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ASYMPTOTIC ANALYSIS OF CONGESTED COMMUNICATION NETWORKS

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This paper is devoted to the mathematical study of a routing problem in telecommunication networks, when the cost function is the average delay of communications. We establish asymptotic expansions for the value function and solutions in the vicinity of a congested nominal problem. The study is strongly related to the one of a partial inverse barrier method for linear programming.

1. Introduction. This paper is devoted to the mathematical study of the problem of minimizing the average delay in low-rate packet-switched networks. This problem is described in detail in the next section. For the moment it suffices to say that the equations are those of a multicommodity flow problem, while the cost function is the sum over the inverse of the residual capacities on each arc, weighted by the capacity of that arc. This is a convex problem that might have several solutions.

We consider the problem of computing the expansion of the value function and solution of this problem, taking as perturbation variables the arc capacities. We assume that the unperturbed problem is such that some arcs are congested, i.e., their residual capacities are null, and we discuss the effect of a variation of capacities that is linear with respect to a scalar parameter t, $t=0$ corresponding to the unperturbed problem. We assume that congestion does not occur for positive t.

This is a particular case of the general problem of computing the expansion of value function and solution of a nonlinear programming problem. This is a very active field, see e.g., the review by Bonnans and Shapiro (1998) and the book by Fiacco (1997). In particular, there exist formulas, based on first-order information, for the marginal value of a convex problem under fairly weak hypotheses. For computing the second-order expansion of the cost and the first-order expansion of solutions, it is necessary to use second-order information. However, most of the theory of expansions of solutions deals with the case of (locally) unique solutions, and the perturbation theory in the case of nonunique solutions, as well as the theory of second-order optimality conditions, is still in infancy. See, however, Bonnans and Ioffe (1995) and Shapiro (1988).

In addition, the main difficulty comes from the fact that the unperturbed problem is singular (due to the congestion). Therefore, we cannot use much the results of the literature; rather, we have to tailor specific estimates.

The results of this paper are obtained by studying an auxiliary problem (P_ε), in which the cost function is the sum of the perturbation parameter t, with weight ε^{-1}, and of the average delay function. Therefore, (P_ε) is a problem that arises quite naturally: minimization of some compromise between the cost of increasing capacities and improvement of the average delay. The advantage of this technical transformation is to ease comparisons

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with the literature devoted to the central path, i.e., the set of points that minimize a given barrier function. In the last section we reformulate the results so far obtained in terms of the original formulation.

The central path is one of the basic concepts of the theory of interior point algorithms (see Gonzaga 1992, Nesterov and Nemirovski 1994). A local analysis of central paths associated with general penalty functions may be found in Auslender, Cominetti, and Haddou (1997). The perturbation problem studied here is similar (although not identical) to a partial inverse barrier method (PIB) in linear programming. That is, a method in which a penalty term, proportional to the inverse of slack variables, is applied to some inequalities, but not all. The inverse barrier method was introduced in Carroll (1961) and is studied in den Hertog, Roos, and Terlaky (1994). A possible motivation of partial barrier methods is as follows. For a large scale optimization problem with a small number of coupling variables (i.e., a recourse problem), it might be of interest to apply a penalty barrier to the constraints on the coupling variables when only a small number of constraints of the decoupled problems is active (and therefore it is not efficient to deal explicitely with all of them).

The paper is organized as follows. The next section is devoted to the presentation of the problem. In §3 we obtain some preliminary estimates and prove some convergence results. Asymptotic expansions of the solutions and the cost function of the auxiliary problem (P_ε) are given in §§4 and 5. In §6 we discuss some extensions. In the concluding section, we rewrite the main results depending on the perturbation parameter.

NOTATIONS. Let $x \in \mathbb{R}^n$. The relation $x > 0$ means $x_i > 0$, $i = 1, 2, \dots, n$, while $x \geq 0$ means $x_i \geq 0$, $i = 1, 2, \dots, n$. We denote x^+ the nonnegative part of x, i.e., $x_i^+ := \max(x_i, 0)$, $i = 1, 2, \dots, n$. Let J be a subset of $I := \{1, \dots, p\}$. We denote $\bar{J} := I \setminus J$ and $x_J := \{x_i,\ i \in J\}$. By $\|.\|$ we mean the Euclidean norm $\|.\|_2$ in the space given by the context.

The notation $x(\varepsilon) = O(\varepsilon)$ (resp. $x(\varepsilon) = o(\varepsilon)$) means that there is a constant K (dependent on problem data) such that for every $\varepsilon > 0$, $\|x(\varepsilon)\| \leq K\varepsilon$ (resp. $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \|x(\varepsilon)\| = 0$). We will also denote $x(\varepsilon) \approx \varepsilon$ when $x(\varepsilon) = O(\varepsilon)$ and $\|x(\varepsilon)\|^{-1} = O(\varepsilon)$. Finally let (P) be an optimization problem. By $F(P)$, $S(P)$, and $v(P)$ we denote, respectively, the feasible set, the optimal set, and the optimal value.

2. Minimizing the average delay in low-rate packet-switched networks.

We now describe the problem of minimizing the average delay in low-rate packet-switched networks; see Kleinrock (1972), Chifflet, Mahey, and Reyner (1994) when small changes can be made in capacities. Such a network can be modeled as an oriented graph $G = (N, L)$, where N and L are the nodes and arcs sets, with capacities γ_i on each arc l_i. We denote $|N| = n$ and $|L| = p$, and associate with a demand d^k of value v^k, between a source and sink nodes o^k and s^k, a commodity k represented by a vector $x^k \in \mathbb{R}^p$. Each component x_i^k is the amount of commodity k that flows through the link l_i. The problem is then

$$(2.1)\qquad \begin{aligned} &\operatorname*{Min}_x \sum_{1 \leq i \leq p} \frac{\gamma_i}{x_i^0}, \quad \gamma_i - \sum_{1 \leq k \leq K} x_i^k = x_i^0, \quad i = 1, \dots, p, \\ &Ax^k = b^k, \quad k = 1, \dots, K, \quad x^k \geq 0, \quad k = 0, \dots, K. \end{aligned}$$

Here A is the incidence matrix of the graph G, and $b^k \in \mathbb{R}^n$ is such that

$$b_i^k = -v^k \quad \text{if } i = o^k, \quad b_i^k = v^k \quad \text{if } i = s^k, \quad b_i^k = 0 \quad \text{otherwise.}$$

The cost function represents the sum of global delays occurring for packets sent through the network. The domain of definition of this function is the set of vectors $x = x^0, \ldots, x^K$ such that each component of x^0 is positive.

We assume in this paper that the set of xs that satisfy the linear constraints is nonempty, but at the same time that the network is congested, i.e., no such vector x is such that each component of x^0 is positive. Therefore, there exists no feasible point for the above problem. It makes sense, however, to perform an asymptotic analysis with respect to small variations of the arc capacities that allow feasible points to be provided.

To obtain an expansion of the solution, we assume that the capacity on arc i is now augmented of an amount $t\omega_i$, where $t \geq 0$ is a scalar number. We suppose that congestion does not occur for positive t. The considered problems are

$$(PR_t) \quad \begin{aligned} &\operatorname*{Min}_{x} \sum_{1\leq i\leq p} \frac{\gamma_i + t\omega_i}{x_i^0}, \quad \gamma_i + t\omega_i - \sum_{1\leq k\leq K} x_i^k = x_i^0, \quad i = 1,\ldots,p, \\ &Ax^k = b^k, \quad k = 1,\ldots,K, \quad x^k \geq 0, \quad k = 0,\ldots,K, \end{aligned}$$

for any value (relatively small) of the additional variable t.

It is not easy, however, to make a sensitivity analysis of the above problem. Therefore, we prefer to perform a sensitivity analysis of a different problem, whose solutions are strongly related to those of (PR_t). So, consider the following family of problems, where $\varepsilon > 0$ is a perturbation parameter, and t is now a variable to be optimized:

$$(P_\varepsilon) \quad \begin{aligned} &\operatorname*{Min}_{x,t} \ t + \varepsilon \sum_{1\leq i\leq p} \frac{\gamma_i + t\omega_i}{x_i^0}, \quad \gamma_i + t\omega_i - \sum_{1\leq k\leq K} x_i^k = x_i^0, \quad i = 1,\ldots,p, \\ &Ax^k = b^k, \quad k = 1,\ldots,K, \quad t \geq 0, \quad x^k \geq 0, \quad k = 0,\ldots,K. \end{aligned}$$

To see that these problems are strongly linked to problems (PR_t), observe that, for a fixed ε, if $(x(\varepsilon), t(\varepsilon))$ is a solution of (P_ε), then $x(\varepsilon)$ is a solution of (PR_t) for $t = t(\varepsilon)$. Furthermore, we will establish a precise expansion for $t(\varepsilon)$ for ε close to zero (see Lemma 4.2 i); then solving (PR_t) will be "in some way" (see the conclusion section) equivalent to solving (P_ε). These problems also have their own interest because congestion and capacity investment costs appear in their objective functions.

Let us denote the cost function of the above problem as

$$f_\varepsilon(x,t) := t + \varepsilon \sum_{1\leq i\leq p} \frac{\gamma_i + t\omega_i}{x_i^0}.$$

This problem may be interpreted as a penalization of the perturbation parameter t. Observe that this problem is very close to an inverse barrier method applied to the linear program obtained by taking $\varepsilon = 0$, the barrier being applied only to the coupling constraints.

3. Preliminary estimates. In this section, we prove some preliminary convergence results for the problems (P_ε) and establish a first connection between the two parameters t and ε. Consider the linear programming problem obtained by setting $\varepsilon = 0$ in (P_ε):

$$(LP) \quad \begin{aligned} &\operatorname*{Min}_{x,t} \ t, \quad \gamma_i + t\omega_i - \sum_{1\leq k\leq K} x_i^k = x_i^0, \quad i = 1,\ldots,p, \\ &Ax^k = b^k, \quad k = 1,\ldots,K, \ t \geq 0, \quad x^k \geq 0, \quad k = 0,\ldots,K. \end{aligned}$$

We assume throughout the paper that the following assumptions are satisfied:

(i) the optimal value $v(LP)$ is equal to 0, and

(ii) the following congestion condition is satisfied:

$$\exists j \in I \text{ such that } \forall (x,0) \in S(LP), \quad x_j^0 = 0.$$

With $(x,t) \in F(LP)$ we associate the sets of active constraints

$$I^k(x,t) := \{i \in I : x_i^k = 0\}, \quad k = 0, \dots, K.$$

The sets of constraints that are active for all solutions of (LP) are denoted

$$I_0^k := \bigcap_{(x,0) \in S(LP)} I^k(x,0), \quad k = 0, \dots, K.$$

These sets determine the solution set $S(LP)$ and its relative interior by

$$S(LP) = \{(x,0) \in F(LP) : I_0^k \subset I^k(x,0), \ k = 0, \dots, K\},$$

$$\operatorname{ri} S(LP) = \{(x,0) \in F(LP) : I^k(x,0) = I_0^k, \ k = 0, \dots, K\}.$$

The family of problems (P_ε) can be interpreted as a partial inverse barrier (PIB) method for solving (LP). Indeed, the second part of the cost function f_ε implicitly implies that x^0 must be nonnegative. We can then consider it as a barrier applied to the components x^0. The results of this section are very closed to those concerning the inverse barrier method introduced in Carroll (1961) and studied in den Hertog, Roos, and Terlaky (1994). Convergence of this well-known method and some other penalty and barrier methods to a particular optimal solution is also proved in Auslender, Cominetti, and Haddou (1997).

We define the *set of partial centers* of $S(LP)$ as the set of solutions of the convex optimization problem

$$(R) \qquad \underset{(x,0) \in S(LP)}{\operatorname{Min}} \sum_{i \notin I_0^0} \frac{\gamma_i}{x_i^0}.$$

We introduce this terminology by analogy with the concepts of analytic centers (see Auslender, Cominetti, and Haddou 1997). The word "partial" comes from the fact that only some constraints are penalized.

Using the fact that $S(LP)$ is nonempty and bounded, it may be checked by standard arguments that $S(R)$ is nonempty, bounded, closed, and convex. Since $F(R) = S(LP)$, every partial center $(x,0)$ satisfies

$$x_i^k = 0 \quad \text{if } i \in I_0^k, \ k = 0, \dots, K.$$

Furthermore, since the cost function of (R) is strictly convex with respect to components x_i^0 for $i \notin I_0^0$, all partial centers have the same positive components x_i^0 for $i \notin I_0^0$.

To begin our discussion, we recall the celebrated Hoffmann's lemma (Hoffmann 1952).

LEMMA 3.1. *Let $C := \{z \in \mathbb{R}^n : Ez = e, \ Gz \le g\}$ be a nonempty set. There exists a constant $\delta > 0$ depending only on the matrices E and G such that*

$$\forall z \in \mathbb{R}^n, \quad \operatorname{dist}(z, C) \le \delta(\|(Gz - g)^+\| + \|Ex - e\|).$$

By $(x(\varepsilon), t(\varepsilon))$ we denote an arbitrary solution of (P_ε). The next lemma gives the order of magnitude of $t(\varepsilon)$.

LEMMA 3.2. *One has* $t(\varepsilon) \approx \sqrt{\varepsilon}$.

PROOF. Let $(x, 0) \in S(LP)$ and define $(y_\varepsilon, t_\varepsilon)$ by

$$y_\varepsilon^k = x^k \quad \text{for } k = 1, \ldots, K, \quad t_\varepsilon = \sqrt{\varepsilon} \quad \text{and} \quad y_\varepsilon^0 = x^0 + \sqrt{\varepsilon}\omega.$$

Obviously $(y_\varepsilon, t_\varepsilon) \in F(P_\varepsilon)$. Therefore

$$t(\varepsilon) \le v(P_\varepsilon) \le f_\varepsilon(y_\varepsilon, t_\varepsilon) = O(\sqrt{\varepsilon}). \tag{3.1}$$

On the other hand, applying Lemma 3.1 to $C = S(LP)$ and $z = (x(\varepsilon), t(\varepsilon))$, we obtain that

$$\exists \delta_1 > 0 \quad \text{such that } \operatorname{dist}((x(\varepsilon), t(\varepsilon)), S(LP)) = \delta_1 t(\varepsilon).$$

In particular, for the components $x_{I_0^0}^0(\varepsilon)$, we can conclude that

$$|x_i^0(\varepsilon)| \le \delta_1 t(\varepsilon), \quad i \in I_0^0.$$

It follows that

$$O(\sqrt{\varepsilon}) \ge v(P_\varepsilon) = f_\varepsilon(x(\varepsilon), t(\varepsilon)) \ge \varepsilon \sum_{i \in I_0^0} \frac{\gamma_i}{x_i^0(\varepsilon)} \ge \frac{|I_0^0|}{\delta_1} \frac{\varepsilon}{t(\varepsilon)},$$

and then $t(\varepsilon) \approx \sqrt{\varepsilon}$. □

LEMMA 3.3. *The solution sets* $\{S(P_\varepsilon)\}_{\varepsilon > 0}$ *are uniformly bounded for* ε *close to* 0, *and any limit-point* $(\bar{x}, \bar{t})$ *of* $\{(x(\varepsilon), t(\varepsilon))\}_{\varepsilon > 0}$ *with* $(x(\varepsilon), t(\varepsilon)) \in S(P_\varepsilon)$, *is a partial center.*

PROOF. By Lemma 3.2, $t(\varepsilon) \approx \sqrt{\varepsilon}$. In particular, for ε small enough, $t(\varepsilon) \le 1$ and

$$S(P_\varepsilon) \subset Z := \{(x, t) : 0 \le t \le 1 \text{ and } 0 \le x^k \le \gamma + \omega \quad k = 0, \ldots, K\}.$$

As Z is bounded, uniform boundedness of $S(P_\varepsilon)$ follows.

Let $(\bar{x}, \bar{t})$ be a limit-point of $\{(x(\varepsilon), t(\varepsilon))\}_{\varepsilon > 0}$. As $F(LP)$ is closed and $t(\varepsilon) \approx \sqrt{\varepsilon}$, $\bar{t} = 0$ and $(\bar{x}, \bar{t}) = (\bar{x}, 0) \in S(LP)$. Now let $\varepsilon_k \downarrow 0$ be such that there exists $\{(x(\varepsilon_k), t(\varepsilon_k))\} \in S(P_{\varepsilon_k})$ converging to $(\bar{x}, 0)$. Let $(\hat{x}, 0) \in \operatorname{ri} S(LP)$. Then

$$(\hat{x}, 0) = \lim_{k \to +\infty} (\hat{x}(\varepsilon_k), t(\varepsilon_k)) \quad \text{with } \hat{x}(\varepsilon_k) := x(\varepsilon_k) + \hat{x} - \bar{x}.$$

As $(\hat{x}, 0) \in \operatorname{ri} S(LP)$, we have $x_i^0(\varepsilon_k) = \hat{x}_i^0(\varepsilon_k)$ whenever $i \in I_0^0$ and $(\hat{x}(\varepsilon_k), \hat{t}(\varepsilon_k)) \in F(LP)$ for ε_k small enough.

From $f_{\varepsilon_k}(x(\varepsilon_k), t(\varepsilon_k)) \le f_{\varepsilon_k}(\hat{x}(\varepsilon_k), t(\varepsilon_k))$, we deduce that

$$\sum_{i \notin I_0^0} \frac{\gamma_i + t(\varepsilon_k)\omega_i}{x_i^0(\varepsilon_k)} \le \sum_{i \notin I_0^0} \frac{\gamma_i + t(\varepsilon_k)\omega_i}{\hat{x}_i^0(\varepsilon_k)}. \tag{3.2}$$

Passing to the limit in (3.2), we obtain that $\bar{x}_i^0 > 0$ whenever $i \notin I_0^0$ and

$$\sum_{i \notin I_0^0} \frac{\gamma_i}{\bar{x}_i^0} \le \sum_{i \notin I_0^0} \frac{\gamma_i}{\hat{x}_i^0}.$$

This proves that $(\bar{x}, 0)$ is a partial center. □

4. Asymptotic expansion of the solutions. Let $(x^*, 0)$ be a fixed partial center. For each $(x(\varepsilon), t(\varepsilon)) \in S(P_\varepsilon)$, we define $(\tau(\varepsilon), d(\varepsilon))$ by

$$\tau(\varepsilon) := \varepsilon^{-1/2} t(\varepsilon) \quad \text{and} \quad d(\varepsilon) := \varepsilon^{-1/2}(x(\varepsilon) - x^*).$$

Our purpose in this section is to prove that $\{(\tau(\varepsilon), d^0(\varepsilon))\}$ converges. We will consider the other components of $\{d(\varepsilon)\}$ in the last section.

The first result will concern $(\tau(\varepsilon), d^0_{I_0^0}(\varepsilon))$ corresponding to the *principal part* in the cost function. Indeed, by Lemmas 3.2 and 3.3

$$(4.1) \qquad \frac{f_\varepsilon(x(\varepsilon), t(\varepsilon))}{\sqrt{\varepsilon}} = \frac{t}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} \sum_{i \in I} \frac{\gamma_i + t(\varepsilon)\omega_i}{x_i^0(\varepsilon)} = \tau(\varepsilon) + \sum_{i \in I_0^0} \frac{\gamma_i}{d_i^0(\varepsilon)} + O(\sqrt{\varepsilon}).$$

Formula (4.1) suggests to consider the convex optimization problem

$$(U) \qquad \begin{cases} \operatorname*{Min}_{d,\tau} & \tau + \sum_{i \in I_0^0} \frac{\gamma_i}{d_i^0}, \\ \text{s.t.} & Ad^k = 0, \qquad k = 1, \dots, K, \\ & \tau\omega_i - \sum_{1 \le k \le K} d_i^k = d_i^0, \quad i = 1, \dots, p, \\ & d^k_{I^k(x^*,0)} \ge 0, \qquad k = 0, \dots, K, \\ & d^0_{I_0^0} \ge 0, \ \tau \ge 0. \end{cases}$$

Here again, the domain of definition of the cost function is those (d, τ) such that each component of $d^0_{I_0^0}$ is positive.

Note that for each $\varepsilon > 0$, $(d(\varepsilon), \tau(\varepsilon))$ is feasible for (U). Furthermore, the cost function of (U) is strictly convex with respect to $d^0_{I_0^0}$. Hence, $d^0_{I_0^0}$ and consequently τ are constant over $S(U)$ whenever $S(U)$ is nonempty. The next lemma describes two essential properties of problem (U) that will be useful for proving the expansion concerning the principal part.

LEMMA 4.1. *The feasible set $F(U)$ is a convex cone and, whenever $(\hat{d}, \hat{\tau}) \in S(U)$, we have*

$$\hat{\tau} = \sum_{i \in I_0^0} \frac{\gamma_i}{\hat{d}_i^0}.$$

PROOF. The first property concering $F(U)$ is obvious. Let $(\hat{d}, \hat{\tau}) \in S(U)$ and $\lambda > 0$. Then $(\lambda\hat{d}, \lambda\hat{\tau})$ belongs to $F(U)$. Consider the auxiliary problem

$$\operatorname*{Min}_{\lambda > 0} \lambda\hat{\tau} + \sum_{i \in I_0^0} \frac{\gamma_i}{\lambda\hat{d}_i^0}.$$

This problem is strictly convex and its optimal value is $v(U)$, while its unique solution is $\lambda = 1$. The optimality condition for $\lambda = 1$ achieves the proof. □

The next lemma gives the expansions of $t(\varepsilon)$ and $x^0_{I_0^0}(\varepsilon)$ for ε close to 0.

LEMMA 4.2. *The optimal set $S(U)$ is nonempty. Let $(d^*, \tau^*) \in S(U)$, then*
(i) $t(\varepsilon) = \tau^*(\varepsilon) + o(\varepsilon)$,
(ii) $x^0_{I_0^0}(\varepsilon) = \sqrt{\varepsilon}(d^*)^0_{I_0^0} + O(\varepsilon)$.

PROOF. By Lemma 3.2, $\tau(\varepsilon)$ is bounded. Applying Lemma 3.1 to the linear constraints defining $F(LP)$ and those defining $S(LP)$ implies that $d^0_{I^0_0}(\varepsilon)$ is also bounded. Let $(\tilde{\tau}, \tilde{d}^{\,0}_{I^0_0})$ be a limit-point of $\{(\tau(\varepsilon), d^0_{I^0_0}(\varepsilon))\}$.

For each $\varepsilon > 0$ define $F_\varepsilon := \{(\tau, d) \in F(U) : (\tau, d)^0_{I^0_0} = (\tau(\varepsilon), d^0_{I^0_0}(\varepsilon))\}$. Applying Theorem 4.1 in Bonnans and Launay (1991) to F_ε, we can construct a sequence $(\tau_b(\varepsilon), d_b(\varepsilon)) \in F_\varepsilon$ satisfying $\|(\tau_b(\varepsilon), d_b(\varepsilon))\| \le \|(\tau(\varepsilon) - \tilde{\tau}, d(\varepsilon) - \tilde{d}^{\,0}_{I^0_0})\| \to 0$. This new sequence is bounded and passing to the limit, we can prove that all its limit points belong to $F(U)$ and verify $(\bar{\tau}, \bar{d}^0_{I^0_0}) = (\tilde{\tau}, \tilde{d}^0_{I^0_0})$. For each (d, τ) in $F(U)$, define

$$(\bar{x}(\varepsilon), \bar{t}(\varepsilon)) := (x^*, 0) + \sqrt{\varepsilon}(d, \tau).$$

As this vector belongs to $F(P_\varepsilon)$, we have

$$t(\varepsilon) + \varepsilon \sum_{i \in I} \frac{\gamma_i + t(\varepsilon)\omega_i}{x^0_i(\varepsilon)} \le \bar{t}(\varepsilon) + \varepsilon \sum_{i \in I} \frac{\gamma_i + \bar{t}(\varepsilon)\omega_i}{\bar{x}^0_i(\varepsilon)}.$$

Dividing by $\sqrt{\varepsilon}$, we obtain

$$\tau(\varepsilon) - \tau + \sum_{i \in I^0_0} \left[\frac{\gamma_i}{d^0_i(\varepsilon)} - \frac{\gamma_i}{d^0_i} \right]$$

$$\le \sqrt{\varepsilon} \sum_{i \in I^0_0} \left[\frac{\tau\omega_i}{d^0_i} - \frac{\tau(\varepsilon)\omega_i}{d^0_i(\varepsilon)} \right] + \sqrt{\varepsilon} \sum_{i \notin I^0_0} \left[\frac{\gamma_i + \bar{t}(\varepsilon)\omega_i}{(x^*)^0_i + \sqrt{\varepsilon} d^0_i} - \frac{\gamma_i + t(\varepsilon)\omega_i}{(x^*)^0_i + \sqrt{\varepsilon} d^0_i(\varepsilon)} \right],$$

and then

$$\tau(\varepsilon) - \tau + \sum_{i \in I^0_0} \left[\frac{\gamma_i}{d^0_i(\varepsilon)} - \frac{\gamma_i}{d^0_i} \right] \le \sqrt{\varepsilon} \sum_{i \in I^0_0} \left[\frac{\tau\omega_i}{d^0_i} - \frac{\tau(\varepsilon)\omega_i}{d^0_i(\varepsilon)} \right] + o(\sqrt{\varepsilon}). \tag{4.2}$$

This inequality proves that $\bar{d}^0_{I^0_0} > 0$ and passing to the limit, we obtain that

$$\bar{\tau} + \sum_{i \in I^0_0} \frac{\gamma_i}{\bar{d}^0_i} \le \tau + \sum_{i \in I^0_0} \frac{\gamma_i}{d^0_i}.$$

Therefore $(\bar{d}, \bar{\tau}) \in S(U)$ so that $S(U) \ne \emptyset$, and $(\bar{d}^0_{I^0_0}, \bar{\tau}) = ((d^*)^0_{I^0_0}, \tau^*)$.

i. For $\varepsilon > 0$, using the definition of $S(P_\varepsilon)$, we have

$$(d(\varepsilon), \tau(\varepsilon)) \in \underset{(d,\tau) \in F(U)}{\operatorname{argmin}} \ \tau + \sum_{i \in I^0_0} \frac{\gamma_i + \sqrt{\varepsilon}\tau\omega_i}{d^0_i} + \sqrt{\varepsilon} \sum_{i \notin I^0_0} \frac{\gamma_i + \sqrt{\varepsilon}\tau\omega_i}{(x^*)^0_i + \sqrt{\varepsilon} d^0_i}. \tag{4.3}$$

Since $\lambda(d(\varepsilon), \tau(\varepsilon)) \in F(U)$ for each $\lambda > 0$, we have

$$1 \in \underset{\lambda > 0}{\operatorname{argmin}} \ \lambda\tau(\varepsilon) + \sum_{i \in I^0_0} \frac{\gamma_i + \sqrt{\varepsilon}\lambda\tau(\varepsilon)\omega_i}{\lambda d^0_i(\varepsilon)} + \sqrt{\varepsilon} \sum_{i \notin I^0_0} \frac{\gamma_i + \sqrt{\varepsilon}\lambda\tau(\varepsilon)\omega_i}{(x^*)^0_i + \sqrt{\varepsilon}\lambda d^0_i(\varepsilon)}.$$

The first-order optimality condition implies after simplification that

$$\tau(\varepsilon) - \sum_{i \in I^0_0} \frac{\gamma_i}{d^0_i(\varepsilon)} + \sqrt{\varepsilon} \sum_{i \notin I^0_0} \frac{\sqrt{\varepsilon}\tau(\varepsilon)\omega_i(x^*)^0_i - \sqrt{\varepsilon}\gamma_i d^0_i(\varepsilon)}{[(x^*)^0_i + \sqrt{\varepsilon} d^0_i(\varepsilon)]^2} = 0. \tag{4.4}$$

As $(\sqrt{\varepsilon}d^0(\varepsilon), \sqrt{\varepsilon}\tau(\varepsilon))$ converges to 0 by Lemma 3.2 and Lemma 3.3, we get

$$\tau(\varepsilon) = \sum_{i\in I_0^0} \frac{\gamma_i}{d_i^0(\varepsilon)} + o(\sqrt{\varepsilon}). \tag{4.5}$$

Since $(d(\varepsilon), \tau(\varepsilon)) \in F(U)$, applying (4.2) for $(d,\tau) = (d^*, \tau^*)$ and using the fact that $(d^*, \tau^*) \in S(U)$, we have

$$0 \le \tau(\varepsilon) - \tau^* + \sum_{i\in I_0^0} \left[\frac{\gamma_i}{d_i^0(\varepsilon)} - \frac{\gamma_i}{(d^*)_i^0}\right] \le \sqrt{\varepsilon} \sum_{i\in I_0^0} \left[\frac{\tau^*\omega_i}{(d^*)_i^0} - \frac{\tau(\varepsilon)\omega_i}{d_i^0(\varepsilon)}\right] + o(\sqrt{\varepsilon}).$$

Then, the convergence of $(d_{I_0^0}^0(\varepsilon), \tau(\varepsilon))$ to $((d^*)_{I_0^0}^0, \tau^*)$ implies that

$$0 \le \tau(\varepsilon) - \tau^* + \sum_{i\in I_0^0} \left[\frac{\gamma_i}{d_i^0(\varepsilon)} - \frac{\gamma_i}{(d^*)_i^0}\right] \le o(\sqrt{\varepsilon}).$$

By Lemma 4.1 and (4.5), we conclude that

$$0 \le 2\tau(\varepsilon) - 2\tau^* + o(\sqrt{\varepsilon}) \le o(\sqrt{\varepsilon}).$$

Point (i) follows.

ii. We can write $S(U)$ as follows

$$S(U) = \{(d,\tau) \in F(U), \quad d_{I_0^0}^0 = (d^*)_{I_0^0}^0 \quad \text{and} \quad \tau = \tau^*\}.$$

We know that $(d(\varepsilon), \tau(\varepsilon)) \in F(U)$ and $(d_{I_0^0}^0(\varepsilon), \tau(\varepsilon))$ converges to $((d^*)_{I_0^0}^0, \tau^*)$. This implies that $x_{I_0^0}^0 = (d^*)_{I_0^0}^0 \sqrt{\varepsilon} + o(\sqrt{\varepsilon})$. We have to show that the remainder term is of order $O(\varepsilon)$. In other words, we have to prove that

$$\|(\tilde{d}^*)_{I_0^0}^0 - d_{I_0^0}^0(\varepsilon)\| \le = O(\sqrt{\varepsilon}). \tag{4.6}$$

By Lemma 3.1, for each $\varepsilon > 0$ there exists $(\tilde{d_\varepsilon}, \tau^*) \in S(U)$ such that

$$\|\tilde{d_\varepsilon} - d(\varepsilon)\| \le \delta(\|(d^*)_{I_0^0}^0 - d_{I_0^0}^0(\varepsilon)\| + |\tau^* - \tau(\varepsilon))|) = o(1), \tag{4.7}$$

where δ is a positive constant independent of ε. Applying (4.2) with $(d,\tau) = (\tilde{d_\varepsilon}, \tau^*)$, we have

$$\tau(\varepsilon) - \tau^* + \sum_{i\in I_0^0} \left[\frac{\gamma_i}{d_i^0(\varepsilon)} - \frac{\gamma_i}{(d^*)_i^0}\right] \tag{4.8}$$

$$\le \sqrt{\varepsilon} \left(\sum_{i\in I_0^0} \left[\frac{\tau^*\omega_i}{(d^*)_i^0} - \frac{\tau(\varepsilon)\omega_i}{d_i^0(\varepsilon)}\right] + \sum_{i\notin I_0^0} \left[\frac{\gamma_i + \sqrt{\varepsilon}\tau^*\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon}(\tilde{d_\varepsilon})_i^0} - \frac{\gamma_i + \sqrt{\varepsilon}\tau(\varepsilon)\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon}d_i^0(\varepsilon)}\right] \right).$$

Since $(d(\varepsilon), \tau(\varepsilon) \in F(U)$ and $(d^*, \tau^*) \in S(U)$ and since the cost function of (U) is strongly convex with respect to $d_{I_0^0}^0$, there exists $\beta > 0$ such that

$$\tau(\varepsilon) - \tau^* + \sum_{i\in I_0^0} \left[\frac{\gamma_i}{d_i^0(\varepsilon)} - \frac{\gamma_i}{(d^*)_i^0}\right] \ge \beta\|(d^*)_{I_0^0}^0 - d_{I_0^0}^0(\varepsilon)\|^2. \tag{4.9}$$

To obtain the order of magnitude $d_{I_0^0}^0(\varepsilon) - (d^*)_{I_0^0}^0$, we will give an upper estimate of the right-hand side of (4.8), which in turn will give an upper estimate of the left-hand side of (4.9), depending only on $d_{I_0^0}^0(\varepsilon) - (d^*)_{I_0^0}^0$ and ε.

We start with the second term in the right-hand side of (4.8). Using (i) and (4.7), for $i \notin I_0^0$, we have

$$\frac{\gamma_i + \sqrt{\varepsilon}\tau^*\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon}(\tilde{d_\varepsilon})_i^0} - \frac{\gamma_i + \sqrt{\varepsilon}\tau(\varepsilon)\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon}d_i^0(\varepsilon)} = \frac{\sqrt{\varepsilon}\tau^*\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon}(\tilde{d_\varepsilon})_i^0} - \frac{\sqrt{\varepsilon}\tau(\varepsilon)\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon}d_i^0(\varepsilon)}$$
$$+ \frac{\gamma_i\sqrt{\varepsilon}(d_i^0(\varepsilon) - (\tilde{d_\varepsilon})_i^0)}{((x^*)_i^0 + \sqrt{\varepsilon}(\tilde{d_\varepsilon})_i^0))((x^*)_i^0 + \sqrt{\varepsilon}d_i^0(\varepsilon))}$$
$$= o(1).$$

Furthermore, using (ii) and since $d_{I_0^0}^0(\varepsilon)$ converges to $(d^*)_{I_0^0}^0 > 0$, we have

$$\frac{\tau^*\omega_i}{(d^*)_i^0} - \frac{\tau(\varepsilon)\omega_i}{d_i^0(\varepsilon)} = \tau^*\omega_i\left[\frac{1}{(d^*)_i^0} - \frac{1}{d_i^0(\varepsilon)}\right] + \frac{o(\sqrt{\varepsilon})}{d_i^0(\varepsilon)} \tag{4.10}$$
$$\leq \tau^*\omega_i\frac{\|(d^*)_{I_0^0}^0 - d_{I_0^0}^0(\varepsilon)\|}{(d^*)_i^0 d_i^0(\varepsilon)} + o(\sqrt{\varepsilon}).$$

Then the inequality (4.8) implies that

$$\tau(\varepsilon) - \tau^* + \sum_{i\in I_0^0}\left[\frac{\gamma_i}{d_i^0(\varepsilon)} - \frac{\gamma_i}{(d^*)_i^0}\right] \leq O(\sqrt{\varepsilon})\|(d^*)_{I_0^0}^0 - d_{I_0^0}^0(\varepsilon)\| + o(\varepsilon). \tag{4.11}$$

The proof of (4.6) is then complete using (4.11) and (4.9). □

The second part of this section is devoted to the analysis of the other components of $\{d^0(\varepsilon)\}_{\varepsilon>0}$. In this study and also when computing the expansion of the cost function, we will meet the following linear and quadratic, respectively, optimization problems:

$$(T) \qquad \operatorname*{Min}_d\left\{-\sum_{i\notin I_0^0}\frac{\gamma_i d_i^0}{(x^{*0}_i)^2},\ (d,\tau^*)\in S(U)\right\},$$

$$(T_1) \qquad \operatorname*{Min}_d\left\{2\sum_{i\notin I_0^0}\frac{\gamma_i(d_i^0)^2}{(x^{*0}_i)^3} - \sum_{i\notin I_0^0}\frac{\tau^*\omega_i d_i^0}{(x^{*0}_i)^2},\ d\in S(T)\right\}.$$

By Lemma 4.2, $t(\varepsilon) = \sqrt{\varepsilon} + o(\varepsilon)$ and $x(\varepsilon)$ is a solution of the following optimization problem in which $t(\varepsilon)$ is viewed as a given data:

$$(\tilde{P}_\varepsilon) \qquad \begin{cases} \operatorname*{Min}_x & \displaystyle\sum_{i\notin I_0^0}\frac{\gamma_i + t(\varepsilon)\omega_i}{x_i^0}, \\ \text{s.t.} & Ax^k = b^k, \qquad k = 1,\dots,K, \\ & \gamma_i + t(\varepsilon)\omega_i - \displaystyle\sum_{1\leq k\leq K} x_i^k = x_i^0, \quad i = 1,\dots,p, \\ & x^k \geq 0, \qquad k = 0,\dots,K, \\ & x_{I_0^0}^0 = x_{I_0^0}^0(\varepsilon) = \sqrt{\varepsilon}d_{I_0^0}^* + O(\varepsilon). \end{cases}$$

REMARK. The above problem can be interpreted as a perturbation of the limit problem

$$(\tilde{P}) \qquad \begin{cases} \operatorname*{Min}_x & \sum_{i\notin I_0^0} \frac{\gamma_i}{x_i}, \\ \text{s.t.} & Ax^k = b^k, \qquad k=1,\dots,K, \\ & \gamma_i - \sum_{1\le k\le K} x_i^k = x_i^0, \quad i=1,\dots,p, \\ & x^k \ge 0, \qquad k=0,\dots,K, \\ & x_{I_0^0}^0 = 0, \end{cases}$$

that satisfies $S(\tilde{P}) \times \{0\} = S(R)$.

LEMMA 4.3. *The sequence* $\{d^0(\varepsilon)\}_{\varepsilon>0}$ *is bounded.*

PROOF. We first prove that $S(T) \neq 0$. Since (T) is a feasible linear program, we only need to prove that $v(T) > -\infty$. Suppose the contrary; then there exists a feasible descent direction y satisfying:

$$-\sum_{i\notin I_0^0} \frac{y_i^0}{(x^{*0}_i)^2} < 0, \quad y_{I_0^0}^0 = 0, \quad y_{I^k(x^*,0)}^k \ge 0, \quad k=0,\dots,K,$$

$$Ay^k = 0, \quad k=1,\dots,K, \qquad -\sum_{1\le k\le K} y_i^k = y_i^0, \quad i=1,\dots,p.$$

Then, for $\mu > 0$ small enough, we have

$$x^\mu := x^* + \mu y \in F(\tilde{P}) \quad \text{and} \quad \sum_{i\notin I_0^0} \frac{\gamma_i}{(x^\mu)_i^0} < \sum_{i\notin I_0^0} \frac{\gamma_i}{(x^*)_i^0}.$$

This contradicts the optimality of x^* for $(\tilde{P})$. This proves that $S(T) \neq \emptyset$. To prove that $\{d^0(\varepsilon)\}$ is bounded, we will give two estimates of $v(\tilde{P}_\varepsilon)$ depending on ε and $\|(x^0/I_0^0)(\varepsilon) - (x^*)(0/I_0^0)\|$. These two estimates will then imply that $\|x(0/I_0^0)(\varepsilon) - (x^*)(0/I_0^0)\| = O(\sqrt{\varepsilon})$.

Let $d \in S(T)$. Since $t(\varepsilon) = \sqrt{\varepsilon}\tau^* + o(\varepsilon)$, Lemma 3.1 applied to the set $F(\tilde{P}_\varepsilon)$ ensures that

$$\operatorname{dist}(x^* + \sqrt{\varepsilon} d, F(\tilde{P}_\varepsilon)) = O(\varepsilon).$$

So there exists a vector $\tilde{x}(\varepsilon) := x^* + \sqrt{\varepsilon} d + O(\varepsilon)$ belonging to $F(\tilde{P}_\varepsilon)$. Therefore, an upper bound of $v(\tilde{P}_\varepsilon)$ is

$$(4.12) \qquad v(\tilde{P}_\varepsilon) = \sum_{i\notin I_0^0} \frac{\gamma_i + t(\varepsilon)\omega_i}{x_i^0(\varepsilon)} \le \sum_{i\notin I_0^0} \frac{\gamma_i + t(\varepsilon)\omega_i}{(x^*)_i^0 + \sqrt{\varepsilon} d_i^0 + O(\varepsilon)},$$

$$\le \sqrt{\varepsilon} v(T) + \sum_{i\notin I_0^0} \left(\frac{\gamma_i}{(x^*)_i^0} + \sqrt{\varepsilon} \frac{\tau^* \omega_i}{(x^*)_i^0} \right) + O(\varepsilon).$$

We now compute a lower bound of $v(\tilde{P}_\varepsilon)$. By Lemma 4.2,

$$(d(\varepsilon), \tau(\varepsilon)) \in F(U), \quad \tau(\varepsilon) = \tau^* + o(\sqrt{\varepsilon}) \quad \text{and} \quad d_{I_0^0}^0(\varepsilon) = (d^*)_{I_0^0}^0 + O(\sqrt{\varepsilon}).$$

Applying again Lemma 3.1, there exists $(\bar{d}(\varepsilon), \tau^*) \in S(U)$ such that

$$\bar{d}(\varepsilon) - d(\varepsilon) = O(\sqrt{\varepsilon}). \tag{4.13}$$

For $i \notin I_0^0$ and ε small enough, since $x_i^0(\varepsilon)$ converges to $(x^*)_i^0 > 0$, we have

$$\frac{t(\varepsilon)\omega_i}{x_i^0(\varepsilon)} = \frac{\sqrt{\varepsilon}\tau^*\omega_i}{x_i^0(\varepsilon)} + o(\varepsilon),$$

and

$$\frac{\gamma_i}{x_i^0(\varepsilon)} = \frac{\gamma_i}{(x^*)_i^0} - \frac{\gamma_i\sqrt{\varepsilon}d_i^0(\varepsilon)}{((x^*)_i^0)^2} + \frac{2\gamma_i[x_i^0(\varepsilon) - (x^*)_i^0]^2}{((x^*)_i^0)^3} + o(|x_i^0(\varepsilon) - (x^*)_i^0|^2)$$

$$\geq \frac{\gamma_i}{(x^*)_i^0} - \frac{\gamma_i\sqrt{\varepsilon}d_i^0(\varepsilon)}{((x^*)_i^0)^2} + \frac{\gamma_i[x_i^0(\varepsilon) - (x^*)_i^0]^2}{((x^*)_i^0)^3}.$$

Then,

$$v(\tilde{P}_\varepsilon) \geq \sum_{i \notin I_0^0} \left(\frac{\gamma_i}{(x^*)_i^0} + \sqrt{\varepsilon}\left[\frac{-\gamma_i d_i^0(\varepsilon)}{((x^*)_i^0)^2} + \frac{\tau^*\omega_i}{x_i^0(\varepsilon)} \right] + \frac{\gamma_i[x_i^0(\varepsilon) - (x^*)_i^0]^2}{((x^*)_i^0)^3} \right) + o(\varepsilon),$$

and by (4.13), a lower bound for $v(\tilde{P}_\varepsilon)$ is

$$v(\tilde{P}_\varepsilon) \geq \sqrt{\varepsilon}v(T) + \sum_{i \notin I_0^0} \left(\frac{\gamma_i}{(x^*)_i^0} + \frac{\gamma_i[x_i^0(\varepsilon) - (x^*)_i^0]^2}{((x^*)_i^0)^3} + \sqrt{\varepsilon}\frac{\tau^*\omega_i}{x_i^0(\varepsilon)} \right) + O(\varepsilon). \tag{4.14}$$

Thus (4.12) and (4.14) yield to

$$\sum_{i \notin I_0^0} \frac{\gamma_i(x_i^0(\varepsilon) - (x^*)_i^0)^2}{((x^*)_i^0)^3} \leq \sqrt{\varepsilon} \sum_{i \notin I_0^0} \left(\frac{\tau^*\omega_i}{(x^*)_i^0} - \frac{\tau^*\omega_i}{x_i^0(\varepsilon)} \right) + O(\varepsilon). \tag{4.15}$$

For $i \notin I_0^0, (x^*)_i^0 > 0$ and then there exist two constants $\alpha > 0$ and $\beta \geq 0$ independent of ε such that

$$\alpha\|x_{\overline{I_0^0}}^0(\varepsilon) - (x^*)_{\overline{I_0^0}}^0\|^2 \leq \beta\sqrt{\varepsilon}\|x_{\overline{I_0^0}}^0(\varepsilon) - (x^*)_{\overline{I_0^0}}^0\| + O(\varepsilon), \tag{4.16}$$

completing the proof. □

Lemma 4.4. *The sequence $\{d^0(\varepsilon)\}_{\varepsilon>0}$ is convergent.*

Proof. By Lemma 4.3, $d^0(\varepsilon)$ has at least a limit point. Let $(\bar{d}, \tau^*) \in F(U)$ be such that $\bar{d}^0$ is a limit point of $d^0(\varepsilon)$. Such a vector $\bar{d}$ exists by applying Lemma 3.1 to $\overline{F(U)}$. By Lemma 4.2, $(\bar{d}, \tau^*) \in S(U)$. If $S(U)$ is reduced to a singleton the conclusion follows. Otherwise, let $(\tilde{d}, \tau^*) \in S(U)$ be such that $\tilde{d} \neq \bar{d}$. Consider a sequence $(d^0(\varepsilon_k), \tau(\varepsilon_k))$ converging to $(\bar{d}^0, \tau^*)$. Set

$$\tilde{d}(\varepsilon_k) = \tilde{d} - \bar{d} + d(\varepsilon_k).$$

We have $\tilde{d}_{I_0^0}^0(\varepsilon_k) = d_{I_0^0}^0(\varepsilon_k)$ and $\lim_{\varepsilon_k \downarrow 0}(\tilde{d}(\varepsilon_k), \tau(\varepsilon_k)) = (\tilde{d}, \tau^*)$.

Choosing $(d, \tau) = (\tilde{d}(\varepsilon_k), \tau(\varepsilon_k))$ in (4.2), we have

$$\sum_{i \notin I_0^{*0}} \frac{\gamma_i + t(\varepsilon_k)\omega_i}{x^{*0}_i + \sqrt{\varepsilon_k}d_i^0(\varepsilon_k)} \leq \sum_{i \notin I_0^0} \frac{\gamma_i + t(\varepsilon_k)\omega_i}{x^{*0}_i + \sqrt{\varepsilon_k}\tilde{d}_i^0(\varepsilon_k)} \tag{4.17}$$

and then

$$\sum_{i\notin I_0^0}\left[-\frac{\gamma_i\sqrt{\varepsilon_k}d_i^0(\varepsilon_k)}{(x^{*0}_i)^2}+\frac{2\gamma_i\varepsilon_k(d_i^0(\varepsilon_k))^2}{(x^{*0}_i)^3}+\frac{\sqrt{\varepsilon_k}t(\varepsilon_k)\omega_i[\tilde{d}_i^0-\bar{d}_i^0]}{(x^{*0}_i)^2}\right]$$

$$\leq\sum_{i\notin I_0^0}\gamma_i\left[-\frac{\sqrt{\varepsilon_k}\tilde{d}_i^0(\varepsilon_k)}{(x^{*0}_i)^2}+\frac{2\varepsilon_k(\tilde{d}_i^0(\varepsilon_k))^2}{(x^{*0}_i)^3}\right]+o(\varepsilon_k).$$

We obtain, after a simple computation,

$$\sum_{i\notin I_0^0}\left[2\gamma_i\frac{(d_i^0(\varepsilon_k))^2-(\tilde{d}_i^0(\varepsilon_k))^2}{(x^{*0}_i)^3}+\frac{\tau^*\omega_i[\tilde{d}_i^0-\bar{d}_i^0]}{(x^{*0}_i)^2}\right]\leq\sum_{i\notin I_0^0}\frac{\gamma_i(\bar{d}_i^0-\tilde{d}_i^0)}{\sqrt{\varepsilon_k}(x^{*0}_i)^2}+o(1). \tag{4.18}$$

Since the left-hand side is bounded and $\lim_{k\to+\infty}\frac{1}{\sqrt{\varepsilon_k}}=+\infty$, it follows that

$$\sum_{i\notin I_0^0}\frac{\gamma_i[\bar{d}_i-\tilde{d}_i]}{(x^{*0}_i)^2}\geq 0,\quad\text{for all }(\tilde{d},\tau^*)\in S(U).$$

This inequality proves that $\bar{d}$ belongs to $S(T)$. Choosing $\tilde{d}\in S(T)$ in (4.18) and passing to the limit, we prove that $\bar{d}$ is also a solution of T_1. The proof is then complete since d^0 is constant over $S(T_1)$. □

Finally we can state a theorem that summarizes all results of this section:

THEOREM 4.5. *Every path* $\{(x(\varepsilon),t(\varepsilon))\}$ *such that* $x(\varepsilon),t(\varepsilon))\in S(P_\varepsilon)$ *satisfies*

$$t(\varepsilon)=\tau^*\sqrt{\varepsilon}+o(\varepsilon),$$

$$x^0_{I_0^0}(\varepsilon)=\sqrt{\varepsilon}(d^*)^0_{I_0^0}+O(\varepsilon),$$

$$x^0_{\bar{I}_0^0}(\varepsilon)=(x^*)^0_{\bar{I}_0^0}+\sqrt{\varepsilon}(d^*)^0_{\bar{I}_0^0}+o(\sqrt{\varepsilon}),$$

where $(x^*,0)\in S(R)$, $(d^*,\tau^*)\in S(U)$ *and* $d^*\in S(T_1)$.

5. Asymptotic expansion of the cost function. In this section we study the cost function $v(\cdot):\varepsilon\to v(P_\varepsilon)$ and give an asymptotic expansion for ε close to 0. The method that we follow is reminiscent of asymptotic techniques in Torralba (1996) based on epi-convergence of functions theory; i.e. Attouch (1984). Here is the main result of this section.

THEOREM 5.1. *The function* $v(\cdot)$ *satisfies*

$$v(P_\varepsilon)=\sqrt{\varepsilon}v(U)+\varepsilon\left(v(R)+\sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0}\right)+o(\varepsilon),$$

where $(\tau^*,d^*)\in S(U)$.

PROOF. Let α be a nonnegative real number that we will choose next. Set

$$\tilde{f}_\varepsilon(x,t):=\begin{cases}\dfrac{f_\varepsilon(x,t)-\alpha\sqrt{\varepsilon}}{\varepsilon} & \text{if }(x,t)\in F(LP)\\ +\infty & \text{else.}\end{cases} \tag{5.1}$$

This rescaling is motivated by the expansion of the solutions. Indeed, the obtained expansion suggests that a part of the cost function is of order $\sqrt{\varepsilon}$. More about similar rescaling techniques in optimization problems can be found in Torralba (1996). Observe first that solving (P_ε) is equivalent to solving the scaled problem $\mathrm{Min}_{x,t}\, \tilde{f}_\varepsilon(x,t)$.

Let $\{(x(\varepsilon_k), t(\varepsilon_k))\}$, with $((x(\varepsilon_k), t(\varepsilon_k)) \in S(P_{\varepsilon_k}))$, be a sequence converging to a partial center $(\tilde{x}, 0)$. Such a sequence exists by Lemma 3.3. Consider another sequence $\{(\bar{x}_{\varepsilon_k}, \bar{t}_{\varepsilon_k})\}$ converging to $(\tilde{x}, 0)$ and defined by

$$\bar{x}_{\varepsilon_k} = \tilde{x} + \sqrt{\varepsilon_k} d^* \quad \text{and} \quad \bar{t}_{\varepsilon_k} = \tau^* \sqrt{\varepsilon_k},$$

where (τ^*, d^*) is a fixed solution of (U). For each ε_k small enough $(\bar{x}_{\varepsilon_k}, \bar{t}_{\varepsilon_k}) \in F(P_\varepsilon)$, hence

$$\tilde{f}_{\varepsilon_k}(x(\varepsilon_k), t(\varepsilon_k)) \le \tilde{f}_{\varepsilon_k}(\bar{x}_{\varepsilon_k}, \bar{t}_{\varepsilon_k}). \tag{5.2}$$

Let us study both sides of this inequality. We first give a lower estimate of the left-hand side. We have

$$\tilde{f}_{\varepsilon_k}(x(\varepsilon_k), t(\varepsilon_k)) = \frac{t(\varepsilon_k)}{\varepsilon_k} - \frac{\alpha}{\sqrt{\varepsilon_k}} + \sum_{i\in I_0^0}\left(\frac{\gamma_i}{x_i^0(\varepsilon_k)} + \frac{t(\varepsilon_k)\omega_i}{x_i^0(\varepsilon_k)}\right) + \sum_{i\notin I_0^0}\frac{\gamma_i + t(\varepsilon_k)\omega_i}{x_i^0(\varepsilon_k)}$$

$$= \frac{1}{\sqrt{\varepsilon_k}}\left[\tau(\varepsilon_k) + \sum_{i\in I_0^0}\frac{\gamma_i}{d_i^0(\varepsilon_k)} - \alpha\right] + \sum_{i\in I_0^0}\frac{\tau(\varepsilon_k)\omega_i}{d_i^0(\varepsilon_k)} + \sum_{i\notin I_0^0}\frac{\gamma_i + t(\varepsilon_k)\omega_i}{x_i^0(\varepsilon_k)}.$$

(The notations $\tau(\varepsilon_k)$ and $d(\varepsilon_k)$ are those used in §3.)

Since $(d(\varepsilon_k), \tau(\varepsilon_k)) \in F(U)$, choosing $\alpha = v(U)$ we have

$$\tilde{f}_{\varepsilon_k}(x(\varepsilon_k), t(\varepsilon_k)) \ge \sum_{i\in I_0^0}\frac{\tau(\varepsilon_k)\omega_i}{d_i^0(\varepsilon_k)} + \sum_{i\notin I_0^0}\frac{\gamma_i + t(\varepsilon_k)\omega_i}{x_i^0(\varepsilon_k)}. \tag{5.3}$$

Passing to the limit and using Lemma 4.2, we obtain

$$\liminf_{\varepsilon_k\downarrow 0} \tilde{f}_{\varepsilon_k}(x(\varepsilon_k), t(\varepsilon_k)) \ge \sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0} + \sum_{i\notin I_0^0}\frac{\gamma_i}{\tilde{x}_i^0} = \sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0} + v(R). \tag{5.4}$$

Now, consider the right-hand side of (5.2); again with $\alpha = v(U)$ we have

$$\tilde{f}_{\varepsilon_k}(\bar{x}_{\varepsilon_k}, \bar{t}_{\varepsilon_k}) = \frac{\bar{t}_{\varepsilon_k}}{\varepsilon_k} - \frac{v(U)}{\sqrt{\varepsilon_k}} + \sum_{i\in I_0^0}\frac{\gamma_i}{(\bar{x}_{\varepsilon_k})_i^0} + \sum_{i\in I_0^0}\frac{\bar{t}_{\varepsilon_k}\omega_i}{(\bar{x}_{\varepsilon_k})_i^0} + \sum_{i\notin I_0^0}\frac{\gamma_i + \bar{t}_{\varepsilon_k}\omega_i}{(\bar{x}_{\varepsilon_k})_i^0}$$

$$= \frac{1}{\sqrt{\varepsilon_k}}\left[\tau^* + \sum_{i\in I_0^0}\frac{\gamma_i}{(d^*)_i^0} - v(U)\right] + \sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0} + \sum_{i\notin I_0^0}\frac{\gamma_i + \bar{t}_{\varepsilon_k}\omega_i}{(\bar{x}_{\varepsilon_k})_i^0}.$$

Since $(\tau^*, d^*) \in S(U)$ and $(\bar{x}_{\varepsilon_k}, \bar{t}_{\varepsilon_k})$ converges to the partial center $(\tilde{x}, 0)$, the last equality becomes

$$\tilde{f}_{\varepsilon_k}(\bar{x}_{\varepsilon_k}, \bar{t}_{\varepsilon_k}) = \sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0} + \sum_{i\notin I_0^0}\frac{\gamma_i + \bar{t}_{\varepsilon_k}\omega_i}{(\bar{x}_{\varepsilon_k})_i^0} = \sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0} + v(R) + o(1).$$

Passing to the limit and using (5.2), we have

$$\limsup_{\varepsilon_k\downarrow 0} \tilde{f}_{\varepsilon_k}(x(\varepsilon_k), t(\varepsilon_k)) \le \sum_{i\in I_0^0}\frac{\tau^*\omega_i}{(d^*)_i^0} + v(R). \tag{5.5}$$

Combining (5.2), (5.4), and (5.5), we obtain the result. □

6. Extensions. In this section we consider a special path $\{(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon)) \in S(P_\varepsilon)\}$ converging to some given partial center $(\tilde{x}, 0)$. For this special path we establish an asymptotic expansion near 0 of all components of the solutions and not only x^0.

Let $(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon))$ be defined by

$$\tilde{t}(\varepsilon) := \operatorname*{argmin}_{(x,t)\in S(P_\varepsilon)} t \tag{6.1}$$

and

$$\tilde{x}(\varepsilon) := \operatorname*{argmin}_{(x,\tilde{t}(\varepsilon))\in S(P_\varepsilon)} g(x) := \sum_{1\le k\le K} \sum_{i\in I} (x_i^k - \tilde{x}_i^k)^2. \tag{6.2}$$

LEMMA 6.1. *$\{(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon))\}$ is well defined and* $\lim_{\varepsilon\downarrow 0} \tilde{x}(\varepsilon) = \tilde{x}$.

PROOF. Since $S(P_\varepsilon)$ is closed and bounded, $\tilde{t}(\varepsilon)$ exists and is unique. Then, the strict convexity of f_ε (resp. g) with respect to the components in I^0 (resp. in I^k, $1 \le k \le K$) implies the uniqueness of $\tilde{x}(\varepsilon)$. Furthermore, it is already proved in Lemma 3.3 that $\{(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon))\}$ is bounded and all its limit points belong to $S(R)$.

Let $(\bar{x}, 0) = \lim_{k\to+\infty} (\tilde{x}(\varepsilon_k), \tilde{t}(\varepsilon_k))$ for some sequence $\{\varepsilon_k\}$ converging to 0. Then

$$(\tilde{x}, 0) = \lim_{k\to+\infty} (x(\varepsilon_k), t(\varepsilon_k)), \text{ where } (x(\varepsilon_k), t(\varepsilon_k)) := (\tilde{x}(\varepsilon_k), \tilde{t}(\varepsilon_k)) + (\tilde{x}, 0) - (\bar{x}, 0).$$

Since $\bar{x}^0 = \tilde{x}^0$, we have

$$t(\varepsilon_k) = \tilde{t}(\varepsilon_k) \quad \text{and} \quad \bar{x}^0(\varepsilon_k) = \tilde{x}^0(\varepsilon_k).$$

Then (6.2) gives

$$\sum_{1\le k'\le K} \sum_{i\in I} (\tilde{x}_i^{k'}(\varepsilon_k) - \tilde{x}_i^{k'})^2 \le \sum_{1\le k'\le K} \sum_{i\in I} (x_i^{k'}(\varepsilon_k) - \tilde{x}_i^{k'})^2.$$

Passing to the limit we obtain that $\bar{x} = \tilde{x}$. $\square$

Similarly to the previous section let us set

$$d(\varepsilon) := \varepsilon^{-1/2}(\tilde{x}(\varepsilon) - \tilde{x}), \quad \tau(\varepsilon) := \varepsilon^{-1/2}\tilde{t}(\varepsilon).$$

By Theorem 4.5, $\tilde{t}(\varepsilon) = \sqrt{\varepsilon}\tau^* + o(\varepsilon)$ and $\tilde{x}(\varepsilon)$ is a solution of the perturbed problem

$$(\tilde{PS}_\varepsilon) \quad \begin{cases} \operatorname*{Min}_x \sum_{1\le k\le K} \sum_{i\in I} (x_i^k - \tilde{x}_i^k)^2, & \\ \text{s.t. } Ax^k = b^k, & k = 1,\dots,K, \\ \gamma_i + \tilde{t}(\varepsilon)\omega_i - \sum_{1\le k\le K} x_i^k = x_i^0, & i = 1,\dots,p, \\ x \ge 0,\ x^0 = \tilde{x}^0 + d^{*0}\sqrt{\varepsilon} + o(\sqrt{\varepsilon}). & \end{cases}$$

The corresponding limit problem is

$$(\tilde{PS})\quad \begin{cases} \operatorname*{Min}_x \sum_{1\le k\le K} \sum_{i\in I} (x_i^k - \tilde{x}_i^k)^2, \\ \text{s.t. } Ax^k = b^k, \qquad k=1,\dots,K, \\ \quad \gamma_i - \sum_{1\le k\le K} x_i^k = x_i^0, \quad i=1,\dots,p, \\ \quad x \ge 0,\ x^0 = \tilde{x}^0. \end{cases}$$

REMARK. Note that we have $F(\widetilde{PS})\times\{0\} = S(R)$ and $S(\widetilde{PS}) = \{\tilde{x}\}$. Using Lemma 3.1, there exists $\hat{x}(\varepsilon)\in F(\widetilde{PS}_\varepsilon)$ such that

$$\operatorname{dist}(\hat{x}(\varepsilon),\tilde{x}) = O(\sqrt{\varepsilon})$$

and in particular

$$\sum_{1\le k\le K}\sum_{i\in I}(\hat{x}_i^k(\varepsilon) - \tilde{x}_i^k)^2 = O(\varepsilon).$$

This obviously implies that

$$v(\widetilde{PS}_\varepsilon) = \sum_{1\le k\le K}\sum_{i\in I}(\tilde{x}_i^k(\varepsilon) - \tilde{x}_i^k)^2 = O(\varepsilon),$$

and consequently

$$\operatorname{dist}(\tilde{x}(\varepsilon),\tilde{x}) = O(\sqrt{\varepsilon}),$$

or equivalently, the sequence $\{d(\varepsilon)\}_{\varepsilon>0}$ is bounded.

LEMMA 6.2. *$\tilde{x}(\varepsilon) = \tilde{x} + d^*\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$, with*

$$\{d^*\} = \operatorname*{argmin}_{d\in S(T_1)} \sum_{1\le k\le K}\sum_{i\in I}(d_i^k)^2. \tag{6.3}$$

PROOF. By Theorem 4.5, $\lim_{\varepsilon\downarrow 0}\tau(\varepsilon) = \tau^*$ and any limit point of $\{d(\varepsilon)\}$ belongs to $S(T_1)$. Let $\tilde{d}$ be one of these limit points and $\bar{d}$ be another element of $S(T_1)$. We can write, for some sequence $\{\varepsilon_k\}$ converging to 0, that

$$\tilde{d} = \lim_{k\to+\infty}\tilde{d}(\varepsilon_k) \quad\text{and}\quad \bar{d} = \lim_{k\to+\infty} d(\varepsilon_k) \quad \text{with } d(\varepsilon_k) = \tilde{d}(\varepsilon_k) - \tilde{d} + \bar{d}.$$

Rewriting the definition of $\tilde{x}(\varepsilon_k)$, we obtain, since $\tilde{d}^0(\varepsilon_k) = \bar{d}^0(\varepsilon_k)$,

$$\sum_{1\le k\le K}\sum_{i\in I}(\tilde{d}_i^k(\varepsilon_k))^2 \le \sum_{1\le k\le K}\sum_{i\in I}(\bar{d}_i^k(\varepsilon_k))^2.$$

Passing to the limit we obtain the desired result. □

7. Conclusion. Until now all our results depend on the artificial variable ε and not on the perturbation parameter t. Fortunately, the parameter t is strongly linked to ε and we can reinterprete all our results concerning the problem (PR_t) using only t. Let x^* be a fixed partial center and denote, respectively, $x(t,\omega)$ and $v(t,\omega)$ the nearest solution of (PR_t) to x^* and the optimal value of (PR_t).

We first prove some technical lemmas and then state the main result of this section.

LEMMA 7.1. *Let* $\varepsilon > 0$ *small enough and* $(x(\varepsilon), t(\varepsilon)) \in S(P_\varepsilon)$. *Then*

$$v(t(\varepsilon), \omega) = \frac{\tau^*(v(U) - \tau^*)}{t(\varepsilon)} + v(R) + \sum_{i \in I_0^0} \frac{\tau^* \omega_i}{(d^*)_i^0} + o(1),$$

$$x_{I_0^0}^0(t(\varepsilon), \omega) = \frac{t(\varepsilon)}{\tau^*}(d^*)_{I_0^0}^0 + O(t^2(\varepsilon)),$$

$$x(t(\varepsilon), \omega) = (x^*) + \frac{t(\varepsilon)}{\tau^*} d^* + o(t(\varepsilon)),$$

where $(d^*, \tau^*) \in S(U)$ *and* $d^* \in S(T_1)$.

PROOF. Since $x(t(\varepsilon), \omega) = x(\varepsilon)$, these expansions are direct consequences of Theorem 4.5 and Theorem 5.1. □

LEMMA 7.2. *The function* $v(., \omega)$ *is decreasing.*

PROOF. Let t and t' be two real numbers such that $0 < t < t'$. If x is an optimal solution for (PR_t), we can define a feasible point x' for $(PR_{t'})$ by

$$(x')^k = x^k \quad \text{for } k \geq 1 \quad \text{and} \quad (x')^0 = x^0 + (t' - t)\omega.$$

Then, we have

$$v(t', \omega) \leq \sum_{i \in I} \frac{\gamma_i + t\omega_i + (t' - t)\omega_i}{x_i^0 + (t' - t)\omega_i} < \sum_{i \in I} \frac{\gamma_i + t\omega_i}{x_i^0} = v(t, \omega),$$

where we use the inequality $\dfrac{a + \delta}{b + \delta} < \dfrac{a}{b}$ if $a > b$ and $\delta > 0$. □

LEMMA 7.3. *Let* $\varepsilon > 0$ *small enough and* $(x(\varepsilon), t(\varepsilon)) \in S(P_\varepsilon)$, *then there exists a constant* $M > 0$ *such that*

$$\forall t > 0, \quad \text{if } t - t(\varepsilon) = o(\sqrt{\varepsilon}) \text{ then } |v(t, \omega) - v(t(\varepsilon), \omega)| \leq M \frac{|t - t(\varepsilon)|}{\varepsilon}.$$

PROOF. (i) We consider first the situation $t \geq t(\varepsilon)$. In this case, $v(t(\varepsilon), \omega) \geq v(t, \omega)$. Using the optimality of $(x(\varepsilon), t(\varepsilon))$, we have

$$t(\varepsilon) + \varepsilon v(t(\varepsilon), \omega) \leq t + \varepsilon v(t, \omega).$$

The two last inequalities yield obviously to

$$0 \leq v(t(\varepsilon), \omega) - v(t, \omega) \leq \frac{t - t(\varepsilon)}{\varepsilon}, \tag{7.1}$$

so that the conclusion is obtained with $M = 1$.

(ii) Suppose now that $t \leq t(\varepsilon)$, then $v(t(\varepsilon), \omega) \leq v(t, \omega)$ and define x by

$$x^k = x^k(\varepsilon) \quad \text{for } k \geq 1 \quad \text{and} \quad x^0 = x^0(\varepsilon) + (t - t(\varepsilon))\omega.$$

Since $x^0_{I_0^0}(\varepsilon) \approx \sqrt{\varepsilon}$ and $x^0_{\overline{I_0^0}} \approx (x^*)^0_{\overline{I_0^0}} > 0$, x is feasible for the problem (PR_t). Therefore,

$$v(t(\varepsilon),\omega) \le v(t,\omega) \le \sum_{i\in I} \frac{\gamma_i + t\omega_i}{x_i^0} \le \sum_{i\in I} \frac{\gamma_i + t(\varepsilon)\omega_i}{x_i^0}$$

$$\le \sum_{i\in I} \frac{\gamma_i + t(\varepsilon)\omega_i}{x_i^0(\varepsilon)} \left(1 - \frac{(t(\varepsilon)-t)\omega_i}{x_i^0(\varepsilon)}\right)^{-1}.$$

Using $\dfrac{t(\varepsilon)-t}{x_i^0(\varepsilon)} = o(1)$, we have

$$v(t(\varepsilon),\omega) \le v(t,\omega) \le v(t(\varepsilon),\omega) + 2\sum_{i\in I} \frac{(\gamma_i + t(\varepsilon)\omega_i)\omega_i |t(\varepsilon)-t|}{(x_i^0)^2(\varepsilon)}.$$

Since for ε sufficiently small $t(\varepsilon) \le 1$, we obtain

$$0 \le v(t,\omega) - v(t(\varepsilon),\omega) \le M\frac{|t-t(\varepsilon)|}{\varepsilon}$$

$$\text{for} \quad M \ge 2\frac{\max_{i\in I}\omega_i(\gamma_i+\omega_i)}{\min_{i\in I_0^0}((d^*)_i^0)^2}. \quad \square$$

Finally we can state the main result of this section.

THEOREM 7.4. *For $t > 0$ small enough, setting $\varepsilon = \dfrac{t^2}{(\tau^*)^2}$, we have*

(i) $v(t,\omega) = \dfrac{\tau^*(v(U)-\tau^*)}{t} + v(R) + \sum_{i\in I_0^0} \dfrac{\tau^*\omega_i}{(d^*)_i^0} + o(1)$, *and*

(ii) *if $(x, t(\varepsilon)) \in S(P_\varepsilon)$ then the vector $\tilde{x}$ defined by*

$$\tilde{x}^k = x^k \quad \text{for} \quad k \ge 1 \quad \text{and} \quad \tilde{x}^0 = x^0 + (t - t(\varepsilon))\omega$$

is $o(1)$ optimal for (PR_t), in the sense that it is feasible and that the associated cost is no more than $v(PR_t) + o(1)$.

PROOF. By Theorem 4.5, we have $t(\varepsilon) = \tau^*\sqrt{(\varepsilon)} + o(\varepsilon)$. Therefore $t - t(\varepsilon) = o(\varepsilon) = o(t^2)$. Combining with Lemma 7.1 i) and Lemma 7.3, we obtain i. For proving ii, one can follow exactly the second part of the proof for Lemma 7.3. $\square$

References

Attouch, H. 1984. *Variational convergences for functions and operators.* Pitman Advanced Publishing Program, Boston, MA.

Auslender, A., R. Cominetti, M. Haddou. 1997. Asymptotic analysis for penalty and barrier methods in convex and linear programming. *Math. Oper. Res.* **22** 43–62.

Bonnans, J. F., A. Ioffe. 1995. Quadratic growth and stability in convex programming problems with multiple solutions. *J. Convex Anal.* **2** 41–57.

——, G. Launay. 1991. On the stability of sets defined by a finite number of equalities and inequalities. *J. Optim. Theory Appl.* **70** 417–428.

——, A. Shapiro. 1998. Optimization problems with perturbations, a guided tour. *SIAM Rev.* **40** 202–227.

Carroll, C. W. 1961. The created response surface technique for optimizing nonlinear restrained systems. *Oper. Res.* **9** 169–184.

Chifflet, J., P. Mahey, V. Reynier. 1994. Proximal decomposition for multicommodity flow problems with convex costs. *Telecommunication System* **3** 1–10.

Fiacco, A. V. 1997. *Mathematical programming with perturbations.* Marcel Dekker, New York.

den Hertog. D., C. Roos, T. Terlaky. 1994. Inverse barrier methods for linear programming. *R.A.I.R.O. Rech. Oper.* **28** 135–163.

Gonzaga, C. C. 1992. Path following methods for linear programming, *SIAM Rev.* **34** 167–227.

Hoffmann, A. 1952. On approximate solutions of systems of inequalities. *J. Res. National Bureau Standards, Sect. B* **49** 263–265.

Kleinrock, L. 1964. *Communications, Nets, Stochastic message flow and delay*. McGraw Hill, New York.

Nesterov, Y., A. Nemirovski. 1994. Interior-point polynomial algorithms in convex programming. *SIAM Publications*, Philadelphia.

Shapiro, A. 1988. Perturbation theory of nonlinear programs when the set of optimal solutions is not a singleton. *Appl. Math. Optim.* **18** 215–229.

Torralba, D. 1996. Développement asymptotiques pour les méthodes d'approximation par viscosité. *C.R. Acad. Sci. Paris* Série I **t. 322** 123–128.

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