Finally we prove the convergence result of the numerical approximation.

**Theorem 5** Let us assume that \((P_{b})\) is a weakly stable problem on the left and let \( h_{\delta} > 0 \) be as in Theorem 4. Given a family of controls \( \{u_{h,k}\}_{k<h_{\delta}}, \bar{u}_h \) being a solution of \((P_{bh})\), then there exist subsequences \( \{\bar{u}_{h,k}\}_{k \in \mathbb{N}} \), with \( h_k \rightarrow 0 \) as \( k \rightarrow \infty \), and elements \( \bar{u} \in K \) such that \( \bar{u}_{h,k} \rightarrow \bar{u} \) in the weak-* topology of \( L^\infty(\Omega) \). Each one of these limit points is a solution of \((P_{b})\). Moreover we have

\[
\lim_{h \rightarrow 0} J_h(\bar{u}_h) = \inf(P_b).
\]

**Proof.** Let \( \bar{u}_h \) be the state associated to \( u_h \). Since \( \{\bar{u}_h\}_{h<h_{\delta}} \subset K \) and \( K \) is bounded in \( L^\infty(\Omega) \), we can extract a subsequence \( \{\bar{u}_{h,k}\} \) such that \( h_k \rightarrow 0 \) and \( \bar{u}_{h,k} \rightarrow \bar{u} \) weakly-* in \( L^\infty(\Omega) \) for some element \( \bar{u} \in K \). Now we prove that \( \bar{u} \) is a solution of \((P_{b})\). Let \( \bar{y} \) be the state associated to \( \bar{u} \). Since \( \bar{y}_{h,k} \rightarrow \bar{y} \) uniformly in \( \bar{\Omega} \) and \( g(\bar{y}_{h,k}(x_j)) \leq \delta \) for every \( j \in I_h \), it follows that \( g(\bar{y}(x)) \leq \delta \) and therefore \( \bar{u} \) is a feasible control for the problem \((P_{b})\).

Let \( \delta' \in [\delta_0 + \epsilon, \delta_0] \), with \( 0 < \epsilon < \delta - \delta_0 \) fixed, and let \( u_{\delta'} \) be a solution of \((P_{\delta'})\). For each \( h \) let us take \( u_{\delta'} \in K_h \) such that \( u_{\delta'}(x) \rightarrow u_{\delta'}(x) \) for almost every point of \( \Omega \). From the uniform convergence \( y_h(u_{\delta'}(x)) \rightarrow y_{u_{\delta'}} \) and the inequality \( g(y_{u_{\delta'}}(x)) \leq \delta' < \delta \) for all \( x \in \bar{\Omega} \) we deduce the existence of \( h_{\delta'} \geq h_{\delta' + \epsilon} \) such that \( g(y_h(u_{\delta'}(x))) \leq \delta' \) for all \( x \in \bar{\Omega} \) and \( \forall h \leq h_{\delta'} \). Hence \( u_{\delta'} \) is a feasible control for \((P_{bh})\) always that \( h \leq h_{\delta'} \). Then, given \( k_0 > 0 \) such that \( h_k \leq h_{\delta' + \epsilon} \) \( \forall k \geq k_0 \),
we obtain that $J_{h_k}(\overline{u}_{h_k}) \leq J_{h_k}(u_{\delta'/h_k})$ for every $k > k_0$. Using the convexity of $L$ with respect to the third variable we get

$$J(\overline{u}) \leq \lim_{k \to \infty} J_{h_k}(\overline{u}_{h_k}) \leq \lim_{k \to \infty} J_{h_k}(u_{\delta'/h_k}) = J(u_{\delta'}) = \inf(P_{\delta'}).$$

Finally, the feasibility of $\overline{u}$ for $(P_{\delta})$ and the stability condition (4) enables us to conclude that

$$\inf(P_{\delta}) \leq J(\overline{u}) \leq \lim_{\delta' \to \delta} \inf(P_{\delta'}) = \inf(P_{\delta}),$$

which proves that $\overline{u}$ is a solution of $(P_{\delta})$. The rest of theorem is immediate. □

**Remark 3** We can imagine that nonstable problems $(P_{\delta})$ have isolated solutions which are very close to the solutions of problems $(P_{\delta'})$, with $\delta' < \delta$. Obviously these type of problems are not easy to solve numerically. To deal with these cases we must increase $\delta$ when the numerical approximations are carried out, by taking $\delta + \epsilon_h, \epsilon_h \to 0$ as $h \to 0$. In order to find out the region containing the solution of $(P_{\delta})$ we must choose $\epsilon_h$ large enough. However, if $\epsilon_h$ is too large, then the approximation $\overline{u}_h$ could be too far from the solution of $(P_{\delta})$. This equilibrium is possible, though sometimes difficult, because the set of numbers $\delta$ corresponding to nonstable problems is numerable. Anyway to enlarge $\delta$ in the approximations is not a good idea for stable problems because this procedure becomes slower the convergence.

**References**


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